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NEW BRANCH AND BOUND METHOD OVER A BOXED SET OF $\mathbb{R}^N$

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ABSTRACT. We present in this paper the new Branch and Bound method with new quadratic approach over a boxed set (a rectangle) of $\mathbb{R}^n$. We construct an approximate convex quadratics functions of the objective function to fined a lower bound of the global optimal value of the original non convex quadratic problem (NQP) over each subset of this boxed set. We applied a partition and technical reducing on the domain of (NQP) to accelerate the convergence of the proposed algorithm. Finally,we study the convergence of the proposed algorithm and we give a simple comparison between this method and another methods wish have the same principle.

1. INTRODUCTION

We consider the following non convex quadratic problem:

$$\text{(NQP)} \qquad \begin{cases} \min f(x) = \frac{1}{2}x^TQx + d^Tx \\ x \in S \cap (D_f) \end{cases}$$

where:

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$$\begin{aligned}
S &= \left\{x \in \mathbb{R}^n : L_i^0 \leq x_i \leq U_i^0 : i = \overline{1,n}\right\} \\
(D_f) &= \{x \in \mathbb{R}^n : Ax \leq b; x \geq 0\} \\
Q &: \text{ is a real } (n \times n) \text{ non positive symetric matrix} \\
A &: \text{ is a real } (n \times n) \text{ symetric matrix} \\
d^T &= (d_1, d_2, \ldots, d_n) \in \mathbb{R}^n \\
b^T &= (b_1, b_2, \ldots, b_m) \in \mathbb{R}^m
\end{aligned}$$

In our life, every things, every problems is create as a mathematic problems [5], we can also take the quotes of Gualili "The world is created at mathematical language or mathematical problems", specially "quadratic one".

In this paper we present a new rectangle Branch and Bound approach for solving non convex quadratic programming problems were we construct a lower approximate convex quadratic functions of the objective function f over a boxed set of $\mathbb{R}^n$ [2].

This lower approximate function is given to determine a lower bound of the global optimal value of the original problem (NQP) over each subrectangle.

The paper is organised as follows:

In section 1, we give a simple introduction of our studies; in which we give and define the standard form of our problem.

In section 2, we presente a new equivalent forms of the objective function proposed as un lower approximate linear functions of the quadratic form over each rectangle [6]. We can also proposed as an upper approximate linears functions, but we must respect the procedur of calculate the lower and the upper bound of the original principal rectangle S which noted by $S^k = \left[L^k, U^k\right] \subseteq \mathbb{R}^n$in the k-step [4].

In section 3, we define a new lower approximate quadratics functions of the non convex function over a rectangle to calculate a lower bound on the global optimal value of the original no convex problem (NQP) [7].

In section 4, we give a new simple rectangle partitioning method and describe rectangle reducing tactics [3].

In section 5, we present a new Branch and Reduce Algorithm in order to solve the original non convex quadratic problem (NQP).

In section 6, we study the convergence of the proposed Algorithm and we give a simple comparison between this method and other methods which have the same principle [1].

Finally, a conclusion is draw in section 7.

2. THE EQUIVALENT FORMS OF f OVER THE RECTANGLES

In this section we construct and define the equivalent form of the non convex quadratic function which proposed as a lower approximate linear functions over $S^k = \left[L^k, U^k\right]$. This work is proposed to determine the lower bound of the global optimal value of (NQP).

Let $\lambda_{\min}$ and $\lambda_{\max}$ be the min eigenvalue and the max eigenvalue of the matrix Q respectivelly, and we show the number θ that $\theta \geq |\lambda_{\min}|$.

The equivalent linear form of f is given by:

$$\begin{aligned} f(x) &= \left(x - L^K\right)^T (Q + \theta I)\left(x - L^K\right) + d^T x - \theta \sum_{i=1}^{n} x_i^2 \\ &+ 2\left(L^K\right)^T (Q + \theta I)\, x - \left(L^K\right)^T (Q + \theta I)\, L^K, \end{aligned}$$

by the use of the lower bound L^k, and is given by:

$$\begin{aligned} f(x) &= \left(x - U^K\right)^T (Q + \theta I)\left(x - U^K\right) + d^T x - \theta \sum_{i=1}^{n} x_i^2 \\ &+ 2\left(U^K\right)^T (Q + \theta I)\, x - \left(U^K\right)^T (Q + \theta I)\, U^K, \end{aligned}$$

by the use of the upper bound U^k of the rectangle S^k.

In the other hand, we have the following definitions:

Definition 2.1. *Let the function $f : C \subseteq \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ and $S^\circ \subseteq C \subseteq \mathbb{R}^n$ a rectangle, the convex envelope of the function f is given by:*

$$f_i(x_i) = \delta_i x_i + \eta_i : i = \overline{1, n}$$

with:

$$\begin{aligned}\delta_i &= \frac{f_i(U_i^\circ) - f_i(L_i^\circ)}{U_i^\circ - L_i^\circ} : i = \overline{1,n};\\ \eta_i &= f_i(L_i^\circ) - \delta_i L_i^\circ : i = \overline{1,n}.\end{aligned}$$

So, by the use of this definition the convex envelope of the function $h(x) = (-x_j^2)$ over the interval $S_j^k = \left[L_j^k, U_j^k\right]$ is given by the function:

$$\overline{h}(x) = -(U_j^k + L_j^k)x_i + L_j^k U_j^k,$$

which is a linear function, then we get the best linear lower bound of $h(x) = \sum_{j=1}^{n}(-x_j^2)$ given by:

$$\varphi_{S^k}(x) = \sum_{j=1}^{n}(-(U_j^k + L_j^k)x_i + L_j^k U_j^k) = -(U^k + L^k)^T x + (L^k)^T U^k.$$

3. Lower Approximate functions and Error Calculation

By definition, the initial rectangle S^0 is given by:

$$S^0 = \left\{x \in \mathbb{R}^n : L_i^0 \leq x_i \leq U_i^0 : i = \overline{1,n}\right\}.$$

We subdivide this rectangle into two sub-rectangles defined by:

$$\begin{aligned}S_{+1} &= \left\{x \in \mathbb{R}^n : L_s^0 \leq x_s \leq h_s^0 : L_j^0 \leq x_j \leq U_j^0 : j = \overline{1,n} : j \neq s\right\},\\ S_{+2} &= \left\{x \in \mathbb{R}^n : h_s^0 \leq x_s \leq U_s^0 : L_j^0 \leq x_j \leq U_j^0 : j = \overline{1,n} : j \neq s\right\},\end{aligned}$$

where, we calculate the point h_s by a normal rectangular subdivision ($\omega-$subdivision).

