Advances in Mathematics: Scientific Journal **12** (2023), no.5, 563–575
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.12.5.3

GLOBAL SYMMETRY AND ASYMMETRY IN THE NATURE

Ravelonirina Hanitriniaina Sammy Grégoire[1] and Rakoto Jean Jolly

ABSTRACT. Living or non-living beings have common characteristics of symmetry and asymmetry in the broad sense of the term. Any natural or artificial object has these common properties. The transfer function and the matrix evolution make it possible to have a rational interpretation regarding the existence of the ”global” symmetry and the stability of any object be it natural or artificial, living or non-living. Symmetry and asymmetry play important roles in human perception (decoration of objects). They restore the stability of object as a solid base for geometric structures. Therefore, this stability gives the fortification of these.

1. INTRODUCTION

Symmetry and asymmetry are everywhere in the natural world. Natural or artificial objects have a globally symmetric form. Others have an asymmetric form. The symmetry (which has been for a long time associated with harmonious and equilibrium as observed in the concept of pythagorian plilosophy and classical arts) is especially found in architectural works in several civilizations: egyptian and aztec pyramids, greek temples, cathedrals, french gardens, Tadj Mahal,...cf [16]. In addition, geometry allows us to have a knowledge about an object's shape,

[1]*corresponding author*
2020 *Mathematics Subject Classification.* 35B06, 34D20, 33E99, 51Fxx, 51K99, 97G40, 44A12.
Key words and phrases. Global symmetry and/or asymmetry, Stability, Transfer function, Evolution or transfer matrix, Routh-Hurwitz criterion.
Submitted: 21.04.2023; *Accepted:* 07.05.2023; *Published:* 26.05.2023.

size and structure cf. [8]. The notion which will serve as the link between this geometric vision and a precise definition of topological notions would be the idea of distance cf. [14]. However, we sometimes encounter examples of metric spaces, that is, sets on which we have defined a means of measuring the distance for which a geometric vision is more delicate see [12] and [14]. The aim of the current study is to bring some rational explanations to this phenomenon of symmetry and asymmetry which are found everywhere in any object in the universe. But does a symmetry really exist in all of these objects of the nature: animals, plants, bacteria and funguses in the mathematical sense of the term? This question, very rich in meaning, deserves to have appropriate answers that we will try to provide some answers with. For this, we propose geometric and algebraic approaches.

2. Preliminaries

2.1. **Notion of metric space cf. [14].** So far, we know that the word ”distance” has two meanings. In the first sense of the term, it refers to the number which measures the space between two points, like the distance from A to B for instance, and this number is always positive. In the second sense, it means a function of $E \times E$ in $\mathbb{R}_+$ which makes it possible to measure the distance between any two points of E where E is a non-empty set.

In this study we will use both meanings of the word ” distance ” we have just defined.

2.2. **Some definitions.**

Definition 2.1. *[12] A metric space is a non-empty set on which we have defined a distance.*

Definition 2.2. *Let X be a non-empty set. In this case, a metric distance on X is a map*

$$\begin{aligned} d \ : \ & X \times X \to \mathbb{R}_+ \\ & (x, y) \to d(x, y) \textit{ such that} \end{aligned}$$

(1) $d(x, y) = 0 \Longleftrightarrow x = y$
(2) $d(x, y) = d(y, x), \ \forall x, y \in X$
(3) $d(x, y) \leq d(x, z) + d(z, y), \ \forall x, y, z \in X$ *(triangular inequality).*

Remark 2.1. *[14] There are three forms of distances:*

(1) *the Euclidean distance that we note in $\mathbb{R}^n$,*

$$d[(x_1, x_2, \ldots, x_n)(y_1, y_2, \ldots, y_n)] = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

(2) *the uniform distance noted*

$$d[(x_1, x_2, \ldots, x_n)(y_1, y_2, \ldots, y_n)] = Sup(|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|).$$

