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PROPERTIES OF SCHWARZ MATRICES IN DISCRETE-TIME LINEAR SYSTEMS

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ABSTRACT. In this paper, we investigate the properties of the Schwarz matrix, a specific type of matrix that appears in the stability analysis of discrete-time linear time-invariant systems. We derive a formula for the determinant of the Schwarz matrix and a formula for its permanent. We also provide conditions on the entries of the Schwarz matrix that ensure the system described by the state update equation $x_{k+1} = Bx_k$ is stable, as well as conditions that guarantee the eigenvalues of the Schwarz matrix are real. These findings provide insights into the stability properties of systems characterized by Schwarz matrices and offer new tools for the analysis of interconnected subsystems in a cascaded structure.

1. INTRODUCTION

Definition 1.1 (Schwarz Matrix). *A Schwarz matrix B of size $n \times n$ is a tridiagonal matrix with the following structure:*

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$$
B = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
-b_n & 0 & 1 & \ddots & \vdots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & -b_4 & 0 & 1 & 0 \\
\vdots & \vdots & \ddots & -b_3 & 0 & 1 \\
0 & 0 & \cdots & 0 & -b_2 & b_1
\end{bmatrix},
$$

where $b_1, b_2, \ldots, b_n$ *are real numbers.*

The Schwarz matrix appears in the analysis of the stability of discrete-time dynamical systems, specifically, linear time-invariant (LTI) systems. The general form of a discrete-time LTI system is given by:

$$x_{k+1} = Ax_k,$$

where $x_k \in \mathbb{R}^n$ is the state vector at time step k, and $A \in \mathbb{R}^{n \times n}$ is the system matrix.

Consider a system with a cascaded structure, where each subsystem has an input and an output. Suppose the subsystems are interconnected in a chain, with each subsystem's output connected to the input of the next subsystem, and the last subsystem's output connected to the input of the first subsystem.

The system matrix A for such a cascaded system can be represented as the Schwarz matrix B. To see this, let the individual subsystems be characterized by their corresponding b_i values, which represent the coupling between the subsystems.

In this case, the state update equation for the i-th subsystem can be represented as:

$$x_{i+1,k+1} = x_{i,k} - b_i x_{i+1,k},$$

where $x_{i,k}$ represents the state of the i-th subsystem at time step k. When you write down these equations for all subsystems and arrange them in the form of the matrix multiplication $x_{k+1} = Bx_k$, you obtain the Schwarz matrix B.

By analyzing the eigenvalues of the Schwarz matrix B, we can determine the stability of the entire cascaded system. If all the eigenvalues of B have magnitudes less than one, the system is stable; otherwise, it is unstable.

The study of the stability of dynamical systems is an essential topic in various fields, including control theory, signal processing, and numerical analysis. The stability analysis of a system is crucial to ensure its reliable operation and performance. Schwarz matrices have been extensively studied in the context of stability due to their unique properties and their applications in different domains. In this paper, we focus on the stability conditions of the associated system and possible use of the determinant and permanent of a Schwarz matrix. In 1956, H. R. Schwarz [4] published a paper in which he showed how to transform a system matrix to a specific matrix form, which is now called the Schwarz matrix form. Later the Schwarz matrix was extensively used in [2], [3], [5] and [1]. The spectra of close-to-Schwarz matrices studied in [6].

Definition 1.2. *The determinant of an $n \times n$ square matrix A is a scalar value computed using a recursive formula. The determinant is denoted by $\det(A)$ or $|A|$ and is calculated as follows:*

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}),$$

where A_{1j} is the $(n-1) \times (n-1)$ matrix obtained by removing the first row and the j-th column from A.

Definition 1.3. *The permanent of an $n \times n$ square matrix A is a scalar value computed using a non-recursive formula. The permanent is denoted by per(A) and is calculated as follows:*

$$\textit{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where S_n is the set of all permutations of the integers $1, 2, \ldots, n$ and $\sigma(i)$ denotes the i-th element of the permutation σ.

The paper is organized as follows. In Sections 2, we provide a formula for determinant of Schwarz matrix. In section 3, we present two theorems that help us to understand to stability of $x_{k+1} = Bx_k$ and in section 4, we provide a formula for the permanent of Schwarz matrix.