3.1. The lower approximate linear function of f over the rectangle S^K:

The best lower approximate linear function of f over the rectangle S^K is given in the following theorem:

Theorem 3.1. ***[3]***: *Let the function $f : C \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ and the rectangle $S^0 \subseteq \mathbb{R}^n$ where $C \subseteq S^0 \subseteq \mathbb{R}^n$, the lower approximate linear function of f is given by:*

$$\begin{aligned} L_{S^K}(x) &= \left(a_{S^K}\right)^T x + b_{S^K}, \\ U_{S^K}(x) &= \left(\overline{a_{S^K}}\right)^T x + \overline{b_{S^K}}, \end{aligned}$$

where:

$$\begin{aligned} a_{S^K} &= d + 2(Q+\theta I)L^K - \theta(L^K + U^K), \\ b_{S^K} &= -(L^K)^T(Q+\theta I)L^K + \theta(L^K)^T(U^K), \\ \overline{a_{S^K}} &= d + 2(Q+\theta I)U^K - \theta(L^K + U^K), \\ \overline{b_{S^K}} &= -(U^K)^T(Q+\theta I)U^K + \theta(L^K)^T(U^K). \end{aligned}$$

3.2. **The new lower approximate quadratic convex function of f over the rectangle S^K:** We use the preceding lower approximate linear function of f over the rectangle S^K to define the new lower approximate quadratic convex function of f over the same rectangle by:

Definition 3.1.

$$L_{quad}(x) := L_{S^K}(x) - \frac{1}{2}K\left(U^K - x\right)\left(x - L^K\right),$$

and:

$$U_{quad}(x) := U_{S^K}(x) - \frac{1}{2}K\left(U^K - x\right)\left(x - L^K\right),$$

where:

- *K is a positive real number given by the spectral radius of the matrix $(Q+\theta I)$, $\theta \geq |\lambda_{\min}|$,*
- *$L_{S^K}(x)$ the best lower approximate linear function of f over the rectangle S^K.*

3.3. **The New Lower Approximate Linear Function of f over the Rectangle S^K.** By the use of the preceding new lower approximate quadratic function of f over the rectangle S^K we can define the new lower approximate linear function of f over the same rectangle by:

Definition 3.2.

$$\widetilde{L}_{quad}(x) := L_{S^K}(x) - \frac{1}{8}Kh^2,$$

and:

$$\widetilde{U}_{quad}(x) := U_{S^K}(x) - \frac{1}{8}Kh^2,$$

with:

$$h := \left\| U^K - L^K \right\|.$$

3.3.1. ***The relation between the convex quadratic approximation and the linear one***. We have the following theorem:

Theorem 3.2. *The tow following inequality are satisfied:*

$$\begin{aligned}\widetilde{L}_{quad}(x) &:= L_{S^K}(x) - \frac{1}{8}Kh^2 \leq L_{quad}(x) \leq f(x),\\ \widetilde{U}_{quad}(x) &:= U_{S^K}(x) - \frac{1}{8}Kh^2 \leq U_{quad}(x) \leq f(x),\end{aligned}$$

for all $x \in (D_f) \cap S^K$ *and* $h := \left\| U^K - L^K \right\|$ *and* $\left\| \frac{\partial^2 f(x)}{\partial x^2} \right\| \leq K$ *(the regularity condition)*.

Proof. Let the function $g_1 : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ defined by:

$$\begin{aligned} g_1(x) &= \widetilde{L}_{quad}(x) - L_{quad}(x)\\ &= L_{S^K}(x) - \frac{1}{8}Kh^2 - (L_{S^K}(x) - \frac{1}{2}K\left(U^K - x\right)\left(x - L^K\right))\\ &= \frac{1}{2}K(-x^2 + (L^K + U^K)x - L^K U^K - \frac{1}{4}\left\| U^K - L^K \right\|^2). \end{aligned}$$

Passing to the first derivation of g_1,then, we get:

$$\frac{\partial g_1}{\partial x}(x) = \frac{1}{2}K(-2x + (L^K + U^K)).$$

Thus:

$$(\frac{\partial g_1}{\partial x}(x) = 0) \Longleftrightarrow (x = \frac{(L^K + U^K)}{2}).$$

The critical point of the function $g_1$is the middle point of the edge $\left[L^K, U^K\right]$, in the other hand, the function g_1 is concave, immediately, it reaches here max at the middle point $x^* = \frac{(L^K+U^K)}{2}$ of $\left[L^K, U^K\right]$, then we have:

$$g_1(x) \leq \max\left\{g_1(x) : x \in (D_f) \cap S^K\right\} = g_1(x^*) = 0.$$

Then,

$$\widetilde{L}_{quad}(x) - L_{quad}(x) \leq 0.$$

In the other hand, we define the function $g_2 : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ given by:

$$\begin{aligned} g_2(x) &= f(x) - L_{quad}(x) \\ &= f(x) - (L_{S^K}(x) - \frac{1}{2}K\left(U^K - x\right)\left(x - L^K\right)). \end{aligned}$$

Passing to the first derivation of g_2, then, we get:

$$\begin{aligned} \frac{\partial g_2}{\partial x}(x) &= \frac{\partial f}{\partial x}(x) - \frac{\partial L_{S^K}}{\partial x}(x) + \frac{1}{2}K\frac{\partial}{\partial x}((U^K - x)\left(x - L^K\right)) \\ &= \frac{\partial f}{\partial x}(x) - a_{S^K} + \frac{1}{2}K\frac{\partial}{\partial x}(-x^2 + (U^K + L^K)x - L^K U^K \\ &= \frac{\partial f}{\partial x}(x) - a_{S^K} + \frac{1}{2}K(-2x + (U^K + L^K)). \end{aligned}$$

Then, passing to the second derivation:

$$\frac{\partial^2 g_2}{\partial x^2}(x) = \frac{\partial^2 f}{\partial x^2}(x) - K.$$

We have the condition:

$$\frac{\partial^2 f(x)}{\partial x^2} \leq K \text{ (the regularity condition)}.$$

Then, we obtain:

$$\frac{\partial^2 g_2}{\partial x^2}(x) \leq 0.$$

Thus, the function g_2 is concave over S^K,and by this we have:

$$g_2(x) \geq \min\left\{g_2(x) : x \in S^K\right\} = \min\left\{g_2(L^K), g_2(U^K)\right\} = 0.$$

Then:

$$(g_2(x) = f(x) - L_{quad}(x) \geq 0) \Longrightarrow L_{quad}(x) \leq f(x).$$

Finally, we get:

$$\widetilde{L}_{quad}(x) \leq L_{quad}(x) \leq f(x) : x \in S^K.$$

The same thing when we use the upper bound $U_{quad}(x)$ with the equivalent linear form of the objective function f and we obtain:

$$\widetilde{U}_{quad}(x) \leq U_{quad}(x) \leq f(x) : x \in S^K.$$

□

3.4. **Approximation errors:** We can estimate the approximation error by the distance between the non convex objective function f and here lower approximation functions.

3.4.1. ***The linear approximation error:*** Is presented by the distance between the function f and here new lower approximate linear function $\widetilde{L}_{quad}$ over the boxed set S^K, then we have the following proposition:

Proposition 3.1. *Let the function $f : C \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ where $C \subseteq S^0 \subseteq \mathbb{R}^n$ and $\theta \geq |\lambda_{\min}|$ for this the matrix $(Q+\theta I)$ be semi-positive, then we have:*

$$\begin{aligned}
\max_{x\in S^K\cap\left(D_f\right)} \left\{\left|f(x)-\widetilde{L}_{quad}(x)\right|\right\} &\leq \left(\rho\left(Q+\theta I\right)+\theta+\frac{1}{8}K\right)\left\|U^K-L^K\right\|^2, \\
\max_{x\in S^K\cap\left(D_f\right)} \left\{\left|f(x)-\widetilde{U}_{quad}(x)\right|\right\} &\leq \left(\rho\left(Q+\theta I\right)+\theta+\frac{1}{8}K\right)\left\|U^K-L^K\right\|^2.
\end{aligned}$$

Proof. We have:

$$\begin{aligned}
& f(x)-\widetilde{L}_{quad}(x) \\
= & \left(x-L^K\right)^T\left(Q+\theta I\right)\left(x-L^K\right)+d^Tx-\theta\sum_{i=1}^n x_i^2 \\
& +2\left(L^K\right)^T\left(Q+\theta I\right)x-\left(L^K\right)^T\left(Q+\theta I\right)L^K-\left(L_{S^K}(x)-\frac{1}{8}Kh^2\right) \\
= & \left(x-L^K\right)^T\left(Q+\theta I\right)\left(x-L^K\right)+d^Tx-\theta\sum_{i=1}^n x_i^2 \\
& +2\left(L^K\right)^T\left(Q+\theta I\right)x-\left(L^K\right)^T\left(Q+\theta I\right)L^K \\
& -((d+2(Q+\theta I)L^K-(L^K+U^K))^Tx+(-(L^K)^T(Q+\theta I)L^K+(L^K)^TU^K)) \\
& +\frac{1}{8}Kh^2 \\
= & \left(x-L^K\right)^T\left(Q+\theta I\right)\left(x-L^K\right)+\frac{1}{8}Kh^2+\theta((L^K+U^K)^Tx-x^Tx-(L^K)^TU^K).
\end{aligned}$$

