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A STUDY ON t^2-SYMMETRIC RINGS

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ABSTRACT. In this article our attempt is to study the ring theoretic properties of t^2-symmetric and strongly t^2-symmetric rings of tripotent elements of a ring. Let R be a ring and t be a tripotent element of R, then R is said to be t^2-symmetric if $abc = 0$ implies $acbt^2 = 0$ for all $a, b, c \in R$. It has been proved that R is a t^2-symmetric ring if and only if t^2 is left semicentral and t^2Rt^2 is a symmetric ring. We also introduce the strongly t^2-symmetric ring and establish various properties of it.

1. INTRODUCTION

Throughout this article, all rings are associative with identity unless otherwise stated. Let R be a ring, we denote $Z(R)$ and $N(R)$ the centre and the set of all nilpotent elements of R respectively. Also $M_n(R)$ denotes the $n \times n$ upper triangular matrix ring over R. For a ring R, an element t is called tripotent if $t^3 = t$, the set of all tripotent elements is denoted by $T(R)$. Clearly, every idempotent is tripotent but the converse is not true. For example let $R = M_2(R)$, then $t = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is a tripotent element in R but not idempotent.

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A ring is usually called reduced if it has no nilpotent elements other than zero. Following Lambek [3], a ring R is called symmetric if $abc = 0$ implies $acb = 0$ for all $a, b, c \in R$. Later on, Anderson and Comillo [1], used the term ZC_3 for symmetric ring. The investigation of symmetric ring is also covered by G. Marks [4]. Ouyang et al. [7], generalised the concept of symmetric rings and they defined weak symmetric i.e., a ring R is said to be weak symmetric if $abc \in N(R)$ implies $acb \in N(R)$ for all $a, b, c \in R$. Another generalisation of symmetric rings is central symmetric rings, that is, a ring R is said to be central symmetric [2] if $abc = 0$ implies $bac \in Z(R)$ for any $a, b, c \in R$. Wei [8] introduces generalised weakly symmetric rings which further expands the idea of symmetric rings. According to Meng and Wei [5], a ring R is called (strongly) e-symmetric if $abc = 0$ implies ($aceb = 0$) $acbe = 0$, for any $a, b, c \in R$; e is an idempotent element of R and also they recently studied some important properties of it (see [6]).

In this paper, we extend and generalize the structure of e-symmetric rings defined by F. Meng et al. [5] using the concept of non-zero tripotent elements of the ring. The objective is to study and to define a new type of ring called t^2-symmetric ring using the concept of non-zero tripotent element in a ring. Also we introduce a strong condition on this notion and we call it strongly t^2-symmetric ring. And various properties of (strongly) t^2-symmetric rings are estblished.

2. t^2-Symmetric and Strongly t^2-symmetric Rings

In this section we introduce the t^2-symmetric and strongly t^2-symmetric rings and study some of its basic properties. We begin with the following definitions.

Definition 2.1. *Let R be a ring and $t \in T(R)$. Then,*

(1) *R is said to be t^2-symmetric if $abc = 0$ implies $acbt^2 = 0$ for all $a, b, c \in R$.*
(2) *R is called strongly t^2-symmetric if $abc = 0$ implies $act^2b = 0$ for all $a, b, c \in R$.*

From the above definition we have seen, whenever $t \in T(R)$ then t^2 must be an idempotent in R but t need not be an idempotent in R. Consider $t = -1$ then $t \in T(R)$, as $(-1)^3 = -1$ and since $((-1)^2)^2 = (-1)^2$, therefore $t^2 \in E(R)$, where $E(R)$ is the set of idempotent elements of a ring R but $t \notin E(R)$, as $(-1)^2 \neq -1$. Thus $t = -1 \in T(R)$ implies that $t^2 = (-1)^2 \in E(R)$ but $t = -1 \notin E(R)$.

Example 1. *Every symmetric ring is t^2-symmetric for any tripotent element t in R, but the converse need not be true. Let us consider $R = M_2(\mathbf{Z_3})$ where $\mathbf{Z_3} = \{-1, 0, 1\}$ is a reduced ring, since every reduced ring is also a symmetric ring by [1, Theorem 1.3], so $\mathbf{Z_3}$ is a symmetric ring. For $t = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \in T(R)$. Let*

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \textit{and} \quad c = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

are in R. Then $abc = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ implies that

$$acbt^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This shows that R is a $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}^2$-symmetric ring. But R is not a symmetric ring, as

$$acb = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Example 2. *Every strongly t^2-symmetric ring is also a t^2-symmetric ring for any tripotent element t of the ring. But the converse need not be true by Example 1, since in Example 1, we have*

$$act^2b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus R is not a strongly $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}^2$-symmetric ring, whereas R is a $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}^2$-symmetric ring.

Remark 2.1.