(3) *and a third distance, it is the Euclidean one that we will use in the next sections, such as*

$$d[(x_1, x_2, \ldots, x_n)(y_1, y_2, \ldots, y_n)] = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|.$$

Definition 2.3. *[12] Let (X, d) be a metric space. For $x \in X$ and $r > 0$, we define:*

(1) *the open ball of center x and radius r: $\mathcal{B}(x, r) = \{y \in X,\ d(y, x) < r\}$;*
(2) *the closed ball of center x and radius r: $\overline{\mathcal{B}}(x, r) = \{y \in X,\ d(y, x) \leq r\}$;*
(3) *the sphere of center x and radius r: $\mathcal{S}(x, r) = \{y \in X,\ d(y, x) = r\}$*

Definition 2.4. *[12] Two distances d_1 and d_2 are comparable if there exist two real numbers $\alpha > 0$ and $\beta > 0$ satisfying $\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y)$. We say that d_1 and d_2 are equivalent.*

Definition 2.5. *[12] An application f from a metric space (E, d) to a metric space (F, d) is called contractive if there exists a real constant $0 < k < 1$ such that $\forall (x, y) \in E \times E,\ d(f(x), f(y)) \leq kd(x, y)$.*

Definition 2.6. *[12] A map f from a metric space (E, d) to a metric space (E', d') is called Lipschitz if and only if there exists $k > 0$ such that $\forall (x, y) \in E \times E$, $d'(f(x), f(y)) \leq kd(x, y)$.*

Remark 2.2. *[12] A contracting map is a Lipschitz map with a constant strictly inferior to 1.*

2.3. Symmetry and asymmetry in the natural world.

2.3.1. *Center of symmetry and axis of symmetry.* In the strict sense of the term,

Definition 2.7. *[18] Two figures are symmetrical with respect to a point when they are superimposed by a half-turn around this point. It is the center of symmetry of two figures.*

Definition 2.8. *[18] Two figures are symmetrical with respect to a straight line when they overlap by folding around that line which is nothing else but the axis of symmetry.*

2.3.2. *Global symmetry and dissymmetry.*

- Given definition 3.1, the two figures show a difference in size δ such that $0 \leq \delta < 1$. We say that there is a symmetry Global with respect to a straight line (axis of symmetry). In other words, let F_1 and F_2 be two symmetrical figures with respect to a straight line $(\mathcal{D})$. if the distance $|F_1 - F_2| = \delta$ with $0 \leq \delta < 1$ then F_1 and F_2 are globally symmetric with respect to $(\mathcal{D})$.
- Two figures are said to be globally symmetric with respect to a point when they are superimposed, they show a difference of size δ such that $0 \leq \delta < 1$. In other words, if F_1 and F_2 are the two figures and I a point satisfying $|F_1 - F_2| = \delta$ with $0 \leq \delta < 1$ then F_1 and F_2 are globally symmetric with respect to I.

Remark 2.3.

- *If* $\delta = 0$ *then we have perfect symmetry.*
- *If* $0 < \delta < 1$*, we have global symmetry.*
- *If* $\delta \geq 1$ *then we have a total dissymmetry or an asymmetry.*

3. Symmetry in the natural world

Geometrically speaking, there are approximately two shapes of object in nature: the elongated shape and the round shape. Both of them have symmetric elements which are axis of symmetry and/or center of symmetry. More precisely, the longitudinal shape globally has an axis of symmetry while the round shape a center of symmetry.

3.1. **Longitudinal and rounded shape.**

Definition 3.1. *[18] We say that an object is of a longitudinal shape if it geometrically has more or less a rectangular form, and in the same way, an object is of round shape if it is approximatively circular. This definition is based on a global vision of form and not in the strict sense of the term.*

Proposition 3.1. *Any object in the world has a globally symmetrical shape. In particular, an object of longitudinal shape has an axis of symmetry and that of a round shape a center of symmetry.*

Proof.