In the other hand, we have:

$$\left(x-L^K\right)\left(U^K-x\right)=(L^K+U^K)^Tx-x^Tx-(L^K)^TU^K.$$

Then we get:

$$f(x) - \widetilde{L}_{quad}(x) = \left(x - L^K\right)^T (Q + \theta I)\left(x - L^K\right) + \frac{1}{8}Kh^2 + \theta\left(x - L^K\right)\left(U^K - x\right).$$

So:

$$\begin{aligned}
&\left\|f(x) - \widetilde{L}_{quad}(x) : x \in S^K \cap (D_f)\right\| \\
= &\max_{x \in S^K \cap \left(D_f\right)} \left\{\left|f(x) - \widetilde{L}_{quad}(x)\right|\right\} \\
= &\left\|\left(x - L^K\right)^T (Q + \theta I)\left(x - L^K\right) + \frac{1}{8}Kh^2 + \theta\left(x - L^K\right)\left(U^K - x\right)\right\|_\infty \\
\leq &\left\|\left(x - L^K\right)^T (Q + \theta I)\left(x - L^K\right)\right\| + \theta\left\|\left(x - L^K\right)\left(U^K - x\right)\right\| + \frac{1}{8}Kh^2 \\
\leq &\left(\rho(Q + \theta I)\left\|U^K - L^K\right\|^2\right) + \theta\left\|U^K - L^K\right\|^2 + \frac{1}{8}Kh^2 \\
\leq &\left(\rho(Q + \theta I) + \theta + \frac{1}{8}K\right)h^2 : h^2 = \left\|U^K - L^K\right\|^2.
\end{aligned}$$

The same thing whene we use the upper bound $U_{quad}(x)$ with the equivalent linear form of the objective function f and we obtain:

$$\left\|f(x) - \widetilde{U}_{quad}(x) : x \in S^K \cap (D_f)\right\| \leq \left(\rho(Q + \theta I) + \theta + \frac{1}{8}K\right)h^2 : h^2 = \left\|U^K - L^K\right\|^2.$$

Then, the proof is complete. □

3.4.2. ***The quadratic approximation error:*** Is presented by the distance between the function f and here lower approximate quadratic function $\widetilde{L}_{quad}$ over the rectangle S^K, then we have the following proposition:

Proposition 3.2. *let the function $f : C \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ where $C \subseteq S \subseteq \mathbb{R}^n$ and $\theta \geq |\lambda_{\min}|$ for this the matrix $(Q + \theta I)$ be semi-positive, then we have:*

$$\begin{aligned}
\max_{x \in S^K \cap \left(D_f\right)} \{|f(x) - L_{quad}(x)|\} &\leq \left(\rho(Q + \theta I) + \theta + \frac{1}{2}K\right)\left\|U^K - L^K\right\|^2, \\
\max_{x \in S^K \cap \left(D_f\right)} \{|f(x) - U_{quad}(x)|\} &\leq \left(\rho(Q + \theta I) + \theta + \frac{1}{2}K\right)\left\|U^K - L^K\right\|^2.
\end{aligned}$$

Proof. By the definition of the function $L_{quad}(x)$ as well as the meaning of $\varphi_{S^k}(x)$, we have:

$$\begin{aligned} f(x) - L_{quad}(x) &= f(x) - L_{S^K}(x) + \frac{1}{2}K\left(U^K - x\right)\left(x - L^K\right) \\ &= \left(x - L^K\right)^T (Q + \theta I)\left(x - L^K\right) + (\frac{1}{2}K + \theta)(U^K - x)(x - L^K). \end{aligned}$$

Then:

$$\begin{aligned} & \left\|f(x) - L_{quad}(x)\right\|_\infty \\ = \ & \max\left\{f(x) - L_{quad}(x) : x \in S^K \cap (D_f)\right\} \\ \leq \ & \left\|\left(x - L^K\right)^T (Q + \theta I)\left(x - L^K\right)\right\|_\infty + \left\|(\frac{1}{2}K + \theta)(U^K - x)(x - L^K)\right\|_\infty \\ \leq \ & \left(\rho\left(Q + \theta I\right) + \theta + \frac{1}{2}K\right)\left\|U^K - L^K\right\|^2. \end{aligned}$$

The same thing whene we use the lower bound $U_{quad}(x)$ with the equivalent linear form of the objective function f and we obtain:

$$\left\|f(x) - U_{quad}(x)\right\|_\infty \leq \left(\rho\left(Q + \theta I\right) + \theta + \frac{1}{2}K\right)\left\|U^K - L^K\right\|^2.$$

So, the proof is complete. □

3.5. **The quadratic approximate problem (QAP).**

3.5.1. ***Construction of the interpolate problem (IP)***. It's clear that:

$$f(x) \geq \max\left\{L_{quad}(x), U_{quad}(x) : \forall x \in (D_f) \cap S^K\right\} = \gamma(x).$$

This function present the best quadratic lower bound of f, similarly, we construct the following interpolate problem by:

$$\text{(LBP)} \qquad \begin{cases} \alpha_h = \max \widehat{x} \\ \widehat{x} \in \{L_{quad}(x), U_{quad}(x)\} : \forall x \in (D_f) \cap S^K \end{cases}.$$

And the convex quadratic problem define by:

$$\text{(ACQP)} \qquad \begin{cases} \min \alpha_h \\ \forall x \in (X_f) \cap S^K \end{cases}.$$

The question is: what's the relation between the optimal values $f(\widetilde{x})$, $f(x^*)$ and $L_{quad}(\widetilde{x})$?

We have the following proposition:

Proposition 3.3. *Let the function* $f : C \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ *and* $S^0 \subseteq \mathbb{R}^n$ *where* $C \subseteq S \subseteq \mathbb{R}^n$*, we have:*

$$\begin{aligned} 0 &\leq f(\widetilde{x}) - f(x^*) \leq (\rho(Q + \theta I) + \theta + \frac{1}{2}K) \left\|U^K - L^K\right\|^2 \\ L_{quad}(\widetilde{x}) &\leq f^* \leq f(\widetilde{x}), \end{aligned}$$

with $f^* = f(x^*)$ *is the global optimal value of the original problem (NQP) and* $\widetilde{x}$ *be the optimal solution of (ACQP).*

Proof. From the previous proposition, we have:

$$f(x) - L_{quad}(x) \leq \left(\rho\left(Q + \theta I\right) + \theta + \frac{1}{2}K\right) \left\|U^K - L^K\right\|^2 : x \in S^K \cap (D_f).$$

And for $x = \widetilde{x}$:

$$f(\widetilde{x}) - L_{quad}(\widetilde{x}) \leq \left(\rho\left(Q + \theta I\right) + \theta + \frac{1}{2}K\right) \left\|U^K - L^K\right\|^2.$$

Thus:

$$f(\widetilde{x}) - f^* + f^* - L_{quad}(\widetilde{x}) \leq \left(\rho\left(Q + \theta I\right) + \theta + \frac{1}{2}K\right) \left\|U^K - L^K\right\|^2.$$

And:

$$f(\widetilde{x}) - f^* \leq \left(\rho\left(Q + \theta I\right) + \theta + \frac{1}{2}K\right) \left\|U^K - L^K\right\|^2 + (L_{quad}(\widetilde{x}) - f^*).$$

As well as $L_{quad}(\widetilde{x}) - f^* \leq 0$, we have:

$$0 \leq f(\widetilde{x}) - f^* \leq \left(\rho\left(Q + \theta I\right) + \theta + \frac{1}{2}K\right) \left\|U^K - L^K\right\|^2.$$

In the other hand, we have:

$$\begin{cases} L_{quad}(\widetilde{x}) - f^* \leq 0 \\ f(\widetilde{x}) - f^* \geq 0 \end{cases} \Longrightarrow (L_{quad}(\widetilde{x}) \leq f^* \leq f(\widetilde{x})).$$

□

3.5.2. ***Question: is the solution*** $\widetilde{x}$ ***present the best lower bound of the global optimal solution of (NQP)?***.

