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ON THE NUMBER OF VERTICES OF INTEGER CONVEX POLYTOPES

Ravelonirina Hanitriniaina Sammy Grégoire[1] and Rakoto Jean Jolly

ABSTRACT. The aim of this paper is to propose a polynomial that we call "characteristic polynomial" of an integer convex polytope of dimension d $(d \geq 1)$. This polynomial makes it possible to count the number of vertices of an integer convex polytope without using the Schläfli symbol. We give after that the algebraic characteristics of this polynomial.

1. INTRODUCTION

In the book XIII cf. [2] about his studies of elements, Euclid gives a description and construction of Plato's solids. Unlike Plato, he demonstrated that there are five and only five regular polyhedra known as Plato's polyhedra cf. [5, 6]. The classification and the numerical constraints of these regular polyhedra give rise to the famous formula of Euler: $f - a - s = 2$ where $f,\ a,\ s$ denote respectively the numbers of faces, edges and vertices of a spherical polyhedron cf. [2]. And from the Schläfli symbol, we can calculate $f,\ a$ and s such that $f = \frac{4n}{2m+2n-mn}$, $a = \frac{mf}{2}$, $s = \frac{mf}{n}$ [2] where (m, n) denotes the Schläfli symbols, varying according to regular polyhedra (tetrahedron, cube, icosahedron, octahedron, dodecahedron). And in our work, we proposed a characteristic polynomial, a polynomial resulting from

[1] *corresponding author*
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the special polynomial cf. [7, 8]. This polynomial makes it possible to calculate the number of vertices of an integer convex polytope of dimension d ($d \geq 1$) without using the symbol of Schläfli. The characteristic polynomials of regular polyhedra are obtained for dimensions superior or equals to one ($d \geq 1$). But from the dimension $d = 6$ ($d \geq 6$), the characteristic polynomials appear only if the dimensions of the polytopes follow an arithmetic sequence of reason four and of first term equals to six, defined by: $d_n = 4n+6$, $\forall n \in \mathbb{N}$. And the constant terms of these polynomials define an arithmetic sequence of reason two and of first term equals to four. Let us denote by c_n, the constant terms of these characteristic polynomials ; for $d \geq 6$, we have: $c_n = 2n + 4$, for all $n \in \mathbb{N}$. The most remarkable characteristics of these polynomials give us a great interest in the particular study of this third generation of Ehrhart's polynomial.

2. Preliminaries

2.1. **Ehrhart's polynomial.** In the following development we denote by P the convex polytope with integer vertices, d its dimension ($d \geq 1$) and P° the interior of the polytope P.

Definition 2.1. *[1] For each integer convex polytope with integer vertices P, we can associate an Ehrhart's polynomial f_P which makes it possible to count the number of integer coordinate points inside and on the edge of the polytope.*

Theorem 2.1. *[1] There exists a unique polynomial f_P with real variable n of degree d with rational coefficients such that:*

(1) $f_P(n) = card(nP \cap \mathbb{Z}^d)$ *for all* $n \geq 1$*;*
(2) *Moreover, we have* $f_P(0) = 1$*;*
(3) $f_P(-n) = (-1)^d card(nP^\circ \cap \mathbb{Z}^d)$ *for all* $n \geq 1$ *(reciprocity law).*

Proof. Cf. [3]. □

The coefficients of Ehrhart's polynomials

Proposition 2.1. *[1] Let f_P be an Ehrhart polynomial such that:*

$$f_P(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + 1.$$

Then we have:

$$c_d = vol_d(P) \text{ and } c_{d-1} = \frac{1}{2}\sum_{F\subset P} vol_{d-1}(F).$$

The sum relates to the $(d-1)$*–faces of* P.