(1) *From the Definition 2.1, it is clear that for $t = -1, 1$; R is a symmetric ring if and only if R is a strongly $(-1)^2 = 1$ or $1^2 = 1$-symmetric ring if and only if R is a $(-1)^2 = 1$ or $1^2 = 1$-symmetric ring , as both -1 and 1 are in $T(R)$.*

(2) *Since every idempotent is also a tripotent but every tripotent need not be an idempotent. So let e be an idempotent in R then $e \in T(R)$. Again if $e \in T(R)$*

then e^2 is always an idempotent. Thus every e-symmetric ring [5] is also a e^2-symmetric. But every e^2-symmetric ring need not be an e-symmetric, because in Example 1, we have R is a $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}^2$-symmetric ring but R is not a $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$-symmetric ring, as $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \notin E(R)$.

Now we begin with the following results.

Theorem 2.1. *Let R be a ring and $t \in T(R)$. Then the following conditions are equivalent.*

(1) *R is a t^2-symmetric ring;*
(2) *t^2Rt^2 is a symmetric ring and t^2 is left semicentral.*

Proof.
$(1) \implies (2)$. Suppose R is a t^2-symmetric ring. Let $x \in R$ and $y = (1-t^2)xt^2+t^2$. Then $t^2y = t^2(1-t^2)xt^2 + t^2t^2 = t^2$, since $t \in T(R)$ this implies that $t^2 \in E(R)$, therefore $t^2t^2 = t^2$. Similarly $yt^2 = y$; $y^2 = y$; $t^2yt^2 = t^2$ and $(1-y)yt^2 = 0$. Since R is a t^2-symmetric ring, so $(1-y)t^2yt^2 = 0 \implies (1-y)t^2 = 0 \implies t^2 = yt^2 = y$. Thus $y = (1-t^2)xt^2 + t^2 \implies (1-t^2)xt^2 = 0 \implies xt^2 = t^2xt^2$. Hence t^2 is left semicentral.

Secondly to show t^2Rt^2 is a symmetric ring. Let $a, b, c \in t^2Rt^2$ such that $abc = 0$. Since t^2Rt^2 is a subring of R and R is a t^2-symmetric ring, so we have $acbt^2 = 0$. This implies that $acb = 0$, as $bt^2 = b$. Thus t^2Rt^2 is a symmetric ring.

$(2) \implies (1)$. Suppose the condition (2) holds. Let $a, b, c \in R$ such that $abc = 0$. Then $t^2at^2, t^2bt^2, t^2ct^2 \in t^2Rt^2$ and t^2Rt^2 is a symmetric ring, we have $t^2at^2t^2bt^2t^2ct^2 = 0$ implies $t^2at^2t^2ct^2t^2bt^2 = 0$. Thus $t^2at^2bt^2ct^2 = 0$ implies $t^2at^2ct^2bt^2 = 0$, as $t^2t^2 = t^2$. Since t^2 is left semicentral, so we have $t^2at^2ct^2bt^2 = 0 \implies at^2ct^2bt^2 = 0 \implies act^2bt^2 = 0 \implies acbt^2 = 0$. Thus $abc = 0 \implies acbt^2 = 0$. This shows that R is a t^2-symmetric ring. □

Theorem 2.2. *Let R be a ring such that $t \in T(R)$. Then the following conditions are equivalent.*

(1) *R is a strongly t^2-symmetric ring;*
(2) *t^2Rt^2 is a symmetric ring and $t^2 \in Z(R)$.*

Proof.

$(1) \implies (2)$. Suppose R is a strongly t^2-symmetric ring. For each $a \in R$, consider $x = t^2 + t^2a(1-t^2)$. Then $t^2x = t^2t^2 + t^2t^2a(1-t^2) = x$, as $t^2t^2 = t^2$. Similarly, $xt^2 = t^2$. Also, $x(1-x)t^2 = 0$ and since R is a strongly t^2-reversible ring, so we have $xt^2t^2(1-x) = 0 \implies t^2(1-x) = 0 \implies t^2 = t^2x = x$. This implies that $t^2a(1-t^2) = 0 \implies t^2a = t^2at^2$ for each $a \in R$. Since R is a strongly t^2-symmetric ring so, R is a t^2-symmetric ring. Thus by Theorem 2.1, we have t^2 is left semicentral. So we have $at^2 = t^2at^2$ for each $a \in R$. Therefore $t^2 \in Z(R)$. Again by Theorem 2.1, t^2Rt^2 is a symmetric ring.

$(2) \implies (1)$. Suppose the condition (2) holds. Let $a, b, c \in R$ such that $abc = 0$. Since t^2Rt^2 is a reversible ring. Thus from the second part of Theorem 2.1 we have, $t^2at^2ct^2bt^2 = 0$. Again since $t^2 \in Z(R)$ so, $t^2a = at^2$, $t^2b = bt^2$ and $t^2c = ct^2$ for each $a, b, c \in R$. This implies that $act^2b = 0$. Thus R is a strongly t^2-symmetric ring. □

As a consequence of Theorem 2.1 and Theorem 2.2, we have the following Corollary 2.1.