We have the following proposition:

Proposition 3.4. *Let take the estimate function noted by:*

$$E(x) := f(x) - L_{quad}(x).$$

For all $x \in S^K \cap (D_f)$, the next inequality is satisfied:

$$E(\widetilde{x}) \geq f(\widetilde{x}) - f^*.$$

Proof. We have:

$$\begin{aligned} f(\widetilde{x}) - f^* &= f(\widetilde{x}) - L_{quad}(\widetilde{x}) + L_{quad}(\widetilde{x}) - f^* \\ &= E(\widetilde{x}) + L_{quad}(\widetilde{x}) - f^*. \end{aligned}$$

And, from the previeus proposition we have:

$$L_{quad}(\widetilde{x}) \leq f^* \leq f(\widetilde{x}).$$

So:

$$L_{quad}(\widetilde{x}) - f^* \leq 0.$$

Then:

$$f(\widetilde{x}) - f^* \leq E(\widetilde{x}).$$

□

Lemma 3.1. *If $E(\widetilde{x})$ is a small value, then $f(\widetilde{x})$ is an acceptable approximate value of the global optimal value $f^* = f(x^*)$ over the rectangle S^K. Similarly, we can find that the point $\widetilde{x}$ is the global approximate solution of the global optimal solution x^* of the original problem (NQP) over S^K.*

Proof. We have:

$$f(\widetilde{x}) - f^* \leq E(\widetilde{x}).$$

So, let take that $E(\widetilde{x})$ is a small value we get:

$$f(\widetilde{x}) - f^* \leq E(\widetilde{x}) << \varepsilon \text{ with } \varepsilon \longrightarrow 0.$$

Then:

$$\|f(\widetilde{x}) - f^*\| << \varepsilon.$$

And:

$$\lim_{\varepsilon \longrightarrow 0} \|f(\widetilde{x}) - f^*\| = 0.$$

Immediately, we get that $f(\widetilde{x})$ is an acceptable approximate value of the global optimal value f^*, then we obtain that the point $\widetilde{x}$ is a global approximate solution of the global optimal solution x^* of the original problem (NQP) over the rectangle S^K. □

In the other hand, the rank of the non convex function f over the new rectangle (sub-rectangle) S^K is small then here rank over the initial rectangle S, by this, the value $E(\widetilde{x})$ will be very small.

4. The technical reduction (technical eliminate):

We get to describe the rectangle partition by the following steps:

Step(0): Let $S^K = \left\{x^k \in \mathbb{R}^n : L_i^K \leq x_i^k \leq U_i^K : i = \overline{1,n}\right\}$ with $x^k \in S^K$.

Step(1): We find the point h_s given by:

$$h_s = \max\left\{\left(x_i - L_i^K\right)\left(U_i^K - x_i\right) : i = \overline{1,n}\right\}.$$

Step(2): If $h_s \neq 0$ then we divide the rectangle S^K into two subrectangle on edge $\left[L_s^K, U_s^K\right]$ by the point h_s, **else**, we divide the rectangle S^K into two subrectangle on the longest edge $\left[L_s^K, U_s^K\right]$ by the middle point $\frac{L^K+U^K}{2}$ which is yet noted as h_s.

Step(3): The rest rectangle is yet noted as S^K.

Now, we describe the rectangle reducing tactics to accelerate the convergence of the proposed global optimization algorithm **(ARSR).**

Remark 4.1.
1. All linear constraints of the problem (NQP) are given by:

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i : i = \overline{1,n}.$$

2. The rectangle S^K be also recorded as constraint to be added to the problem (NQP).
3. The minimum and the maximum of each function:

$$\begin{cases} \psi(x_i) = a_{is}x_s \text{: } i = \overline{1,n} \\ x_i \in \left[L_s^K, U_s^K\right] \end{cases}.$$

Are obtaind at the extremes points of the same interval.

Linearity Based Range Reduction Algorithm:

This algorithm is given to reduce and delete the rectangle S^K.

---

program (LBRRA)

Let $I_k^{'} := \{1, 2, 3, \ldots, n\}$ the set of the index, $P_k := P$

for $1 \leq i \leq n$ **do**

compute $rU_i := \sum_{j=1}^{n} \max\left\{a_{ij}L_j^k, a_{ij}U_j^k\right\}$

compute $rL_i := \sum_{j=1}^{n} \min\left\{a_{ij}L_j^k, a_{ij}U_j^k\right\}$

if $rL_i > b_i$ **then**

stop. the problem (NQP) is infeasible over S^K (there are no solution of (NQP) over S^K, because, S^K is deleted From the subrectangle set produced through partitioning of the rectangleS°)

else

if $rU_i < b_i$ **then**

the constraint is redundant.

$I_k^{'} := I_k^{'} - \{i\}$

$P_k := P_k - \left\{x \in \mathbb{R}^n : (a_i)^T x \leq b_i\right\}$

else

for $1 \leq j \leq n$ **do**

if $a_{ij} > 0$ **then**

$U_j^k := \min\left\{U_j^k, \frac{b_i - rL_i + \min\left\{a_{ij}L_j^k, a_{ij}U_j^k\right\}}{a_{ij}}\right\}$

else

$L_j^k := \max\left\{L_j^k, \frac{b_i - rU_i + \max\left\{a_{ij}L_j^k, a_{ij}U_j^k\right\}}{a_{ij}}\right\}$

end if

anddo

end if

end if

enddo

end program

---

5. Algorithm (ARSR): Branch and Bound

program (ARSR)

Initialization: Determine the initial rectangle S^0 where $(\chi_f) \subset S^0$ and suppose that:

$$QLBP_{S^0} := S^0 \cap (\chi_f)$$

iteration k :

if $QLBP_{S^0} \neq \phi$ **then**

solve the quadratic problem (LBP) when $k = 0$

Let x^0 be an optimal solution of (LBP) and $\alpha(S^0)$ be the optimal value acompaned to x^0

$H := \{S^0\}$ (the set of the subrectangle of the initial rectangle S^0)

$\alpha_0 := \min\{\alpha(S^0)\}, \beta_0 := f(x^0)$ (the upper bound of $f(x^*)$)

$k := 0$

while Stop=false do

if $\alpha_k = \beta_k$ **then**

Stop=true (x^k is a global optimal solution of the problem (NQP))

else

we subdivise the rectangle S^kinto two sub-rectangle $\{S_j^k : j = 1, 2\}$

by the proposed algorithm.

for $j = 1, 2$ **do**

applied the Linearity Based Range Reduction Algorithm over the two sub-rectangle $\{S_j^k\}$

the obtained set is yet noted as the rectangle S_j^k

if $S_j^k \neq \phi$ **then**

$(QLBP)_{S_j^k} := \{x \in \mathbb{R}^n : x \in S_j^k \cap (\chi_f)\}$,

solve the quadratic problem $(QLBP)$ when $S^k := S_j^k$

let x^{k_j} be the optimal solution and $\alpha(S_j^k)$ be the optimal value

$H := H \cup \{S_j^k\}$

$\beta_{k+1} := \min\{f(x^k), f(x^{k_j})\}$

$x^k := arg \min \beta_{k+1}$

end if

end for

$H := H - \{S^k\}$

$\alpha_{k+1} := \min_{S \in H}\{\alpha(S)\}$;choose an rectangle $S^{k+1} \in H$

such that $\alpha_{k+1} = \alpha(S^{k+1})$

$k \leftarrow k + 1$;

end if

end do

end if

end program

6. THE CONVERGENCE OF THE ALGORITHM (ARSR)

In this section, we study the convergence of the proposed algorithm **(ARSR)** and we give a simple comparison between the linear approximate and the quadratic one. In the other hand, we give some examples to expline the proposed algorithm.