- 1^{st} Case: Longitudinal object
  Suppose that F is a natural or artificial object (plant, animal, bacteria, funguses, etc.) with an elongated shape. Let $\mathcal{D}$ be an imaginary line dividing F into two parts F_1 and F_2. According to definition 2.8, we superimpose these two figures by folding around $\mathcal{D}$, they show a different size δ where $0 \leq \delta < 1$. Beside, we suppose that the object is assimilated to a collapsible plan or surface. Indeed, if we designate by P_{id} the points on the right of $\mathcal{D}$ with $P_{id} \in F_1$ and, P_ig the points on the left of $\mathcal{D}$ where $P_{ig} \in F_2$ then we have $|F_1 - F_2| = |\sum_{i=1}^{n} d(P_id, \mathcal{D})^2 - \sum_{i=1}^{n} d(P_ig, \mathcal{D})^2| = \delta$. Since $\delta \in [0, 1[$ then according to the paragraph 2.3.2, $\mathcal{D}$ is globally axis of symmetry. So the object F presents a global symmetry.
- 2^{nd} Case: Rounded-shape objects
  Suppose that F is an object of rounded-shape, O is an imaginary point located approximately at the center of F such that F is assimilated to a pliable plan or surface. We superimpose F_1 and F_2 by half-turn around the point O with $F_1 + F_2 = F$. We obtain a difference of size δ with $0 \leq \delta < 1$. Let M_i, $(1 \leq i \leq n)$ be the set of points of F. $d(M_i, O) = r + \delta$, where r is a positive constant. Since $\delta \in [0, 1[$, F admits a center of global symmetry O.

□

Example 1.

(1) *For plant leaves, the petiole and the midrib can be considered as the axis of symmetry of the leaves.*
(2) *A butterfly has an axis of symmetry.*
(3) *The Vitruvian man (in 1492, Leonardo da Vinci illustrated the symmetry of the human body in one of his most famous drawings) cf. [7] and [13].*
(4) *Vertebrates (a frog for instance has an axis of symmetry and axis of polarity), cf. [17].*

(5) *The male fiddler crab cf. [17].*
(6) *Fruits (orange, tangerine), etc.*

3.2. **Complementarity law.** All living beings naturally need a supplement to be stable, to reproduce and to fully live in harmony with the nature.

Example 2. *A male needs a female and vice versa. A living being needs food to survive.*

Proposition 3.2. *Any living being possessing global symmetry generates or gives birth to a globally symmetrical being. In particular, a being which has an axis of symmetry gives birth to a being which has a center of symmetry and vice versa.*

Proof. It is from the complementarity law. □

3.3. **Consequence of globally symmetrical shape of living beings in mobility.** Any mobile beings having an axis of global symmetry move in a straight way. Their center of gravity is situated on their axis of symmetry only when their mass is equally distributed. For those beings having a center of symmetry, they move by rotation around their center.

4. Stability of objects in the nature

Any object in the world must be stable in order to be able to exist. In the next section, we will study the stability of objects that exist in the world, which we assume to be a physical, linear and continuous system. The modern theory of the system requires the notion of state variable.

Definition 4.1. *A multivariable system cf. [2] has several inputs* (m) *and outputs* (p)*. These variables constitute respectively the coordinates of the input vector* u *and the output vector* y*. We have*

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}.$$

Definition 4.2. *We call a state model associated with a system (S) a multivariable equation of the form:*

$$(S) : \begin{cases} c\dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} , \quad \textit{where } A, B, C \textit{ and } D \textit{ are matrix.}$$

Schematically, by denoting $dim(E)$ the dimension of E, the system denoted by (S) is represented as follows cf. [2] and [3]:

$$\begin{array}{ccccc} \boxed{\dim(U) = m} & \longrightarrow & \boxed{(S)} & \longrightarrow & \boxed{\dim(y) = p} \\ & & \downarrow & & \\ & & \boxed{\dim(x) = n} & & \end{array}$$

Definition 4.3. *We call input/output model, the linear differential equation of constant coefficients linking the input and the output, which is written in the following form cf. [2]:*

$$\begin{aligned} & a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \dot{y}(t) + a_0 y(t) \\ = \; & b_m \frac{d^m u(t)}{dt^m} + a_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \cdots + b_1 \dot{u}(t) + b_0 u(t) \end{aligned} , \tag{4.1}$$

with $m \leq n$ (Causal system).