Proof. Cf. [1]. □

Examples 1. *Cf. [7]*

- $\underline{d=2}$*: Let be a triangle with vertices* $(0,0)$*,* $(2,0)$ *and* $(0,2)$*. Then*

$$f_P(n) = card(nP\cap \mathbb{Z}^2) = \binom{n+2}{2} = \frac{1}{2}n^2 + \frac{3}{2}n + 1,$$

where $\binom{n}{k}$ *denotes the combination of* k *elements chosen from among the* n *elements (*$n \geq k$*) .*

- $\underline{d=3}$*: Let be a tetrahedron with vertices* $(0,0,0)$*,* $(0,1,1)$*,* $(1,0,1)$ *and,* $(1,1,0)$*. Then*

$$\begin{aligned} f_P(n) &= card(nP\cap \mathbb{Z}^3)\\ &= \delta_0\binom{n+3}{3} + \delta_1\binom{n+2}{3} + \delta_2\binom{n+1}{3} + \delta_3\binom{n}{3}. \end{aligned}$$

with $\delta_0 = \delta_2 = 1$ *and* $\delta_1 = \delta_3 = 0$*. Also,* $f_P(n) = \frac{1}{3}n^3 + n^2 + \frac{5}{3}n + 1$ *and* $f_P(-n) = (-1)^3 card(nP^\circ \cap \mathbb{Z}^3) = \frac{1}{3}n^3 - n^2 + \frac{5}{3}n - 1$.

2.2. **Family of Ehrhart's polynomials [7, 8].** Let P be a convex polytope with integer vertices, and d its dimension ($d \geq 1$). The family of Ehrhart polynomials of degree k and parameter m, variable n, denoted $g_{m,d,k}(n)$ is written cf. [4]:

$$g_{m,d,k}(n) = \prod_{j=d-k+1}^{d}(n+j) + \prod_{j=0}^{k-1}(n-j),$$

where d is the dimension of P, with:

- $k = \frac{d}{2}$, if d is pair;
- $k = \lfloor\frac{d+1}{2}\rfloor$, if d is odd.

The polynomial $g_{m,d,k}(n)$ can also be written in the form

$$\begin{aligned} g_{m,d,k}(n) &= (m+1)n^k + \ldots + J_d \\ &= a_k n^k + a_{k-1} n^{k-1} + \ldots + J_d, \end{aligned}$$

with $a_k = m+1$: the coefficient of the term of the highest degree and

$$J_d = \prod_{j=d-k+1}^{d} j = g_{m,d,k}(0) = \prod_{i=0}^{k-1} c_{k,i} = d(d-1)\ldots(d-k+1)$$

the constant term of the family of these polynomials.

Examples 2. *Cf. [7]*

- $m=2,\ d=2,\ k=2.$
  Let be a polytope of dimension 3. *The family of the associated polynomials is:*

$$g_{2,3,2}(n) = 3n^2 + 3n + 6.$$

- $m=9,\ d=7,\ k=4.$
  Let be a polytope of dimension 7. *Then the corresponding family of Ehrhart polynomials is*

$$g_{9,7,4}(n) = 10n^4 + 32n^3 + 278n^2 + 584n + 840.$$

2.3. **Special polynomial [7, 8].**

Definition 2.2. *Let us denote by* $p_k(n)$ *the polynomials resulting from the family of Ehrhart polynomials of variables* n*, of degree* k *without a constant term whose coefficients are the factors of the decomposition of the constant term* J_d *of* $g_{m,d,k}(n)$ *into products of decreasing factors. We write*

$$p_k(n) = c_{k,0} n^k + c_{k,1} n^{k-1} + \ldots + c_{k,k-1} n = \sum_{i=0}^{k-1} c_{k,i} n^{k-i}.$$

Definition 2.3. *We call a special polynomial noted* p_s*, the polynomial which is written:*$p_s(n) = p_k(n) + a_0$ *where* $a_0 \equiv J_d \mod (d+k)$.

Example 1. *[7]*

- $d=3$, $k=2$.
  Let be a polytope P *of dimension* 3. *The associated special polynomial is* $p_s(n) = 3n^2 + 2n + 1$.

- $d = 7$, $k = 4$.
  Let be a polytope P of dimension 7, its special polynomial is $p_s(n) = 7n^4 + 6n^3 + 5n^2 + 4n + 1$.

Remark 2.1.

- *The degree of the family of Ehrhart polynomials and of the special polynomial is inferior to the degree of Ehrhart's polynomial for the same dimension.*
- *The family of Ehrhart's polynomials and the special polynomial have the same degree but coefficients that can be different for the same dimension.*

3. Characteristic polynomials

3.1. Integer convex polytopes and characteristic polynomials.

Definition 3.1. *A simplex of dimension i is the convex hull of $(i+1)$ points in $\mathbb{R}^n$ not located in an affine hyperplane, each of its faces of dimension $(i-1)$ is also a simplex and has exactly i vertices.*

Definition 3.2.

