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PROBLEM FOR A HYPERBOLIC FRACTIONAL EQUATION WITH THE NON-DEGENERATE INTEGRAL BOUNDARY CONDITIONS

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ABSTRACT. In this work, we consider a hyperbolic fractional problem with non-degenerate integral boundary conditions. By using the method of Faedo-Galerkin, we demonstrate the existence of a solution of the considered problem by passing to the limit. A new result is given by proving the uniqueness of the solution based on assumptions for a similar problem.

1. INTRODUCTION

Nowadays, various fractional problems have been actively studied and one can find many papers dealing with them.

In this article we consider a standard equation problem and a non-local condition. In this problem the integral condition is nonlocal of second degenerate kind and transforms into a first kind. This has a significant impact on the method of investigating solvency. We focus our attention on a hyperbolic fractional equation with non-degenerate integral boundary conditions. It is well known that classical methods like those in [4], [2] [5] and [3] widely used to prove the solvency of initial boundary problems fail when applied to problems with integral conditions. Today some methods have been advanced to overcome the difficulties resulting

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from integral conditions. These methods are different and the choice of a concrete method depends on some form of integral condition. We propose a new approach which allows to prove a unique solvability of the nonlocal problem with degenerate integral condition; the Faedo-Galerkin method [7] and using the article [10]. This method consists in proving the existence and uniqueness of the solution using the following techniques: searching for $<< approximate >>$ solutions, establishing on these approximate solutions a priori estimates to guarantee the weak convergence of the solution , pass to the limit thanks to compactness properties (in nonlinear terms) and finally show the uniqueness of the solution.

Definition 1.1. *The fractional Caputo derivative of order $\alpha > 0$ of a function u is given by:*

$$D_t^{\alpha} v(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-x)^{n-\alpha-1} v^{(n)}(x) dx,$$

where $\Gamma(n-\alpha) = \int_0^{\infty} t^{n-\alpha-1} e^{-t} dt$ *and* $n-1 \leq \alpha \leq n$*,*$n \in \mathbb{N}^*$.

We consider the fractional hyperbolic equation

$$D_t^{\alpha} v(x,t) - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial}{\partial x} v(x,t)\right) + b(x,t)v(x,t) = g(x,t), \quad 1 \leq \alpha \leq 2. \tag{1.1}$$

The goal of this work is to find a function $v = v(x,t)$ solution of equation (1.1) in $Q_T = (0,l) \times (0,T)$ with $l, T < \infty$, satisfying the initial conditions:

$$v(x,0) = \varphi_1(x), \qquad D_t^{\alpha-1} v(x,0) = \varphi_2(x), \tag{1.2}$$

with the boundary condition

$$\frac{\partial v}{\partial x}(0,t) = 0, \tag{1.3}$$

and the integral condition

$$\int_0^l K(x) v(x,t) dx = 0. \tag{1.4}$$

In this work we focus on the spatial integral condition of which we give an example

$$\int_0^l K(x) v(x,t) dx = 0, \tag{1.5}$$

$$\frac{\partial v}{\partial x}(l,t) + \int_0^l K(x)v(x,t)dx = 0. \tag{1.6}$$

The condition (1.5) is a non local condition of the first type, (1.6) is a non local condition of the second type. The type of a non-local integral condition depends on the presence or absence of a term containing a trace of the sought solution or its derivative outside the integral. Problems with non-local conditions are studied. Motivated on these types of conditions, we suggest a new approach to the problem (1.1)-(1.4).

2. Auxiliary assumptions, notations and assertions

We are now able to formulate the problem in a precise way, to study this problem, we will need the following hypotheses:

$(H_1):\ a,b \in C^1(\bar{Q}_T), \quad a_1(x,t) \leq a(x,t) \leq a_2(x,t),\ a_1(x,t), a_2(x,t) > 0;$
$(H_2):\ g \in C(\bar{Q}_T);$
$(H_3):\ K \in C^2([0,l]),\ K(l) > 0,\ K^{'}(0) = 0,\ K^{'}(l) = 0;$
$(H_4):\ \int_0^l K(x)v(x,0)dx = 0,\ \int_0^l K(x)D_t^{\alpha-1}v(x,0)dx = 0;$
$(H_5):\ \forall x,y \in Q_T, h_1(x,t) \leq H(x,t) + b(x,t)K(l) \leq h_2(x,t),\ h_1(x,t), h_2(x,t) > 0.$