Corollary 2.1. *Let R be a ring and $t \in T(R)$. Then R is a strongly t^2-symmetric ring if and only if R is a t^2-symmetric ring and $t^2 \in Z(R)$.*

Proof. It directly follows from Theorem 2.1 and 2.2. □

Theorem 2.3. *Let R be a ring and $t \in T(R)$. Then following are equivalent.*

(1) *R is a symmetric ring;*
(2) *R is a t^2-symmetric and $(1-t^2)$-symmetric ring.*

Proof.

$(1) \implies (2)$. It is obvious.

$(2) \implies (1)$. Let the condition (2) holds. Let $a, b, c \in R$ such that $abc = 0$. Then $acb(1-t^2) = 0$, as R is a $(1-t^2)$-symmetric ring. This implies $acb = acbt^2$. Again R is a t^2-symmetric ring, so we have $acbt^2 = 0$. It follows that $acb = 0$. Thus R is a symmetric ring. □

Theorem 2.4. *Let R be a ring and $t \in T(R)$. Then following are equivalent.*

(1) *R is a symmetric ring;*
(2) *R is a strongly t^2-symmetric and $(1-t^2)R(1-t^2)$ is a symmetric ring.*

Proof.

$(1) \implies (2)$. It is obvious.

$(2) \implies (1)$. Let the condition (2) holds. Since $t \in T(R)$ so, t^2 is an idempotent. Again since R is a strongly t^2-symmetric ring, then by Theorem 2.2, $t^2 \in Z(R)$ and t^2Rt^2 is a symmetric ring. Since t^2 is a central idempotent, we have $t^2Rt^2 \cong R/(1-t^2)R(1-t^2)$ and $(1-t^2)R(1-t^2) \cong R/t^2Rt^2$. This implies $R/(1-t^2)R(1-t^2)$ and R/t^2Rt^2 are symmetric rings, as t^2Rt^2 and $(1-t^2)R(1-t^2)$ are symmetric rings. Thus $R/((1-t^2)R(1-t^2) \cap t^2Rt^2)$ symmetric ring. But $((1-t^2)R(1-t^2) \cap t^2Rt^2) = 0$, hence R is a symmetric ring. □

Extending [6, Proposition 4.1], we have the following Theorem 2.5.

Theorem 2.5. *Let R be a ring and $t \in T(R)$ and each $r \in R$. Then we have the following results:*

(1) *$M_2(R)$ is a $\begin{pmatrix} -1 & r \\ 0 & 0 \end{pmatrix}^2$-symmetric ring if and only if R is a symmetric ring, where $\begin{pmatrix} -1 & r \\ 0 & 0 \end{pmatrix} \in T(M_2(R))$.*

(2) *$M_2(R)$ is a $\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}^2$-symmetric ring if and only if R is a t^2-symmetric ring, where $\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \in T(M_2(R))$.*

(3) *$M_2(R)$ is a $\begin{pmatrix} t & t \\ 0 & 0 \end{pmatrix}^2$-symmetric ring if and only if R is a t^2-symmetric ring, where $\begin{pmatrix} t & t \\ 0 & 0 \end{pmatrix} \in T(M_2(R))$.*

Proof. (1) Suppose $M_2(R)$ is a $\begin{pmatrix} -1 & r \\ 0 & 0 \end{pmatrix}^2$-symmetric ring. Let $a, b, c \in R$ such that $abc = 0$. Then we have $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Since $M_2(R)$ is a

$\begin{pmatrix} -1 & r \\ 0 & 0 \end{pmatrix}^2$-symmetric ring, so we get,

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & r \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \implies \begin{pmatrix} acb & -acbr \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus we get, $acb = 0$. This implies R is a symmetric ring.

Conversely let R is a symmetric ring and $A = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}$, $B = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}$ and $C = \begin{pmatrix} a_3 & b_3 \\ 0 & c_3 \end{pmatrix} \in M_2(R)$ such that $ABC = 0$. This implies $a_1a_2a_3 = 0$ and $c_1c_2c_3 = 0$. Since R is symmetric ring so we have $a_1a_3a_2 = 0$ and $c_1c_3c_2 = 0$. Now

$$ACB\begin{pmatrix} -1 & r \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} a_1a_3a_2 & -a_1a_3a_2r \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $M_2(R)$ is a $\begin{pmatrix} -1 & r \\ 0 & 0 \end{pmatrix}^2$-symmetric ring.

(2). Since $M_2(R)$ is a $\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}^2$-symmetric ring so by part(1), we have

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \implies \begin{pmatrix} acbt^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}/$$

Thus we get, $acbt^2 = 0$. This implies R is a t^2-symmetric ring.

For converse part, since R is a t^2-symmetric ring, then by part (1) we have

$$ACB\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} a_1a_3a_2t^2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $M_2(R)$ is a $\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}^2$-symmetric ring.

Similarly we can prove (3). $\square$

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