- *A polytope in $\mathbb{R}^n$ is the convex hull of a finite number of points in $\mathbb{R}^n$.*
- *A convex polytope is integer if the coordinates of its vertices are integers.*

Example 2. *The five regular convex polyhedra known as Plato's polyhedra are: the cube, the tetrahedron, the icosahedron, the octahedron and the dodecahedron.*

3.2. Characteristic polynomials.

Definition 3.3. *We call a polynomial characteristic of an integer convex polytope P of dimension d, the polynomial denoted p_c makes it possible to count the number of vertices of this convex polytope, defined by:*

- *For $1 \leq d \leq 2$, p_c is of degree k , with k integer such that*
  - *If d is pair, the degree of p_c is $k = \frac{d}{2}$ and $p_c(n) = kn^k + k + 1$ and k divides $2k + 1$.*
  - *If d is odd, the degree of p_c is $k = \lfloor \frac{d+1}{2} \rfloor$, and $p_c(n)$ is written by: $p_c(n) = kn^k + k$ and $k \mid 2k$ (k divides the sum of the coefficients of p_c).*
- *For $d \geq 3$, the degree of p_c is $k - 1$, k integer such that*

- *if d is odd,*

$$p_c(n) = k + \sum_{i=0}^{k-1} (k-i) n^{k-1-i}, \text{ for all } n, k \geq 1$$

$$\text{and} \quad k \mid k + \sum_{i=0}^{k-1} (k-i) \ (k \text{ divides the sum of the coefficients of } p_c);$$

- *otherwise,*

$$p_c(n) = (k-1) + \sum_{i=0}^{k-1} (k-i) n^{k-1-i}, \text{ for all } n, k \geq 1$$

$$\text{and} \quad k \mid (k-1) + \sum_{i=0}^{k-1} (k-i) \ (k \text{ divides the sum of the coefficients of } p_c).$$

Example 3. *Let be a convex polytope of dimension d ($d \geq 1$)*

- *If P is a segment of extremities A and B. $\dim P = 1$, that is to say: $d = 1$, the degree of the characteristic polynomial p_c is $\frac{2}{2} = 1$, $p_c(n) = n + 1$, $1 \mid 2$.*
- *If P is a polygon (for example a regular triangle), $d = 2$ and $k = 1$, $p_c(n) = n + 2$, and $1 \mid 3$.*
- *If P is a polyhedron (cube, tetrahedron, octahedron, icosahedron, dodecahedron), $d = 3$, $k = 2$, $p_c(n) = 2n + 2$, and $2 \mid 4$.*

3.3. **Evaluation of p_c for some values of n, for $d = 3$.** To know the number of vertices of some polyhedra, we evaluate the characteristic polynomial of the polyhedron for some values of n. $p_c(n) = 2n + 2$, polyhedron P of dimension $d = 3$.

- If $n = 1$, $p_c(1) = 4$, then P is a tetrahedron with 4 vertices.
- If $n = 2$, $p_c(2) = 6$, then P is an octahedron (6 vertices).
- If $n = 3$, $p_c(3) = 8$, then P is a hexahedron or cube which has 8 vertices.
- If $n = 5$, $p_c(5) = 12$, then P is an icosahedron (12 vertices).
- If $n = 9$, $p_c(9) = 20$, then P is a dodecahedron which has 20 vertices.

Remark 3.1. *The degree of the characteristic polynomial is inferior to the degree of the special polynomial for the same dimension d ($d \geq 3$).*

Proposition 3.1. *The dimensions of convex polytopes which allow to have characteristic polynomials define an arithmetic sequence* $(d_n)_{n\in\mathbb{N}}$ *of reason* 4 *and first term* 6 *from dimension* d *(*$d \geq 6$*),* $d_n = 4n + 6$*,* $\forall\, n \geq 0$*.*

Proof. Easy, just use the induction reasoning. □

Proposition 3.2. *Let* P *be a convex polytope of dimension* d *(*$d \geq 1$*) with integer vertices. The roots of the characteristic polynomials of* P *satisfy the Conjecture of Beck and al., That is: "All the roots* α_i *of* $p_c(n)$ *satisfy the following relation:* $-d \leq Re(\alpha_i) \leq d - 1$*, for all* i*, where* $\alpha_i \in \mathbb{C}$ *and* $Re(\alpha_i)$ *denote the real part of* α_i*" cf. [4].*