We denote by

$$H(x,t) = (a(x,t)K^{'}(x))_x - K(x)b(x,t).$$

Lemma 2.1. *Under the hypotheses* $(H_1)-(H_4)$ *the non-local condition* (1.4) *is equivalent to the dynamic condition*

$$K(l)a(l,t)\frac{\partial v}{\partial x}(l,t) + \int_0^l H(x,t)v(x,t)dx + \int_0^l K(x)g(x,t)dx = 0. \tag{2.1}$$

Proof. Let $v(x,t)$ be a solution of the problem (1.1) satisfying the conditions (1.3) and (1.4). Taking the differential of the relation (1.4) with respect at t we get

$$\int_0^l K(x)D_t^\alpha v(x,t)dx = 0. \tag{2.2}$$

Let

$$D_t^\alpha v(x,t) = g(x,t) + \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial v}{\partial x}(x,t)\right) - b(x,t)v(x,t)$$

and substituting this expression' in (2.2), we get:

$$\int_0^l K(x)g(x,t)dx + \int_0^l K(x)\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial}{\partial x}v(x,t)\right)dx - \int_0^l K(x)b(x,t)v(x,t)dx = 0. \tag{2.3}$$

Integrating by parts the second term on the left-handin (2.3) and using the conditions $(H_1)-(H_5)$ we obtain:

$$\int_0^l K(x)\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial v}{\partial x}(x,t)\right)dx = K(l)a(l,t)\frac{\partial v}{\partial x}(l,t) + \int_0^l \frac{\partial}{\partial x}\left(a(x,t)K^{'}(x)\right)v(x,t)dx. \tag{2.4}$$

Substituting (2.4) in (2.3), we get:

$$\int_0^l K(x)g(x,t)dx + K(l)a(l,t)v_x(l,t) + \int_0^l (a(x,t)K^{'}(x))_x v(x,t)dx - \int_0^l K(x)b(x,t)v(x,t)dx = 0, \tag{2.5}$$

Substituting the expression of $H(x,t)$ in this last relation we have:

$$K(l)a(l,t)v_x(l,t) + \int_0^l H(x,t)v(x,t)dx + \int_0^l K(x)g(x,t)dx = 0. \tag{2.6}$$

□

The conclusion of this lemma allows us to pass to the nonlocal problem with the dynamic condition (2.1). Note that this condition includes $v_x(l,t)$. This fact makes it possible to use a technique presented in the continuation of this work namely the method of compactness and Faedo-Galerkin.

3. Preliminaries

The method used consists in transforming the inhomogeneous conditions of the problem into homogeneous conditions by introducing new functions $w(x,t)$ which is the particular solution and $u(x,t)$ which is the homogeneous one verifying

$$\begin{cases} D_t^{\alpha} u(x,t) - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial}{\partial x}u(x,t)\right) + b(x,t)u(x,t) = f(x,t) \\ u(x,0) = 0, \\ D_t^{\alpha-1}u(x,0) = 0 \\ \frac{\partial u}{\partial x}(0,t) = 0 \\ \int_0^l K(x)u(x,t)dx = 0. \end{cases} \tag{3.1}$$

The solution of the problem (1.1) is therefore of the form

$$u(x,t) = v(x,t) + w(x,t),$$

where

$$\begin{aligned} &w(x,t) = \varphi_1(x)t + \varphi_2(x) \\ &g(x,t) = f(x,t) - \left[D_t^{\alpha} w(x,t) - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial}{\partial x}w(x,t)\right) + b(x,t)w(x,t)\right], \end{aligned} \tag{3.2}$$

$$\begin{aligned} g(x,t) = f(x,t) - &\left[D_t^{\alpha}(\varphi_1(x)t + \varphi_2(x)) - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial}{\partial x}\left(\varphi_1(x)t + \varphi_2(x)\right)\right)\right. \\ &\left. + b(x,t)(\varphi_1(x)t + \varphi_2(x))\right]. \end{aligned} \tag{3.3}$$

Thus the solution of the problem (1.1) is therefore of the form:

$$\begin{cases} v(x,0) = -\varphi_2(x) \\ D_t^{\alpha-1}v(x,0) = -D_t^{\alpha-1}w(x,0). \end{cases} \tag{3.4}$$