The convergence of the proposed algorithm:

The proposed algorithm in **section 5** is different from the one in ref [3] in lower-bounding (quadratic approximation), and added to the rectangle-reducing strategy. We will prove that the proposed algorithm be convergent.

Theorem 6.1. *If the proposed algorithm terminates in finite steps, then a global optimal solution of the problem (NQP) is obtained when the algorithm terminates.*

Proof. Let the result out coming when the algorithm terminate be x^k, then, immediately we have $a_{x=}B_k$ when terminating at the- k-step, so x^k is a global optimal solution of the problem(NQP). □

Theorem 6.2. *If the algorithm generates an infinite sequence $\left\{x^k\right\}_{k\in\mathbb{N}^*}$, then every accumulation piont x^* of this sequence is a global optimal solution of the problem (NQP)(i.e: the global optimal solution is not unique).*

Proof. Let x^* be an accumulation point of the sequence $\left\{x^k\right\}_{k\in\mathbb{N}^*}$ and let $\left\{x_p^k\right\}_{k\in\mathbb{N}^*,p\in\mathbb{N}^*}$ be a subsequence of the sequence $\left\{x^k\right\}_{k\in\mathbb{N}^*}$ converging to x^*. obviously in the proposed algorithm, the lower sequence $\{a_k\}_{k\in\mathbb{N}^*}$ is mono-increase and the upper sequence $\{B_k\}_{k\in\mathbb{N}^*}$ is mono-decrease, and we have:

$$\alpha_k = l_{quad}(x^k), B_k = f\left(x^k\right).$$

We can write:

$$\alpha_k = l_{quad}(x^k) \leq \min_{x\in S_k} f(x) \leq B_k = f\left(x^k\right).$$

So, both $\{x_k\}_{k\in\mathbb{N}^*}$ and $\{B_k\}_{k\in\mathbb{N}^*}$ are convergent and:

$$\lim_{k\to\infty} B_k = \lim_{q\to\infty} B_{k_q} = \lim_{k\to\infty} f\left(x^k\right) = \lim_{q\to\infty} f\left(x^{k_q}\right) = f\left(x^*\right).$$

Without loss of generality, we assume that x^{k_q} is the solution of the problem (LBP) on S_{k_q} which satisfies $S_{k_{q+1}} \subset S_{k_q}, q \geq 1$, by the proprieties of the proposed rectangle partition which is exhaust, i.e.:

$$\lim_{q\to\infty} S_{k_q} = x^*.$$

We have:

$$0 \leq B_{k_q} - \alpha_{k_q} = f\left(x_q^k\right) - l_{quad}(x_q^k) \leq \left(\rho\left(Q+\theta I\right) + \theta + \frac{1}{2}K\right) \left\|U_q^K - L_q^K\right\|^2.$$

Then:

$$\lim_{q\to 0}(f\left(x_q^k\right) - l_{quad}(x_q^k)) = \lim_{q\to 0}(B_{k_q} - \alpha_{k_q}) = 0.$$

Thus, we have:

$$\lim_{q\to 0}(B_{k_q} - \alpha_{k_q}) = \lim_{q\to 0}(\alpha_{k_q} - B_{k_q} - (B_{k_q} - \alpha_{k_q})) = 0.$$

So:

$$\lim_{k\to 0}\alpha_k = \lim_{q\to 0}\alpha_{k_q} = \lim_{q\to 0}(B_{k_q} - (B_{k_q} - \alpha_{k_q})) = \lim_{q\to 0} B_{k_q},$$

and:

$$\lim_{k\to 0}\alpha_k = \lim_{q\to 0} B_{k_q} = \lim_{q\to\infty} f\left(x^{k_q}\right) = f\left(x^*\right).$$

Therefore, the point x^* is an global optimal solution of the problem (NQP). □

6.1. **The type and rank of convergence:** The proposed algorithm converge to the approximate solution of the optimal global solution of (NQP) with a quadratic vitesse over S^K.

In this method, the rank of the non convex function f over the rectangle S^K will be lower then his rank over the initial one S°, thus immediately give that the value $E(\widetilde{x})$ is very small.

By this result, the solution point $\widetilde{x}$ is an global approximate solution to the global optimal solution x^*over S^K.

To accelerate the convergence of the proposed algorithm we used the technical of partitioning and reducing where in every step we eliminate a rectangle and a linear constraint, and this give us a rectangle smaller then the initial one and we denoted it by S^K.

7. Comparison Between "Branch and Bound" and " Method (DCT)"

7.1. **Method (DCT).** In this section, we present a global method noted by "the dual canonical transformation method (DCT)", this method transforms a non convex quadratic problem with linear constraints (NP-hard problem) to a algebraic system easy to resolve. This system is obtained by the use of the canonical duality notion wish we give the same KKT points of the two problems.

Let take the non convex quadratic optimization problem given by:

$$\begin{cases} \min f(x) = \frac{1}{2}x^T Q x - d^T x \\ Ax \le b; \quad x \ge 0 \end{cases} \quad \text{where} \quad \begin{array}{c} Q \in \mathbb{R}^{n\times n} \text{ indefinite matrix} \\ A \in \mathbb{R}^{n\times m} \text{ arbitrary matrix} \\ b, x \text{ vertex of } \mathbb{R}^n \end{array}$$

The fundamental idea of this method is in the chose of the operator:

$$\Lambda(x) : \mathbb{R}^n \to \mathbb{R}^m.$$

By this the objective function f be write as the following canonical form:

$$f(x) = \Phi(x, \Lambda(x)).$$

Define over the set $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}$ in the condition that the function be canonic at every one (point) x and y.

We need the following definitions:

Remark 7.1. *The canonical function $\Phi(x, \Lambda(x))$ can represent by:*

$$\Phi(x, \Lambda(x)) = \overline{W}(y) - \overline{F}(y) : y \in \mathbb{R}^m.$$

This function is defined over $\mathbb{R}^m \times \mathbb{R}^n$ to $\mathbb{R}$.