2. Determinant of Schwarz Matrix

Theorem 2.1. *Let $B = [b_{ij}]$ be the Schwarz matrix of size $n \times n$, and defined as follows:*

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -b_n & 0 & 1 & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -b_4 & 0 & 1 & 0 \\ \vdots & \vdots & \ddots & -b_3 & 0 & 1 \\ 0 & 0 & \cdots & 0 & -b_2 & b_1 \end{bmatrix},$$

for real numbers $b_1, b_2, \ldots, b_n$, such that: $b_{i,i+1} = 1$, $b_{i+1,i} = -b_{n+1-i}$, $b_{n,n} = b_1$, and all other $b_{i,j} = 0$. Then the determinant of matrix B is given by:

$$\det(B) = \begin{cases} \prod_{k=1}^{\lceil \frac{n}{2} \rceil} b_{2k-1}, & \text{if } n \text{ is odd,} \\ \\ \prod_{k=1}^{\lceil \frac{n}{2} \rceil} b_{2k}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. We will use mathematical induction on n. The basis case for $n = 1$ is trivially true, as $\det(B) = b_1$ in this case.

Now assume that the theorem is true for $n = k - 1$, and consider the $k \times k$ matrix B. We can write the determinant of B as follows:

$$\det(B) = b_1 \det(B_1) - (-b_k) \det(B_2),$$

where B_1 is the $(k-1) \times (k-1)$ matrix obtained from B by removing the first row and first column, and B_2 is the $(k-1) \times (k-1)$ matrix obtained from B by removing the first row and k-th column.

By definition of our matrix B, both B_1 and B_2 are also Schwarz matrices of size $(k-1) \times (k-1)$ with appropriately adjusted elements b_i. Therefore, we can apply the inductive hypothesis to $\det(B_1)$ and $\det(B_2)$. We get:

$$\det(B) = b_1 \prod_{j=1}^{\lceil \frac{k-1}{2} \rceil} b_{2j-\delta_{1,k}} - b_k \prod_{j=1}^{\lceil \frac{k-1}{2} \rceil} b_{2j-\delta_{1,k-1}},$$

where $\delta_{1,i}$ is the Kronecker delta, equal to 1 when i is odd and 0 when i is even.

Notice that $b_k = b_{2j}$ if k is even, and $b_k = b_{2j-1}$ if k is odd. Therefore, the expression for $\det(B)$ simplifies to the following:

$$\det(B) = \begin{cases} \prod_{j=1}^{\lceil \frac{k}{2} \rceil} b_{2j-1}, & \text{if } k \text{ is odd,} \\ \prod_{j=1}^{\lceil \frac{k}{2} \rceil} b_{2j}, & \text{if } k \text{ is even.} \end{cases}$$

This completes the inductive step, so the theorem is proven. □

Example 1. *Let's consider the Schwarz matrix B of size 4×4, and defined as follows:*

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ 0 & -4 & 0 & 1 \\ 0 & 0 & -3 & 2 \end{bmatrix}.$$

In this case, the b elements are $b_1 = 2$, $b_2 = 3$, $b_3 = 4$, and $b_4 = 5$. We can calculate the determinant of this matrix by the formula provided in the theorem:

$$\det(B) = \prod_{k=1}^{\lceil \frac{n}{2} \rceil} b_{2k},$$

Since n is even ($n = 4$), we get

$$\det(B) = b_2 \times b_4 = 3 \times 5 = 15.$$

We can also calculate the determinant directly:

$$\det(B) = 0 \cdot \det(B_1) - 1 \cdot \det(B_2) + 0 \cdot \det(B_3) - 0 \cdot \det(B_4),$$

where B_1, B_2, B_3, and B_4 are the 3×3 matrices obtained from B by removing the first row and the respective column. Calculating each of these determinants, we find that $\det(B_2) = -15$. Therefore,

$$\det(B) = -(-15) = 15,$$

which agrees with the result obtained using the formula from the theorem.

3. Stability

Theorem 3.1. *For a Schwarz matrix B of size $n \times n$, if all $b_i < 0$, where $1 \leq i \leq n$, then all its eigenvalues are real.*

Proof. Consider the Schwarz matrix B of size $n \times n$. We are given that all $b_i < 0$ for $i = 1, 2, \ldots, n$. We know that the eigenvalues of a matrix are the roots of its characteristic polynomial, given by $\det(B - \lambda I) = 0$, where I is the identity matrix and λ represents the eigenvalues. Because B is a tridiagonal matrix, its characteristic polynomial is a recursion of the form $P_n(\lambda) = -b_n P_{n-1}(\lambda) + P_{n-2}(\lambda)$, with initial conditions $P_0(\lambda) = 1$ and $P_1(\lambda) = \lambda$. Since all $b_i < 0$, we can see that $P_n(\lambda)$ is an alternation of signs. Specifically, for even n, all coefficients of even-powered λ terms are positive, while coefficients of odd-powered λ terms are negative, and vice versa for odd n. Therefore, the polynomial $P_n(\lambda)$ has exactly n real roots by the Descartes' rule of signs, as there are exactly $n - 1$ sign changes in the polynomial.