4. Existence and uniqueness of the solution

Definition 4.1. *We define as $W_2^1(Q_T)$ the Hilbert space which consists of all functions $u \in L_2(Q_T)$ such that: $D_t^{\alpha}u(x,t)$, $D_t^{\frac{\alpha}{2}}\frac{\partial u}{\partial x}(x,t)$, $D_t^{\frac{\alpha}{2}}u \in L_2(Q_T)$ with standard*

$$\begin{aligned} \| u \|_E = &\int_0^{\tau}\int_0^l \left(D_t^{\alpha}u(x,t)\right)^2 dxdt + \int_0^{\tau}\int_0^l \left(D_t^{\frac{\alpha}{2}}u_x(x,t)\right)^2 dxdt \\ &+ \int_0^{\tau}\int_0^l \left(D_t^{\frac{\alpha}{2}}u(x,t)\right)^2 dxdt, \end{aligned} \tag{4.1}$$

where $F$is a Hilbert space with the finite norm

$$\|f\|_F = \int_0^{\tau} \int_0^l f^2(x,t)dxdt. \tag{4.2}$$

Consider this problem again

$$\mathcal{L}u \equiv D_t^{\alpha} u(x,t) - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial}{\partial x}u(x,t)\right) + b(x,t)u(x,t) = f(x,t) \tag{4.3}$$

$$u(x,0) = 0 \qquad D_t^{\alpha-1}u(x,0) = 0, \tag{4.4}$$

with boundary and integral conditions

$$\frac{\partial u}{\partial x}(0,t) = 0 \tag{4.5}$$

$$K(l)a(l,t)\frac{\partial u}{\partial x}(l,t) + \int_0^l H(x,t)u(x,t)dx + \int_0^l K(x)f(x,t)dx = 0. \tag{4.6}$$

We denote by

$$W(Q_T) = \left\{u(x,t) : u(x,t) \in W_2^1(Q_T), \qquad D_t^{\alpha}u(x,t) \in L_2(Q_T \cup \Gamma_l)\right\}$$

$$\hat{W}(Q_T) = \{v : v \in W(Q_T)\},$$

where $W_2^1(Q_T)$ is a Sobolev space

$$\Gamma_l = (x,l) : x = l, t \in [0,T].$$

4.1. **Variational formulation.** Multiplying the equation (4.3) by the function $v(x,t)$ and integrating the result from 0 to l and from 0 to T, we obtain:

$$\begin{aligned}&\int_0^T \int_0^l D_t^{\alpha}u(x,t)v(x,t)dxdt - \int_0^T \int_0^l \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial}{\partial x}u(x,t)\right)v(x,t)dxdt\\ &+ \int_0^T \int_0^l b(x,t)u(x,t)v(x,t)dxdt = \int_0^T \int_0^l f(x,t)v(x,t)dxdt.\end{aligned} \tag{4.7}$$

Integrating by parts the second term of the left hand-side of (4.7), we obtain:

$$\begin{aligned}&\int_0^T \int_0^l \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial}{\partial x}u(x,t)\right)v(x,t)dxdt\\ &= \int_0^T \left(a(l,t)\frac{\partial}{\partial x}u(l,t)\right)v(l,t)dt - \int_0^T \int_0^l \left(a(x,t)u(x,t)\frac{\partial}{\partial x}v(x,t)\right)dxdt.\end{aligned} \tag{4.8}$$

Substituting (4.8) in (4.7) and taking into account the lemma (2.1), we obtain:

$$\begin{aligned} &K(l)\int_0^T\int_0^l\Big(D_t^\alpha u(x,t)v(x,t)+a(x,t)\frac{\partial}{\partial x}u(x,t)\frac{\partial}{\partial x}v(x,t)\\ &+b(x,t)u(x,t)v(x,t)\Big)\,dxdt+\int_0^T v(l,t)\int_0^l H(x,t)u(x,t)dxdt\\ &=K(l)\int_0^T\int_0^l f(x,t)v(x,t)dxdt-\int_0^T v(l,t)\int_0^l K(x)f(x,t)dxdt. \end{aligned} \tag{4.9}$$

4.2. **Existence.**

Definition 4.2. *A solution $u\in W(Q_T)$ is said to be a generalized solution of the problem* (4.3) - (4.6) *if $u(x,0)=D_t^{\alpha-1}u(x,0)=0$ and for all $v\in\hat W(Q_T)$ which satisfies the condition* (4.9).