In the other hand, we use the dual $\Lambda-$canonical transformation to calculate the conjugate function of $\overline{F}(y)$ given by:

$$\overline{F}^{\Lambda}(y^*) = \left\{(\Lambda(x))^T y^* - \overline{F}(x) : \Lambda_t^T(x)y^* - D\overline{F}(x) = 0\right\},$$

with:

$$\Lambda_t^T(x) = D\Lambda(x).$$

By the use of this notions, we can construct the associate dual function of f by:

$$f^d(y^*) = \overline{F}^{\Lambda}(y^*) - \overline{W}^*(y^*).$$

7.1.1. ***Method (DCT) for the Non Convex Quadratic Problems:*** We must add the *regularity condition* define by the choice of the parameter $\mu > 0$ in order to guarantee the existence of the global optimal solution, this condition is given by:

$$|x|^2 \leq 2\mu.$$

Then, we have:

$$\text{(PQP)} \qquad \begin{cases} \min f(x) = \frac{1}{2}x^T Qx - d^T x \\ Ax \leq b; \quad x \geq 0 \ ; |x|^2 \leq 2\mu \end{cases}.$$

We can transform the problem (PQP) as:

$$\begin{cases} \min f(x) = \frac{1}{2}x^T Qx - d^T x \\ Ax \leq b; \frac{1}{2}|x|^2 \leq \mu \end{cases},$$

with:

$$A = \begin{pmatrix} A & & & & \\ -1 & -1 & -1 & \cdots & -1 \end{pmatrix} \in \mathbb{R}^{(n+1)\times n} \text{ and } b = \begin{pmatrix} b \\ 0 \end{pmatrix} \in \mathbb{R}^{n+1}.$$

Then, we applied the method (DCT) over the associate parametric problem (PQP) in the place of the non convex quadratic problem (NQP) like follows:

Step(1): The form of the operator $\Lambda(x)$

For this type of problem the canonical geometric operator:

$$\Lambda(x) : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}.$$

Is given by:

$$y = \Lambda(x) = \left(Ax, \frac{1}{2}|x|^2\right) = (\varepsilon, \rho) \in \mathbb{R}^m \times \mathbb{R},$$

and, it's presented as an Vertex-Value application. By this, the domain of (PQP) will be define by:

$$D_{PQP} = \{y = (\varepsilon, \rho) \in \mathbb{R}^m \times \mathbb{R} : \varepsilon \leq b, \rho \leq \mu\}$$

Step(2): The structure of the function $\overline{W}(y)$

In this case, the function $\overline{W}(y)$ is given by the Indicative function of the domain D_{PQP} like follows:

$$\begin{array}{rcl}\overline{W} & : & \mathbb{R}^n \to \mathbb{R} \\ y & \mapsto & \overline{W}(y) = \left\{ \begin{array}{lcr} 0 & \text{if} & y \in D_{PQP} \\ +\infty & & \text{else} \end{array} \right. .\end{array}$$

Then, it's clear that the function $\overline{W}(y)$ is always convex from the propriety of the indicative function. In the other hand, the function $\overline{W}(y)$ is proper and s-lower continuous over the set D_{PQP}.

By this we have:

Step(3): The structure of the function $\overline{W}^*(y^*)$

$$\overline{W}^*(y^*) = \sup_{y \in D_{PQP}} \left\{ \langle y, y^* \rangle - \overline{W}(y) \right\} = \sup_{\varepsilon \leq b} \sup_{\rho \leq \mu} \left\{ (\varepsilon, \rho)^T (\varepsilon^*, \rho^*) - \overline{W}(y) : y \in D_{PQP} \right\}$$

$$= \sup_{\varepsilon \leq b} \sup_{\rho \leq \mu} \left\{ \varepsilon^T \varepsilon^* + \rho^T \rho^* : y \in D_{PQP} \right\} = \left\{ \begin{array}{lcr} \varepsilon^T \varepsilon^* + \rho^T \rho^* & \text{if} & \varepsilon^* \geq 0, \rho^* \geq 0 \\ +\infty & & \text{else} \end{array} \right. .$$

Step(4): **The structure of the function $\overline{F}^\Lambda(y^*)$**

The function $\overline{F}(y)$ is a linear function, and we have:

$$f(x) = \Phi(x, \Lambda(x)) = \overline{W}(y) - \overline{F}(y) : y \in \mathbb{R}^m \times \mathbb{R}.$$

Then, we get:

$$f(x) - \overline{W}(y) = -\overline{F}(y) : y \in \mathbb{R}^m \times \mathbb{R},$$

and for $y \in D_{PQP}$ we have:

$$-f(x) = \overline{F}(y).$$

Immediately, the $\Lambda-$canonical conjugate of the function $\overline{F}(y)$ is define by:

$$\begin{aligned}\overline{F}^\Lambda(y^*) &= \sup_{y \in D_{PQP}} \left\{ y^T y^* - \overline{F}(y) : \Lambda_t^T(x) y^* - D\overline{F}(x) = 0 : x \in D_{PQP} \right\} \\ &= \sup_{y \in D_{PQP}} \left\{ (\Lambda(x))^T y^* - \overline{F}(y) : \Lambda_t^T(x) y^* - D\overline{F}(x) = 0 : x \in D_{PQP} \right\},\end{aligned}$$

and, from the first step we have:

$$y = \Lambda(x) = \left(Ax, \frac{1}{2} |x|^2 \right) = (\varepsilon, \rho) \in \mathbb{R}^m \times \mathbb{R}.$$

Thus:

$$\begin{aligned}
\overline{F}^{\Lambda}(y^*) &= \sup_{y\in D_{PQP}} \left\{(\Lambda(x))^T y^* - \overline{F}(\Lambda(x)) : \Lambda_t^T(x)y^* - D\overline{F}(x) = 0 : x \in D_{PQP}\right\} \\
&= \sup_{y\in D_{PQP}} \left\{\frac{1}{2}x^T(Q+\rho^* I)x - (d - A^T\varepsilon^*)^T x\right\} : x \in D_{PQP} \\
&= \frac{-1}{2}(d - A^T\varepsilon^*)^T(Q+\rho^* I)^{-1}(d - A^T\varepsilon^*),
\end{aligned}$$

with $x = (Q+\rho^* I)^{-1}(d - A^T\varepsilon^*)$.

Step(5): The structure of the dual canonical function $f^d(y^*)$:
From the forth step, we define the dual canonical function by:

$$\begin{aligned}
f^d(y^*) &= \overline{F}^{\Lambda}(y^*) - \overline{W}^*(y^*) \\
&= \frac{-1}{2}(d - A^T\varepsilon^*)^T(Q+\rho^* I)^{-1}(d - A^T\varepsilon^*) - \varepsilon^T\varepsilon^* - \rho^T\rho^* : (\varepsilon^*, \rho^*) \in \mathbb{R}^m \times \mathbb{R}.
\end{aligned}$$

Then, the parametric dual problem is given by:

$$\text{(CPD)} \qquad \begin{cases} \max f^d(\varepsilon^*, \rho^*) \\ \varepsilon^* \geq 0, \rho^* \geq 0,\ \det(Q+\rho^* I) \neq 0 \end{cases}.$$

We can find an equivalence between the primal problem and the dual one, that's given by the following theorem:

Theorem 7.1. *[1]:If $\overline{y^*} = (\overline{\varepsilon^*}, \overline{\rho^*})$ be a (K.K.T) point of the parametric dual problem (CPD) then the vertex*

$$\widetilde{x} = (Q + \overline{\rho^*} I)^{-1}(d - A^T\overline{\varepsilon^*})$$

is a (K.K.T) point of the parametric primal problem (PQP), and we have:

$$f^d(\overline{y^*}) = f(\widetilde{x}).$$

Remark 7.2. *Let take id be the number of the negative distincts eigenvalues of the matrix Q then, the quadratic problem be non convex if $id > 0$.*

7.2. **Convergence Theorem of the Method (DCT):.** We can suppose the question "what's the relation between the optimal solutions of the parametric problem (PQP), the primal problem (NQP) and the parametric dual problem (CPD)?