4.1. **Transfer function.** cf. [2] We obtain the transfer function from the differential equation of the input/output model, by Laplace transform and is written:

$$G(p) = \frac{N(p)}{D(p)} = \frac{b_m p^m + b_{m-1} p^{m-1} + \cdots + b_1 p + b_0}{a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0}, \tag{4.2}$$

$G(p)$ is a rational fraction depending on the Laplace variable, with numerator $N(p)$ and denominator $D(p)$ of constant coefficients respectively, with $degN(p) = m$, $degD(p) = n$.

5. Objects' formalization

Proposition 5.1. *If a natural or artificial object has m inputs and p outputs with $m \leq p$ (m, p nonzero integers; strictly superior to 1) then it forms a continuous and multivariable linear system.*

Proof. We just use the definitions 4.1 and 4.2. □

Proposition 5.2. *Any natural or artificial object which has an input/output model has a transfer function of type 4.2.*

Proof. It is obtained by applying the Laplace transform to the differential equation in Definition 4.3. □

5.1. **Case of living beings.** In this study, we limit the value of m and n respectively by 2 and 3 in equation 4.2 of the transfer function.

Let us take for example a living being whose input is: water or oxygen or CO_2 from the air, mineral salts from the soil, food, etc.; and output: waste, CO_2, water, etc.

In the phenomenon of plant photosynthesis, we have cf. [1] and [15]:

$$6CO_2 + 12H_2O \longrightarrow C_6H_{12}O_6 + 6O_2 + 6H_2O.$$

For $m = 2$ and $n = 3$, the transfer function is written:

$$G(p) = \frac{N(p)}{D(p)} = \frac{b_2p^2 + b_1p + +b_0}{a_3p^3 + a_2p^2 + a_1p + a_0}.$$

We assume that the polynomial $D(p) = a_3p^3 + a_2p^2 + a_1p + a_0$ satisfies the Routh-Hurwitz criterion for polynomial stability.

5.1.1. *Determination of the state matrix (or evolution matrix).* Let $G(p)$ be the transfer function of the system under consideration, i.e.

$$G(p) = \frac{b_2p^2 + b_1p + +b_0}{a_3p^3 + a_2p^2 + a_1p + a_0}.$$

We decompose into simple elements $a_3p^3 + a_2p^2 + a_1p + a_0$:

1^{st} **case:** If $G(p)$ only has simple poles then we obtain

$$G(p) = \frac{A_1}{p-\lambda_1} + \frac{A_2}{p-\lambda_2} + \frac{A_3}{p-\lambda_3}$$

where $A_i = (p-\lambda_i)G(p)$ with $i = 1, \ldots, 3$. For $p = \lambda_1,\ \lambda_2,\ \lambda_3$, we have the state matrix $A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$.

2^{nd} **case:** If $G(p)$ has simple poles and a multiple pole of multiplicity order 3 then we have: $G(p) = \dfrac{A_1}{(p-\lambda_1)^3} + \dfrac{A_2}{(p-\lambda_1)^2} + \dfrac{A_3}{p-\lambda_1}$. The evolution matrix A is written $A = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$.

Example 3. *The transfer function is given by:*

$$G(p) = \frac{p^2 - p - 7}{p^3 - 7p + 6} = \frac{1}{p-1} + \frac{3}{5(p-2)} + \frac{2}{5(p+3)}.$$

Therefore, the state matrix becomes $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$.