The following theorem gives us the existence of the solution

Theorem 4.1. *Under the hypotheses $(H_1)-(H_4)$, the problem* (4.3) *admits a general solution.*

The proof of this theorem is based on the Faedo-Galerkin method which consists in performing the following three steps:

(1) **Finding «approximate» solutions**
(2) **We establish, on these approximate solutions, a priori estimates to guarantee a weak convergence of the approximations**
(3) **We pass to the limit, thanks to compactness properties (in nonlinear terms)**

4.2.1. *Approximation solutions.* Let $w_k(x)\in C^2[0,l]$ be a base in $W_2^1(\Omega)$. We define the approximations

$$u^n(x,t)=\sum_{k=1}^n c_k(t)w_k(x),$$

where $u^n(x,t)$ are the approximate solutions of the Cauchy problem:

$$\begin{aligned} &K(l)\int_0^l\Big(D_t^\alpha u^n(x,t)w_j+au_x^n(x,t)w_j'+bu^n(x,t)w_j\Big)\,dx\\ &+w_j(l)\int_0^l H(x,t)u^n(x,t)dx=K(l)\int_0^l f(x,t)w_jdx-w_j(l)\int_0^l K(x)f(x,t)dx. \end{aligned} \tag{4.10}$$

Multiplying the relation (4.10) by $\frac{1}{K(l)}$ and substituting $u^n(x,t)$ by its expression we obtain the relation:

$$\sum_{k=1}^{n}\Big[\int_0^l \Big(D_t^\alpha c_k(t)w_k(x)w_j + a(x,t)c_k(t)w_k^{'}(x)w_j^{'}$$

$$+b(x,t)c_k(t)w_k(x)w_j\big)\,dx + \frac{w_j(l)}{K(l)}\int_0^l H(x,t)c_k(t)w_k(x)dx\Big] \tag{4.11}$$

$$= \int_0^l f(x,t)w_j dx - \frac{w_j(l)}{K(l)}\int_0^l K(x)f(x,t)dx,$$

$$\sum_{k=1}^{n}\left[A_{kj}(x,l)D_t^\alpha c_k(t) + B_{kj}(x,l)c_k(t)\right] = f_j(t), \tag{4.12}$$

where $A_{kj}(x,l) = \int_0^l w_k(x)w_j(l)$,

$$B_{kj}(x,l) = \int_0^l \Big(a(x,t)w_k^{'}(x)w_j^{'} + b(x,t)w_k(x)w_j\Big)\,dx$$

$$+\frac{w_j(l)}{K(l)}\int_0^l H(x,t)(t)w_k(x)dx$$

$$f_j(t) = \int_0^l f(x,t)w_j dx - \frac{w_j(l)}{K(l)}\int_0^l K(x)f(x,t)dx.$$

To show that this equation (4.12) is solvable with respect to $D_t^\alpha c_k(t)$, we consider the quadratic form

$$q = \sum_{k=1}^{n} A_{kj}\xi_k\xi_j, \tag{4.13}$$

and we note $\sum_{k=1}^{n}\xi_k w_k = \eta$. Substituting A_{kj} by his expression in (4.13) A_{kj}, we get:

$$q = \sum_{k=1}^{n}\int_0^l w_k w_j \xi_k\xi_j dx = \int_0^l |\eta|^2 dx \geq 0. \tag{4.14}$$

As $q = 0$ if and only if $\eta = 0$, and w_k is linearly independent for $k = 1,\ldots,n$; so q is positive definite.

Therefore (4.12) is solvable with respect to $D_t^\alpha c_k(t)$. Thus, we can assert under the $(H_1) - (H_4)$ that the Cauchy problem has a solution for each n and for any base u^n constructed.

4.2.2. *A priori estimate.*

Theorem 4.2. *For any function $u \in E$ there is an a priori estimate*

$$\| u^n \|_E \leq C \| f \|_F, \tag{4.15}$$

where C is a positive constant independent of u.