To give the answer we have this theorem:

Theorem 7.2. *[1]:Let Q a matrix with the index $id > 0$ and $\{\lambda_i\}_{i=\overline{1,p}}$: $p \leq n$ a distincts eigenvalues in the order:*

$$\lambda_1 < \lambda_2 < \ldots < \lambda_{id} < 0 \leq \lambda_{id+1} < \lambda_{id+2} < \ldots < \lambda_p$$

and let $(\overline{\varepsilon^*}, \overline{\rho^*})$ be a K.K.T point of the parametric dual problem (CPD), and:

$$\widetilde{x} = (Q + \overline{\rho^*} I)^{-1}(d - A^T \overline{\varepsilon^*})$$

a K.K.T point of the prametric primal problem (PQP), then we have:

1. *If $\overline{\rho_i^*} > -\lambda_1 > 0$ then, the vertex $(\overline{\varepsilon^*}, \overline{\rho^*})$ is a maximum of $f^d(y^*)$ over D^+_{PQP} if and only if the vertex $\widetilde{x}$ is a minimum of $f(x)$ over D^s_{PQP}, and we write:*

$$f(\widetilde{x}_i) = \min_{x \in D^s_{PQP}} f(x) = \max_{(\varepsilon^*, \rho^*) \in D^+_{PQP}} f^d(\varepsilon^*, \rho^*) = f^d\left(\overline{\varepsilon^*}, \overline{\rho^*}\right)$$

2. *If $0 \leq \overline{\rho_i^*} < -\lambda_{id}$ then, the vertex $(\overline{\varepsilon^*}, \overline{\rho^*})$ is a maximum of $f^d(y^*)$ over D^-_{PQP} if and only if the vertex $\widetilde{x}$ is a global maximum of $f(x)$ over D_{PQP}, and we write:*

$$f(\widetilde{x}_i) = \max_{x \in D_{PQP}} f(x) = \max_{(\varepsilon^*, \rho^*) \in D^-_{PQP}} f^d(\varepsilon^*, \rho^*) = f^d\left(\overline{\varepsilon^*}, \overline{\rho^*}\right)$$

3. *If $0 < \overline{\rho_i^*} < -\lambda_{id}$ then, the vertex $(\overline{\varepsilon^*}, \overline{\rho^*})$ is a minimum of $f^d(y^*)$ over D^i_{PQP} if and only if the vertex $\widetilde{x}$ is a global minimum of $f(x)$ over D_{PQP}, and we write:*

$$f(\widetilde{x}_i) = \min_{x \in D_{PQP}} f(x) = \min_{(\varepsilon^*, \rho^*) \in D^i_{PQP}} f^d(\varepsilon^*, \rho^*) = f^d\left(\overline{\varepsilon^*}, \overline{\rho^*}\right)$$

7.3. **Examples.**

7.3.1. *Example 1.* Let the non convex quadratic function define by:

$$f(x) = (x_1 + 1)^2 + (x_2 + 1)^2 - \frac{5}{2}(x_1 + x_2) - 3\left(x_1^2 + x_2^2\right) - 2.$$

So, we have:

$$\begin{aligned} L_{quad}(x) &= (x_1^2 + x_2^2) + \frac{3}{2}(x_1 + x_2) - 2 \\ \widetilde{L}_{quad}(x) &= \frac{1}{2}(x_1 + x_2) - 2 - \frac{1}{8}(3), \end{aligned}$$

with:

$$\begin{aligned} f(x) &: \text{ broun whith black} \\ L_{quad}(x) &: \text{ red whith yellow} \\ \widetilde{L}_{quad}(x) &: \text{ darkgray whith navy} \end{aligned}$$

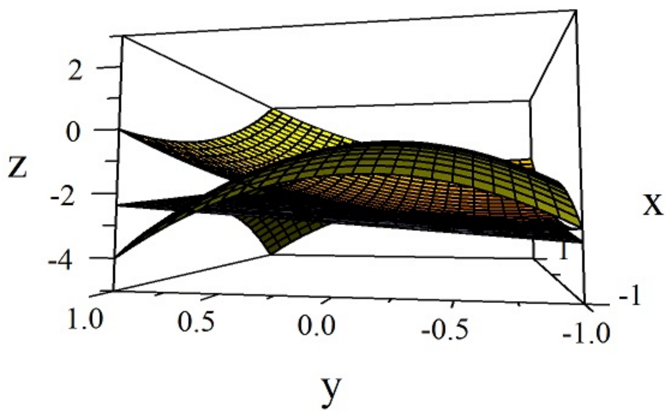


FIGURE 1. The graphic representation of the non convex quadratic function f, the linear approximate function and the convex quadratic lower bound function over the rectangle $[-1, 0] \subseteq \mathbb{R}^n$.

It's clear that the convex quadratic approximate function is between the objective function and the linear approximate one of the same function over he rectangle $S^0 = [-1, 0] \subseteq \mathbb{R}^n$.

7.3.2. ***Example 2.:*** Let take the following quadratic programming problem:

$$\begin{cases} \min f(x) = \frac{1}{2}ax^2 - dx \\ \qquad |x| \leq r \end{cases}.$$

So, if $a \geq 0$ then, the problem be convex and this case is simple to resolve, however, if $a < 0$.

Let $a = -6$, $d = 4$ and $r = 1.5$, then:

$$\begin{cases} \min f(x) = -3x^2 - 4x \\ \qquad |x| \leq 1.5 \end{cases}.$$

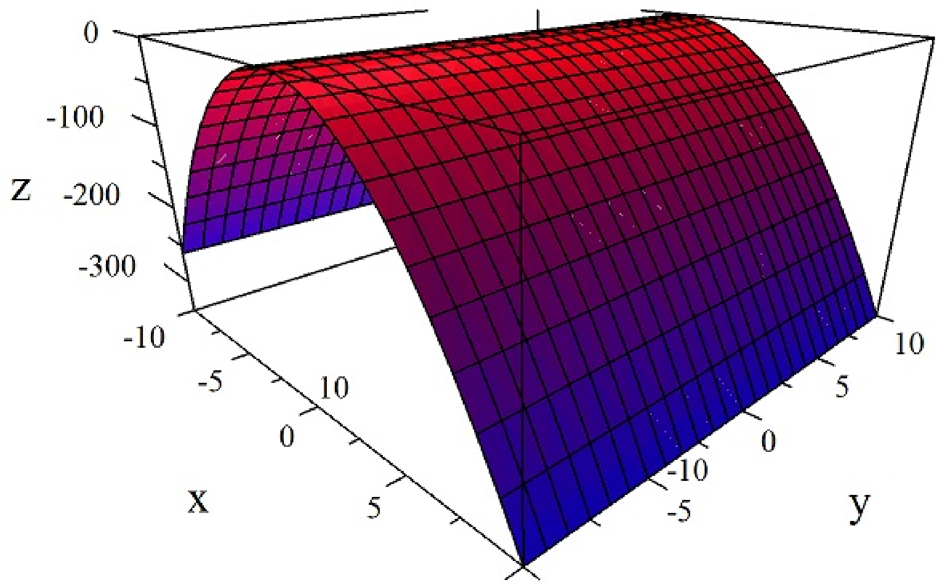


FIGURE 2

Candidate(s) for extrema: $\left\{\frac{4}{3}\right\}$, at $\left\{\left[x=-\frac{2}{3}\right]\right\}$

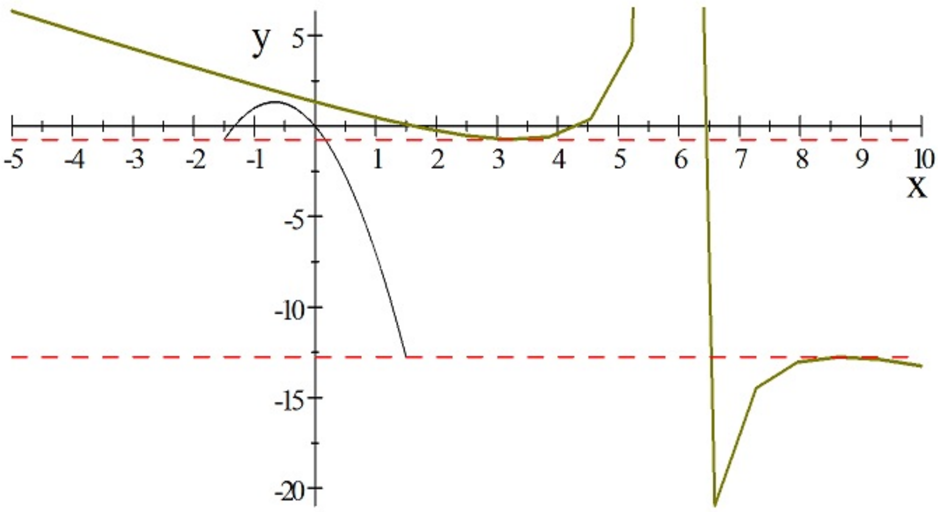
This function accept one and only extrema in the point $x=-\frac{2}{3}$ with the associate value $f(x)=\frac{4}{3}$