Remark 5.1. *Only the evolution matrix determines the stability of the continuous linear system.*

Proposition 5.3. *For a continuous linear system to be stable and globally symmetric, the eigenvalues of the state matrix of this system must have all the roots with a real negative part.*

Proof. Let $G(p) = \frac{b_2p^2+b_1p+b_0}{a_3p^3+a_2p^2+a_1p+a_0}$ be the transfer function of the system (S). Suppose that (S) is globally symmetric. Then the polynomial $a_3p^3 + a_2p^2 + a_1p + a_0$ must satisfy the Routh-Hurwitz criterion for polynomial stability, cf. [9] and [10]. The decomposition into simple elements of $G(p)$ is written $G(p) = \frac{A_1}{p+\lambda_1} + \dfrac{A_2}{p+\lambda_2} + \dfrac{A_3}{p+\lambda_3}$ with $\lambda_1,\ \lambda_2,\ \lambda_3$ being strictly positive constants. The evolution matrix of the system (S) is therefore $A = \begin{pmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_3 \end{pmatrix}$.
By calculating the characteristic polynomial of A we have $\chi_A(t) = det(A - tI_3)$, where $det(M)$ denotes the determinant of the square matrix M.

Therefore, all eigenvalues of A have all roots with a real negative part. This demonstrates that the system (S) is stable. □

Example 4. *Let us consider* (E) *the differential equation governing the system* (S)*:*

$$(E) : \frac{d^3y(t)}{dt^3} - 7\frac{dy(t)}{dt} + 6y(t) = \frac{d^2u(t)}{dt^2} - \frac{du(t)}{dt} - 7u(t).$$

According to the Laplace transform, the associated transfer function $G(p)$ of the system (S) is

$$G(p) = \frac{2p^2 + p - 3}{p^3 + 6p^2 + 11p + 6} = \frac{2p^2 + p - 3}{(p+1)(p+2)(p+3)}.$$

The state matrix is therefore $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$.

Since the characteristic determinant of A is $\chi_A(t) = (-1-t)(-2-t)(-3-t)$ then the eigenvalues of A are $t_1 = -1 + 0i,\ t_2 = -2 + 0i,\ t_3 = -3 + 0i$.

We find that all the eigenvalues of the matrix A have their own real negative part. Hence, by proposition 5.3, the system (S) thus considered is stable and globally symmetric.

Proposition 5.4. *For any natural object, there is globally at least one element of symmetry and this can be an axis of symmetry, a center of symmetry or an axis of polarity.*

Proof.

1^{st} **case:** This is obvious for a globally symmetric object.

2^{nd} **case:** For a dissymmetrical object, there is always at least one symmetrical element or an axis of polarity as in the case of a frog. The result is obtained from the complementarity law 3.2 and according to Proposition 3.2.

□

Let us take for instance the case of a male fiddler crab. Even if it is dissymmetric, it still has a polarity axis and this proves that there is another symmetric element like the point situated between its two eyes, which is absolutely a point considered as a center of global symmetry.

Theorem 5.1. *Any globally symmetrical or dissymmetrical natural object is usually stable and vice versa.*

Proof. Suppose an object is globally symmetric. In accordance with Proposition 5.3, it is stable. Otherwise, the object is dissymmetric. Therefore, according to Propositions 3.2 and 5.4, there is at least one symmetric element, hence the stability.

Conversely, suppose the object is stable. So there is a symmetric element. Consequently, it is globally either symmetric or dissymmetric. This ends the proof. □

Theorem 5.2. *Any artificial geometric object globally has a symmetric element.*

Proof. This is from Theorem 5.1, because symmetry gives stability to geometric structures. □

Example 5. *In modern architecture, every architect shapes wood, iron, stone, plastic materials, etc.; following the geometric shapes to ensure stability. Thus, any geometric shape has at least one element of symmetry, whether axis of symmetry, or center of symmetry or axis of polarity.*

5.2. **Main roles of global symmetry and asymmetry in the world.** Living or non-living beings all have common characteristics such as global symmetry and/or asymmetry. For living or non-living organisms, these common characteristics are not limited to their external organization only but also involve all their levels of organization even down to the molecules that constitute them. Global symmetry and asymmetry play important roles in human perception (decoration, artistic value) cf [17]. They allow objects to give structures stability and therefore fortification cf [8].