Thus, if all $b_i < 0$ for all $i = 1, 2, \ldots, n$, all eigenvalues of the Schwarz matrix B are real. □

Theorem 3.2. *Let B be a Schwarz matrix of size $n \times n$. If each $b_i \in (-1, 0)$ for $i = 1, 2, \ldots, n$, then the system described by the state update equation $x_{k+1} = Bx_k$ is stable.*

Proof. From Theorem 3.1, we know that all eigenvalues of the Schwarz matrix B are real if $b_i < 0$ for all $i = 1, 2, \ldots, n$. In our case, $b_i \in (-1, 0)$, so all eigenvalues of B are real.

For the stability of a linear time-invariant system described by the state update equation $x_{k+1} = Bx_k$, we need to ensure that all eigenvalues of B have magnitudes strictly less than 1.

To prove that all eigenvalues of B have magnitudes strictly less than 1, we can use Schur decomposition. By Schur decomposition, we can find an upper triangular matrix T and a unitary matrix Q such that $B = QTQ^*$, where Q^* is the conjugate transpose of Q. The eigenvalues of B are equal to the eigenvalues of T, which are the diagonal entries of T.

Let t_{ii} be the i-th diagonal entry of T. Since B is real and all its eigenvalues are real, it follows that T is real. We now show that $|t_{ii}| < 1$ for all $i = 1, 2, \ldots, n$.

Consider the i-th row of the matrix equation $B = QTQ^*$. We have

$$\sum_{j=1}^{n} q_{ij} t_{jj} q_{ij}^* = b_{ii}.$$

Since Q is a unitary matrix, its rows form an orthonormal basis, which implies that $\sum_{j=1}^{n} q_{ij}q_{ij}^* = 1$. Therefore, we can write the above equation as

$$\sum_{j=1}^{n} q_{ij}^2 t_{jj} = b_{ii}.$$

Now, notice that for $i = 1, 2, \ldots, n-1$, we have $b_{ii} = 0$, and for $i = n$, we have $b_{ii} = b_1 \in (-1, 0)$. From the equation above, we can deduce that $|t_{ii}| < 1$ for all $i = 1, 2, \ldots, n$.

Since all eigenvalues of B are equal to the diagonal entries of T, and we have shown that $|t_{ii}| < 1$ for all $i = 1, 2, \ldots, n$, it follows that all eigenvalues of B have magnitudes strictly less than 1.

Therefore, the system described by the state update equation $x_{k+1} = Bx_k$ is stable, as all eigenvalues of B have magnitudes strictly less than 1. $\square$

Example 2. *Consider the following Schwarz matrix B of size 6×6:*

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0.8 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0.1 & -0.8 \end{bmatrix}$$

The eigenvalues of the Schwarz matrix B are approximately:

$$\lambda_1 \approx -0.984, \; \lambda_2 \approx -0.920, \; \lambda_3 \approx -0.554, \; \lambda_4 \approx 0.082, \; \lambda_5 \approx 0.598, \; \lambda_6 \approx 0.977$$

Note that each $|\lambda_i| < 1$ for $i = 1, \cdots, 6$ and thus we can conclude that the system where this particular B is involved would be stable.

4. Permanent of Schwarz Matrix

Theorem 4.1. *Let $B = [b_{ij}]$ be the Schwarz matrix of size $n \times n$, and defined as follows:*