Proof. Any special polynomial of dimension d satisfies the Conjecture of Beck and al. cf. [7]. However, the characteristic polynomials of a convex polytope with integer vertices of dimension d are obtained from special polynomials. Thus, by transitivity, the roots of the characteristic polynomials verify the Conjecture of Beck and al. Which completes the proof. □

3.4. **Properties of characteristic polynomials.**

Proposition 3.3. *Let* P *be a convex polytope with integer vertices of dimension* d *(*$d \geq 3$*) and* p_c *the characteristic polynomial of* P*, of degree* $(k-1)$*.*

- $i)$ *The coefficient of the term with the highest degree of* $p_c(n)$ *is equal to* k *and the highest degree is* $(k-1)$*.*
- $ii)$ *The constant term denoted* c_n *of* p_c *is equal to* $k+1$ *and forms an arithmetic sequence of reason* 2 *and of first term equals to* 4*, that is to say:* $c_n = 2n + 4$*, for all* $n \in \mathbb{N}$ *and* $d \geq 6$*,* $k \geq 3$*.*
- $iii)$ *The coefficient of the term with the highest degree of* p_c *divides the sum of the coefficients.*

Proof.
$i)$ and $iii)$. It is sufficient to use the definition of p.
$ii)$ Let c_n be the constant term of p_c and a sequence $(c_n)_{n\in\mathbb{N}}$ with term equals to $c_n = k + 1$, $d \geq 6$, $k \geq 3$. Then

$$p_c(n) = k + \sum_{i=0}^{k-1}(k-i)n^{k-1-i}, \text{ for all } n, k \geq 1.$$

- for $i = k - 1$,

$$\begin{aligned} c_n &= k + [k - (k-2)]n^{k-1-(k-1)} \\ &= k + (k - k + 1)n^{k-1-k+1} \\ c_n &= k + 1 \end{aligned}$$

- for

$$\begin{aligned} k &= 3,\ d = 6,\ \ c_0 = 3 + 1 = 4 \\ k &= 5,\ d = 10,\ c_1 = 5 + 1 = 6 \\ &\vdots \qquad \vdots \qquad\quad \vdots \\ k &= n,\ \ d_n, \qquad c_n = 2n + 4. \end{aligned}$$

By induction on n, we have : $c_n = 2n + 4$.

□

3.5. **Sequence of characteristic polynomials.** Let P be a convex polytope with integer vertices, of dimension d_n such that $d_n = 4n + 6$ and $k_n = 2n + 3$; for all $n \geq 0$, the coefficient of the term with the highest degree of p_c.

Examples 3. *$d_n = 4n + 6$, $k_n = 2n + 3$, for all $n \in \mathbb{N}$.*

- *If $n = 0$,*

$$\begin{aligned} p_0 &= k_0 + \sum_{i=0}^{k_0-1}(k_0 - i)n^{k_0-1-i}, \textit{ for all } \ n \in \mathbb{N}, \\ p_0 &= 3n^2 + 2n + 4. \end{aligned}$$

- *If $n = 1$ then $p_1 = 5n^4 + 4n^3 + 3n^2 + 2n + 6$.*
- . . . .
- *And so on.*

Proposition 3.4. *For any sequence of characteristic polynomials $(p_n)_{n\in\mathbb{N}}$:*

$i)$ *the dominant coefficients form a sequence, denoted $(k_n)_{n\in\mathbb{N}}$, arithmetic of reason 2 and of first term equal to 3 and of general term k_n such that:*

$$k_n = 2n + 3, \ \textit{ for all } \ n \in \mathbb{N},$$

$ii)$ *the constant terms define an arithmetic sequence denoted* $(c_n)_{n\in\mathbb{N}}$ *of reason* 2 *and of first term equals to* 4*:*

$$c_n = 2n + 4.$$

$iii)$
- *the coefficients of the term of degree one to the term of the highest degree of the sequence* $(p_n)_{n\in\mathbb{N}}$ *form an arithmetic progression of reason* 1 *and of first term equals to* 2*.*
- *the degrees of the sequence of polynomials* $(p_n)_{n\in\mathbb{N}}$ *form an arithmetic progression of reason* 1 *and of first term equals to* $k_n + 1$*.*