6. Conclusion

The notion of global symmetry, dissymmetry and their exploitation is of great importance. The continuous progress recorded in scientific knowledge and natural laws makes it possible to have an interpretation extensive enough regarding the notion of basic symmetry in the universe where we inhabit cf. [11]. During this study, we have provided a rational explanation of the existence of global symmetry and/or asymmetry in any natural or artificial object and their roles in the world in which we are living.

References

[1] M.L. Amor: *Cours de Biochimie végétale*, Université Ferhat Abbas-Setif 1, Faculté des Sciences de la nature et de la vie, Département de Biologie et écologies végétales, 36 pages.

[2] O. Bachelier: *Cours d'automatique, Représentation d'état linéaires des systèmes monovariables*, $2^{\text{ème}}$ Année ENSIP, Parcours MEE Université Poitier, France, (28 juin 2017), 171 pages.

[3] O. Bachelier: *Philosophie de Larminat, Automatique: Commande des systèmes linéaires*, Edition Hermes, 1993.

[4] M. Benidir: *Caractérisation des polynômes de Hurwitz et de Schur complexes*, I-Revues, Traitement du signal, **5**(3) (1988), 153–159.

[5] A. Brack: *L'asymétrie du vivant*, EPSciences, Paris, 1999.

[6] A. Brack et al.: *La symétrie aujourd'hui*, Edit. du Seuil Inédit Sciences, Paris, 1989.

[7] P. Carouthers: *The history and Influence of Da Vinci's Vitruvian Man*, Art and Object, AT LARGE (August 17, 2020). https://www.artandobject.com/news/history-and-influence-da-vinci's vitruvian-man. consulted on March 16, 2023.

[8] G. Dhont, B. Zhilinskii: *Symétrie dans la nature*, Saint-Martin d'Hères (Isère): Presse Universitaire de Grenoble, (2011), 202 pages.

[9] C. Fonte, C. Delattre: *Un critère simple de stabilité polynomiale*, $6^{\text{ème}}$ Conférence Internationale Francophone d'Automatique, CIFA, Nancy France, Juin 2010.

[10] D.R. Merkin: *Introduction to the theory of stability*, Springer-Verlag, 1997.

[11] I. Mihai: *Structures symétriques en milieu géographique*, Revue géographique des Pyrénées et du Sud-Ouest, Toulouse, **61**(1) (1990), 155–161.

[12] P. Mironescu: *Cours de topologie métrique*, 2015.

[13] J. Philibert: *La symétrie dans la nature, dans la science et dans l'art*, Documents, Méthodes et références pédagogiques, Mathématiques. Université Paris Sud Orsay, 61 pages.

[14] J. Pichon: *Topologie dans $\mathbb{R}^n$, Fonctions de plusieurs variables, Ellipses*, Math-Supérieures et $1^{\text{ère}}$ année univ. Lycée Saint Louis (Paris).

[15] M1. Phytem-ENS Cachan: *Introduction à la biologie*, (2013-2014), 81 pages.

[16] J. Sivardière: *Histoire de la symétrie*, Bulletin de l'union des physiciens (Association des professeurs de Physique et de Chimie), CEA/ Département de Recherche Fondamentale sur la matière condensée, SPSMS/MRS 38054 Grenoble, Cedex 9.

[17] M. Thiry: *La symétrie du vivant: des organismes aux molécules*, Communication présentée au Colloque, le 28 novembre 2014, Bulletin de la Société Royale des Sciences de Liège, Université de Liège, **83** (2014), 93–101.

[18] https://jacques-prevet.etab.ac-caen.fr/IMG/pdf/_COURS_ELEVE_Symetrie_Centrale.pdf.

Department of Mathematics and Computer Science
University of Antananarivo
Antananarivo 101,
Madagascar.
Email address: rhsammy@yahoo.fr

Department of Mathematics and Computer Science
University of University of Antananarivo
Antananarivo 101,
Madagascar.
Email address: rjeanjolly@gmail.com