Proof. Multiplying each member of (4.10) by $D_t^\alpha c_k(t)$ we get

$$\begin{aligned} & K(l)\int_0^l \Big(D_t^\alpha u^n(x,t) w_j D_t^\alpha c_k(t) + a u_x^n(x,t) w_j^{'} D_t^\alpha c_k(t) \\ & + b u^n(x,t) w_j D_t^\alpha c_k(t)\big)\, dx + w_j(l)\int_0^l H u^n(x,t) D_t^\alpha c_k(t) dx dt \\ & = K(l)\int_0^l f w_j D_t^\alpha c_k(t) dx - w_j(l)\int_0^l K f D_t^\alpha c_k(t) dx. \end{aligned} \tag{4.16}$$

□

Remark 4.1. *We denote by*

$$D_t^\alpha u^n(x,t) = \sum_{k=1}^n D_t^\alpha c_k(t) w_k(x),$$

and

$$D_t^\alpha (u_x^n(x,t)) = \sum_{k=1}^n c_k(t) w_k^{'}(x).$$

Let's apply the sum to each member of (4.16) and use the expression of $u^n(x,t)$

$$\begin{aligned} & K(l)\int_0^l \left(D_t^\alpha u^n(x,t)\right)^2 dx + K(l)\int_0^l a u_x^n(x,t) D_t^\alpha u_x^n(x,t) dx \\ & + K(l)\int_0^l b u^n(x,t) D_t^\alpha u^n(x,t) dx + \int_0^l H u^n(x,t) D_t^\alpha u^n(x,t) dx \\ & = K(l)\int_0^l f D_t^\alpha u^n(x,t) dx - \int_0^l K f D_t^\alpha u^n(x,t) dx. \end{aligned} \tag{4.17}$$

We take the integral from 0 to τ, $\tau \in [0,T]$ with respect to t from the relation (4.17) we get:

$$
\begin{aligned}
&K(l)\int_0^\tau\int_0^l (D_t^\alpha u^n(x,t))^2 dxdt\\
&+K(l)\int_0^\tau\int_0^l a(x,t)u_x^n(x,t)D_t^\alpha u_x^n(x,t)dxdt\\
&+\int_0^\tau K(l)\int_0^l b(x,t)u^n(x,t)D_t^\alpha u^n(x,t)dxdt\\
&+\int_0^\tau\int_0^l H(x,t)u^n(x,t)D_t^\alpha u^n(x,t)dxdt\\
&=\int_0^\tau\int_0^l ((K(l)-K(x))f(x,t))D_t^\alpha u^n(x,t)dxdt.
\end{aligned}
\tag{4.18}
$$

Proposition 4.1. *[?] Let $u(x,t)$ be a function and $1<\alpha<2$, we have:*

$$
\begin{aligned}
u(x,t)D_t^\alpha u(x,t) &= \left(D_t^{\frac{\alpha}{2}}u(x,t)\right)^2,\\
u_x(x,t)D_t^\alpha u_x(x,t) &= \left(D_t^{\frac{\alpha}{2}}u_x(x,t)\right)^2.
\end{aligned}
\tag{4.19}
$$

Using ε - Cauchy inequality:

$$\alpha\beta \le \frac{\varepsilon}{2}\alpha^2 + \frac{1}{2\varepsilon}\beta^2$$

and Poincaré's inequality:

$$\left(\int_0^a D_t^\alpha u(x,t)dx\right)^2 \le \frac{a^2}{2}\int_0^a u^2(x,t)dx,$$

and taking into account the relation (4.19) as well as the previous proposition, we have:

(4.20)

$$K(l)\int_0^\tau\int_0^l a(x,t)u_x^n(x,t)D_t^\alpha u_x^n(x,t)dxdt = K(l)\int_0^\tau\int_0^l a(x,t)\left(D_t^{\frac{\alpha}{2}}u_x^n(x,t)\right)^2 dxdt;$$

$$K(l)\int_0^\tau\int_0^l b(x,t)u^n(x,t)D_t^\alpha u^n(x,t)dxdt = K(l)\int_0^\tau\int_0^l b(x,t)\left(D_t^{\frac{\alpha}{2}}u^n(x,t)\right)^2 dxdt;$$