And, by the use of the dual canonical transformation, we can define the associate dual forme of f by:

$$\begin{aligned} f^d(\rho^*) &= \frac{-1}{2}d(a+\rho^*)^{-1}d-\mu\rho^* = \frac{-1}{2}(16)(-6+\rho^*)^{-1}-\frac{1}{2}(1.5)^2\rho^* \\ &= -(1.125\rho^*+(\frac{8}{\rho^*-6})). \end{aligned}$$

In the other part, the dual canonical problem is given by:

$$\text{(DCP)} \qquad \begin{cases} \max f^d(\rho^*)=-(1.125\rho^*+(\frac{8}{\rho^*-6})) \\ \rho^*\geq 6 \end{cases}.$$



$$\begin{aligned} f(x) &: \text{ black} \\ f^d(\rho^*) &: \text{ broun} \end{aligned}$$

Candidate(s) for extrema: $\{-0.75, -12.75\}$, at $\{[\rho_1^* = 3.3333], [\rho_2^* = 8.6667]\}$.
So, we have the following results:

functions	extremas	candidates for extremas
primal	$-0,6666$	$1,3333$
dual	$3,3333$	$-0,7500$
	$8,6667$	$-12,7500$

With:

$$\begin{aligned} \widetilde{x}_1 &= (a + \overline{\rho_1^*})^{-1} d = -1,4998, \\ \widetilde{x}_2 &= (a + \overline{\rho_2^*})^{-1} d = 1,5000. \end{aligned}$$

Immediately, we have this table:

Dual extremas $\overline{\rho_i^*}$	Primal solutions $\widetilde{x}_i$	Dalues $f(\widetilde{x}_i)$	Dual values
$3,3333$	$-1,4998$	$-0,7490$	$-0,7500$
$8,6667$	$1,5000$	$-12,7500$	$-12,7500$

In the other hand, we find the following results:

$$\overline{\rho_1^*} = 3,3333 < -a = 6,$$

with:

$$f(\widetilde{x_1}) = \min_{x \in D_{PQP}} f(x) = \min_{(\rho^*) \in D^i_{PQP}} f^d(\rho^*) = f^d\left(\overline{\rho_1^*}\right) = -0,75,$$

and:

$$\overline{\rho_2^*} = 8,6667 > -a = 6,$$

with:

$$f(\widetilde{x_2}) = \min_{x \in D^s_{PQP}} f(x) = \max_{(\rho^*) \in D^+_{PQP}} f^d(\rho^*) = f^d\left(\overline{\rho_2^*}\right) = -12,75.$$

So, by the use of the "Branch and Bound method" the convex approximate quadratic form of f is given by:

$$L_{quad}(x) = \frac{1}{2}x^2 + \frac{7}{4}x,$$

and the convex approximate quadratic problem associate to the non convex one is given by:

$$\begin{cases} \min L_{quad}(x) = \frac{1}{2}x^2 + \frac{7}{4}x \\ x \in \left[0, \frac{1}{2}\right] \end{cases},$$

where we applied the reducing and eliminate technic over the initial rectangle$S^\circ := \left[\frac{-1}{2}, \frac{1}{2}\right]$, and we find that the rest rectangle is:$S^1 := \left[0, \frac{1}{2}\right]$.

So, we have this graph:

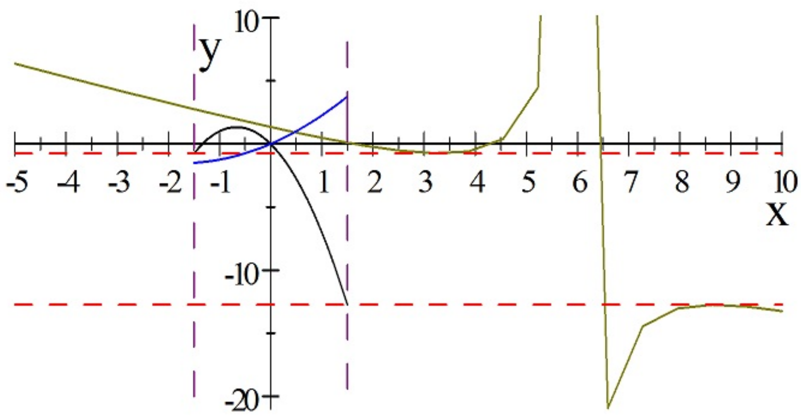


FIGURE 3. The graphic representation of the primal function f, the associate dual function f^d and the convex quadratic approximate function $L_{quad}(x)$

$$\begin{aligned} f(x) &: \text{black} \\ f^d(\rho^*) &: \text{broun} \\ (-12,75) \text{ and } (-0,75) &: \text{lightred} \\ L_{quad}(x) &: \text{lightblue} \end{aligned}$$

So, over the rectangle $S^1 := \left[0, \frac{1}{2}\right]$ we find that:

- $(\frac{1}{2})$ is the minimum point of the function f and it is the maximum point of the convex quadratic function L_{quad} and the minimum point of the associate dual function f^d over the rectangle $S^1 := \left[0, \frac{1}{2}\right]$.
- (0) is the maximum point of f and the minimum point of the convex quadratic function L_{quad} and:$f(0) = L_{quad}(0) < f^d(0)$.

8. Conclusion

In this paper we present a new rectangle Branch and Bound approach for solving non convex quadratic programming problems were we propose a new lower approximate convex quadratic functions of the objective quadratic function f over an $n-$rectangle.

This lower approximate is given to determine a lower bound of the global optimal value of the original problem (NQP) over each rectangle.
To accelerate the convergence of the proposed algorithm we used a simple two-partition and reducing technic over the subrectangles S^K in the k-step [3].

In the other hand, we present an other global method to resolve the problem (NQP), this method is "the dual canonical transformation (DCT)". This method transform a non convex quadratic problem to an Algebraic system.
It's always converge to the global optimal solution over the realisable domain which is a compact set of $\mathbb{R}^n$.

The new algorithm B&B where we used the convex quadratic approximation of the non convex quadratic function f over a rectangle $S^K = \left[L^K, U^K\right] \subseteq \mathbb{R}^n$ with $\theta \geq |\lambda_{\min}|$ and it is not empty, convex, close, and bounded (compact) of $\mathbb{R}^n$ is best then the method (DCT) over the relative Interior of the realisable domain of the function wich we optimized.

We can use the Branch and Bound method (Separation and evaluation) where we write the function f like a (DC) form (deference of tow convex functions) and we approximate the concave part by a convex quadratic function by the use of the lower bound or the upper bound of the realisable rectangle S^K which have a very small rank and it's considered as a confianced region, and by this we assured the existence of the optimal global solution of the original problem (NQP).

In the other hand, the "Branch and Bound method" obtain the approximate optimal solution of the optimal global solution of the original problem (NPQ) with a quadratic vitesse of convergence over the realisable set S^K , but the (DCT) method find the optimal global solution over the Sphere of this realisable set S^K.

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