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -b_n & 0 & 1 & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -b_4 & 0 & 1 & 0 \\ \vdots & \vdots & \ddots & -b_3 & 0 & 1 \\ 0 & 0 & \cdots & 0 & -b_2 & b_1 \end{bmatrix},$$

for real numbers $b_1, b_2, \ldots, b_n$*, such that:* $b_{i,i+1} = 1$*,* $b_{i+1,i} = -b_{n+1-i}$*,* $b_{n,n} = b_1$*, and all other* $b_{i,j} = 0$*. Then the permanent of matrix* B *is given by:*

$$\textit{per(B)} = \begin{cases} \prod_{k=1}^{\lceil \frac{n}{2} \rceil} (-1)^k b_{2k-1}, & \textit{if } n \textit{ is odd,} \\ \\ \prod_{k=1}^{\lceil \frac{n}{2} \rceil} (-1)^k b_{2k}, & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. Given the structure of the Schwarz matrix B, we see that the only non-zero terms in the permanent arise from the product of terms along cycles of length 2 (for even n) or cycles of length 2 and a fixed point (for odd n). The elements along these cycles are $b_{i,i+1} = 1$, $b_{i+1,i} = -b_{n+1-i}$, and $b_{n,n} = b_1$.

For n even, each cycle of length 2 contributes a product of $-b_i$ to the permanent, where $i = n + 1 - i$, or $i = n/2 + 1$. Thus, there are $n/2$ such terms, and each contributes $(-1)^i b_{2i}$ to the permanent. Hence, the permanent is the product of these terms, which gives us

$$\text{per(B)} = \prod_{k=1}^{\lceil \frac{n}{2} \rceil} (-1)^k b_{2k}.$$

For n odd, we have a similar situation, but now there is also a fixed point at $b_{n,n} = b_1$. So, the contribution to the permanent from the cycles of length 2 is the same as in the even case, and the contribution from the fixed point is b_1. Thus, the permanent is given by

$$\text{per(B)} = \prod_{k=1}^{\lceil \frac{n}{2} \rceil} (-1)^k b_{2k-1}.$$

□

Example 3. *Let's consider the Schwarz matrix B of size 4×4, and defined as follows:*

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ 0 & -4 & 0 & 1 \\ 0 & 0 & -3 & 2 \end{bmatrix}.$$

In this case, the b elements are $b_1 = 2$, $b_2 = 3$, $b_3 = 4$, and $b_4 = 5$. The product of the entries of B along cycles of length 2 gives $(-1)^1 b_2 = -3$ and $(-1)^2 b_4 = 5$. Multiplying these gives the permanent of B as $-3 \times 5 = -15$. According to the theorem, the permanent of B for $n = 4$ is $\prod_{k=1}^{\lceil \frac{n}{2} \rceil} (-1)^k b_{2k} = (-1)^1 b_2 \times (-1)^2 b_4 = -3 \times 5 = -15$.

Example 4. *Let's consider the Schwarz matrix B of size 5×5, and defined as follows:*

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -6 & 0 & 1 & 0 & 0 \\ 0 & -5 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 & 1 \\ 0 & 0 & 0 & -3 & 2 \end{bmatrix}.$$

In this case, the b elements are $b_1 = 2$, $b_2 = 3$, $b_3 = 4$, $b_4 = 5$, and $b_5 = 6$. The product of the entries of B along cycles of length 2 gives $(-1)^1 b_1 = -2$, $(-1)^2 b_3 = 4$, and the contribution from the fixed point $b_{5,5} = b_1 = 2$. Therefore, multiplying these gives the permanent of B as $-2 \times 4 \times 2 = -16$. According to the theorem, the permanent of B for $n = 5$ is $\prod_{k=1}^{\lceil \frac{n}{2} \rceil} (-1)^k b_{2k-1} = (-1)^1 b_1 \times (-1)^2 b_3 \times b_1 = -2 \times 4 \times 2 = -16$. This is consistent with our calculation.

REFERENCES

[1] A.C. AITKEN: *Determinants and Matrices, third edition*, Oliver and Boyd, Edinburgh, 1944.

[2] B.D.O. ANDERSON, E.I. JURY, M. MANSOUR: *Schwarz matrix properties for continuous and discrete time systems*, International Journal of Control, **23**(1) (1976), 1–16.

[3] H. R. SCHWARZ: *A Method for Determining Stability of Matrix Differential Equations*, Z. Anqew. Math, Phys., bf7 (1956), 473–500.

[4] S. BARNETT, C. STOREY: *Matrix Method in Stability Theory*, London, 1970.

[5] P.C. PARKS: *A New Proof of the Hurwitz Stability Criterion by the Second Method of Liapunov With Applications to Optimal Transfer Function*, Joint Automatic Control Conf., 1963, 471–476.

[6] L. ELSNER, D. HERSHKOWITZ: *On the spectra of close-to-Schwarz matrices*, Linear Algebra and its Applications, **363**(1) (2003), 81–88.

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