$$\int_0^\tau \int_0^l H(x,t)u^n(x,t)D_t^\alpha u^n(x,t)dxdt + K(l)\int_0^\tau \int_0^l b(x,t)u^n(x,t)D_t^\alpha u^n(x,t)dxdt$$
$$= \int_0^\tau \int_0^l \left(b(x,t)K(l) + H(x,t)\right) u^n(x,t)D_t^\alpha u^n(x,t)dxdt;$$
$$= \int_0^\tau \int_0^l \left(b(x,t)K(l) + H(x,t)\right) \left(D_t^{\frac{\alpha}{2}} u^n(x,t)\right)^2 dxdt;$$
$$\int_0^\tau \int_0^l \left(\left(K(l) - K(x)\right) f(x,t)\right) D_t^\alpha u^n(x,t)dxdt$$
$$\leq \frac{\varepsilon}{2}\int_0^\tau \int_0^l \left((K(l) - K(x))f(x,t)\right)^2 dxdt$$
$$+ \frac{1}{2\varepsilon}\int_0^\tau \int_0^l (D_t^\alpha u^n(x,t))^2 dxdt.$$

The relation (4.17) becomes:

$$\begin{aligned} &\left(K(l) - \frac{1}{2\varepsilon}\right)\int_0^\tau \int_0^l \left(D_t^\alpha u^n(x,t)\right)^2 dxdt \\ &+ K(l)\int_0^\tau \int_0^l a(x,t)\left(D_t^{\frac{\alpha}{2}} u_x^n(x,t)\right)^2 dxdt \\ &+ \min\left(K(l) + H(x,t)\right)\int_0^\tau \int_0^l \left(D_t^{\frac{\alpha}{2}} u^n(x,t)\right)^2 dxdt \\ &\leq \frac{\varepsilon}{2}\sup\left(K(l) - K(x)\right)^2 \int_0^\tau \int_0^l f^2(x,t)dxdt. \end{aligned} \tag{4.21}$$

We take $\epsilon > 0$ such as$\left(K(l) - \frac{1}{2\varepsilon}\right) > 0$ and assumptions (H_3) et (H_5). We deduce the existence of a constant C independent of n such that:

$$\begin{aligned} &\int_0^\tau \int_0^l \left(D_t^\alpha u^n(x,t)\right)^2 dxdt + \int_0^\tau \int_0^l \left(D_t^{\frac{\alpha}{2}} u_x^n(x,t)\right)^2 dxdt \\ &+ \int_0^\tau \int_0^l \left(D_t^{\frac{\alpha}{2}} u^n(x,t)\right)^2 dxdt \leq C\int_0^\tau \int_0^l f^2(x,t)dxdt \leq M, \end{aligned} \tag{4.22}$$

where

$$C = \frac{\frac{\epsilon}{2}\sup\left(K(l) - K(x)\right)^2}{\min\left(K(l) - \frac{1}{2\epsilon}, K(l)a_1(x,t), h_1(x,t)\right)}$$

.

$$M = CTl\sup(f^2(x,t)).$$

So we have

$$\|u^n\|_E \quad \leq \quad C\|f\|_F, \tag{4.23}$$

where

$$\|u^n\|_E = \int_0^\tau \int_0^l (D_t^\alpha u^n(x,t))^2 dxdt + \int_0^\tau \int_0^l (D_t^{\frac{\alpha}{2}} u_x^n(x,t))^2 dxdt$$
$$+ \int_0^\tau \int_0^l (D_t^{\frac{\alpha}{2}} u^n(x,t))^2 dxdt,$$

and

$$\|f\|_F = \int_0^\tau \int_0^l f^2(x,t) dxdt.$$

4.2.3. *Passage to limits.* Multiplying equation (4.10) a function $p \in C^1(0,T)$ with $p(T) = 0$ and integrating with respect to $t \in [0,T]$, we get:

$$\begin{aligned} &K(l) \int_0^T p(t) \int_0^l \Big(D_t^\alpha u^n(x,t) w_j + a u_x^n(x,t) w_j^{'} + b u^n(x,t) w_j \Big) dxdt \\ &+ w_j(l) \int_0^T p(t) \int_0^l H u^n(x,t) dxdt = K(l) \int_0^T p(t) \int_0^l f w_j dxdt \\ &- w_j(l) \int_0^T p(t) \int_0^l K f dxdt. \end{aligned} \tag{4.24}$$

We deduce from the a priori estimate that we can extract convergent subsequences (u^ν) and (u^n) such that for $\nu \longrightarrow \infty$ we have:

$$u^\nu \longrightarrow u \qquad \text{in} \qquad (Q_T); \tag{4.25}$$

$$D_t^\alpha u^\nu \longrightarrow D_t^\alpha u \qquad \text{in} \qquad L_2(Q_T \cup \Gamma_l); \tag{4.26}$$

$$D_t^{\alpha-1} u^\nu(x,0) \longrightarrow D_t^{\alpha-1} u, \qquad \text{in} \qquad (0,l). \tag{4.27}$$

All integrals in (4.24) are defined for any function $p \in C^1(0,T), p(T) = 0$. Taking into account that $w_j(x)$ is dense in $W_2^1(0,l)$, we conclude the existence of the solution.