Proof. Immediate. □

Theorem 3.1. *For any integer convex polytope* P *of dimension* d $(d \geq 6)$*, there exists a sequence of characteristic polynomials* $(p_n)_{n\in\mathbb{N}}$ *satisfying* $i)$*,* $ii)$ *and* $iii)$ *in the previous Proposition.*

Proof. Immediate in accordance with the Definition 3.3 and Proposition 3.4. □

Remark 3.2. *Two integer convex polytopes* P *and* P' *of respective dimension* d *and* d' *with* $d < d'$ *can have the same number of vertices of different values of* n*.*

Example 4. *Let* P *and* P' *be two integer convex polytopes of respective dimensions* $d = 3$ *and* $d' = 10$*.*

The characteristic polynomials are:

- *for* P *of dimension* 3 *then* $p_c(n) = 2n + 2$*.*
- $p_c(n) = 5n^4 + 4n^3 + 3n^2 + 2n + 6$*. And for* P' *of dimension* 10 *so we have* $p_c(n) = 5n^4 + 4n^3 + 3n^2 + 2n + 6$*.*

The number of vertices of P *and* P' *for* $n = 9$ *and* $n = 1$ *respectively are:* $p_c(9) = 20\ vertices$*,* P *is then a dodecahedron and* $p_c(1) = 20\ vertices$*,* P *and* P' *have the same number of vertices with different values of* n*.*

4. Conclusion and discussion

For a given integer convex polytope of dimension d where $(d \geq 1)$, there is a relation between the different polynomials obtained from P.

- For the Ehrhart polynomial $f_P(n)$: it is a polynomial whose degree is equal to the dimension d of the integer convex polytope P.

  The polynomial $f_P(n)$ makes it possible to count the number of integer points inside and on the edge of the integer convex polytope P.
- For the family of Ehrhart polynomials $g_{m,d,k}(n)$, Higashitani, in Theorem 2.1 of paragraph 2, pages 620 cf. [4], constructs in a very clever way a convex polytope with integer vertices of dimension d whose Ehrhart polynomial $f_P(n)$ of degree d satisfies a polynomial of degree k such that:

$$g_{m,d,k}(n) = \prod_{j=d-k+1}^{d} (n+j) + m \prod_{j=0}^{k-1} (n-j),$$

  where $m \geq 1,\ d \geq 2\ and\ k = \dfrac{d}{2}$ if d is pair, but $k = \dfrac{d+1}{2}$ if d is odd.

  The family of Ehrhart's polynomials $g_{m,d,k}(n)$ allows Higashitani with his numerical programs to find counterexamples to the conjecture of Beck and al.

  The degree of $g_{m,d,k}(n)$ is inferior to the degree of $f_P(n)$. We say that $g_{m,d,k}(n)$ is a "first generation" polynomial of Ehrhart's polynomial f_P.
- The special polynomial $p_s(n)$ cf. [7,8] is a polynomial resulting from $g_{m,d,k}(n)$ of degree k, of variable n, built from the decomposition into product of decreasing factors of the constant term of the family of Ehrhart's polynomials $g_{m,d,k}(n)$.

  Any special polynomial satisfies the conjecture of Beck and al. cf. [7]. It is a "second generation" polynomial of Ehrhart's polynomial f_P.
- For the characteristic polynomial $p_c(n)$, it is a polynomial of degree $(k-1)$, of variable n, obtained from the special polynomial p_s. This polynomial makes it possible to count the number of vertices of an integer convex polytope P of dimension d ($d \geq 1$).

  It is a "third generation" polynomial of f_P.

Thus, our study leads, at the theoretical level, to the construction of a polynomial model: the characteristic polynomial p_c which should be programmed from an algorithm. And the polynomials, as modeling tools of suitable modeling for the linear system play an important role for their properties in numerically solving an optimization or decision problem. Even today, the design and then the calculation of control laws for aeronautical or space systems (transport plane, war plane, missiles, launchers, satellites, etc.) are often carried out by the techniques from the linear world.

In perspective, we suggest that we can apply the characteristic polynomials in the design of reliable telecommunication networks, for example in network topology: the bus topology.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
UNIVERSITY OF ANTANANARIVO
ANTANANARIVO 101,
MADAGASCAR.
Email address: rhsammy@yahoo.fr

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
UNIVERSITY OF UNIVERSITY OF ANTANANARIVO
ANTANANARIVO 101,
MADAGASCAR.
Email address: rjeanjolly@gmail.com