5. Uniqueness of the solution

Theorem 5.1. *If the probleme* (4.3) *admits a solution, it's unique*

Proof. Suppose u_1 and u_2 are two solutions of the problem (4.3)-(4.6). So let's fix t and for all $w \in W_2^1(0,l)$, $u(x,t) = u_1(x,t) - u_2(x,t)$ satisfying the conditions $u(x,0) = 0$, $D_t^\alpha u(x,0) = 0$ and the identity

$$(5.1)\quad \begin{aligned} &K(l)\int_0^l \Big(D_t^\alpha u(x,t)w(x,t) + a(x,t)\frac{\partial u(x,t)}{\partial x}\frac{\partial w(x,t)}{\partial x} \\ &+b(x,t)u(x,t)w(x,t)\big)\, dxw(l)\int_0^l H(x,t)u(x,t)dx = 0. \end{aligned}$$

For $t \in [0,T]$, let $w(x,t) = D_t^\alpha u(x,t)$ and $\frac{\partial w(x,t)}{\partial x} = D_t^\alpha u_x(x,t)$, the relation (5.1) becomes:

$$(5.2)\quad \begin{aligned} &K(l)\int_0^l \Big((D_t^\alpha u(x,t))^2 + a(x,t)\frac{\partial u(x,t)}{\partial x}D_t^\alpha u_x(x,t) \\ &+b(x,t)u(x,t)D_t^\alpha u(x,t)\big)\, dx + w(l)\int_0^l H(x,t)u(x,t)dx = 0. \end{aligned}$$

By integrating this relation (5.2) of $(0,\tau)$, $\tau \in [0,T]$, taking into account the relation (4.19) and the ϵ -Cauchy inequality we get:

$$(5.3)\quad \begin{aligned} &K(l)\int_0^\tau\int_0^l (D_t^\alpha u(x,t))^2\, dxdt + K(l)\int_0^\tau\int_0^l a(x,t)\left(D_t^{\frac{\alpha}{2}}u_x(x,t)\right)^2 dxdt \\ &+ K(l)\int_0^\tau\int_0^l b(x,t)\left(D_t^{\frac{\alpha}{2}}u(x,t)\right)^2 dxdt + \frac{\epsilon w(l)}{2}\int_0^\tau\int_0^l [H(x,t)]^2dxdt \\ &+ \frac{w(l)}{2\epsilon}\int_0^\tau\int_0^l [u(x,t)]^2dxdt \le 0. \end{aligned}$$

We put

$$A = \min\left(K(l); K(l)a(x,t); K(l)b(x,t); \epsilon w(l); \frac{w(l)}{2\epsilon}\right),$$

and the relation (5.3) becomes:

$$(5.4)\quad \begin{aligned} &A[\int_0^\tau\int_0^l (D_t^\alpha u(x,t))^2\, dxdt + \int_0^\tau\int_0^l \left(D_t^{\frac{\alpha}{2}}u_x(x,t)\right)^2 dxdt \\ &+ \int_0^\tau\int_0^l \left(D_t^{\frac{\alpha}{2}}u(x,t)\right)^2 dxdt + \int_0^\tau\int_0^l [u(x,t)]^2dxdt] \le 0. \end{aligned}$$

Using the a priori estimate we get:

$$\int_0^\tau \int_0^l \left(D_t^\alpha u(x,t)\right)^2 dxdt + \int_0^\tau \int_0^l \left(D_t^{\frac{\alpha}{2}} u_x(x,t)\right)^2 dxdt + \int_0^\tau \int_0^l \left(D_t^{\frac{\alpha}{2}} u(x,t)\right)^2 dxdt + \int_0^\tau \int_0^l [u(x,t)]^2 dxdt \leq 0. \tag{5.5}$$

This implies that $u(x,t) = 0$ hence $u_1(x,t) = u_2(x,t)$. Therefore the solution is unique. □

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