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EXISTENCE AND UNIQUENESS OF THE CAGINALP PHASE-FIELD SYSTEM BASED ON THE CATTANEO LAW

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ABSTRACT. Our aim in this paper is to study the existence and the uniqueness of the Caginalp phase-field system based on the Cattaneo Law,with initial conditions,Dirichlet Boundary Conditions and Regular Potentiels.

1. INTRODUCTION

The Caginalp phase-field model

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \theta \tag{1.1}$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t} \tag{1.2}$$

proposed in [5], has been extensively studied (see, e.g, [2–6,10]). Here, u denotes the order parameter and θ the $(relative)$ temperature.

Furthermore, all physical constants have been set equal to one. This system models, e.g, melting-solidification phenomena in certain classes of materials. The

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Caginalp system can be derived as follows. We first consider the (total) free energy

$$\psi(u,\theta)=\int_{\Omega}\left(\frac{1}{2}|\nabla u|^{2}+f(u)-u\theta-\frac{1}{2}\theta^{2}\right)dx, \tag{1.3}$$

where Ω is the domain occupied by the materiel.

We then define the enthalpy H as

$$H=-\frac{\partial\psi}{\partial\theta}, \tag{1.4}$$

where ∂ denotes a variational derivative, which gives

$$H=u+\theta. \tag{1.5}$$

The governing equations for u and θ are then given by (see [1])

$$\frac{\partial u}{\partial t}=-\frac{\partial\psi}{\partial u}, \tag{1.6}$$

$$\frac{\partial H}{\partial t}+\text{div}, q=0, \tag{1.7}$$

where q is the thermal flux vector. Assuming the classical Fourier Law

$$q=-\nabla\theta, \tag{1.8}$$

we find (1.1) and (1.2).

Now, a drawback of the Fourier Law is the so-called “paradox of heat conduction", namely, it predicts that thermal signals propagate with infinite speed, which, in particular, violates causality (see,e.g. [7] and [14]). One possible modification, in order to correct this unrealistic feature, is the Maxwell-Cattaneo Law.

$$\left(1+\frac{\partial}{\partial t}\right)q=-\nabla\theta. \tag{1.9}$$

In that case, it follows from (1.7) that

$$\left(1+\frac{\partial}{\partial t}\right)\frac{\partial H}{\partial t}-\Delta\theta=0,$$

hence,

$$\frac{\partial^{2}\theta}{\partial t^{2}}+\frac{\partial\theta}{\partial t}-\Delta\theta=\frac{\partial^{2}u}{\partial t^{2}}+\frac{\partial u}{\partial t}. \tag{1.10}$$

This model can also be derived by considering, as in [6] (see also [8–13]), the Caginalp phase-field model with the so-called Gurtin-Pipkin Law

$$q(t) = -\int_0^{+\infty} k(s)\nabla\theta(t-s)ds, \tag{1.11}$$

for an exponentially decaying memory kernel k, namely,

$$k(s) = e^{-s}. \tag{1.12}$$

Indeed, differentiating (1.11) with respect to t and integrating by parts, we recover the Maxwell-Cattaneo Law (1.9).

Now, in view of the mathematical treatment of the problem, it is more convenient to introduce the new variable

$$\alpha = \int_0^t \theta(s)ds, \quad \theta = \frac{\partial\alpha}{\partial t}, \tag{1.13}$$

and we have, integrating (1.10) with respect to $s \in [0,1]$.

$$\frac{\partial^2\alpha}{\partial t^2} + \frac{\partial\alpha}{\partial t} - \Delta\alpha = -\frac{\partial u}{\partial t} - u. \tag{1.14}$$

Our objective in this article is to study a generalization of the Caginalp phase field system based on these two temperature theories and the usual Fourier law with a nonlinear coupling. In particular, we prove the existence and uniqueness of the solutions by proceeding with the a priori estimates and the standard Garlerkin scheme.

2. Setting of the problem

We consider the following initial and boundary value problem:

$$\frac{\partial u}{\partial t} - \Delta u + f(u) = \frac{\partial\alpha}{\partial t} \quad \text{in} \quad \Omega, \tag{2.1}$$

$$\frac{\partial^2\alpha}{\partial t^2} + \frac{\partial\alpha}{\partial t} - \Delta\alpha = -\frac{\partial u}{\partial t} - u \quad \text{in} \quad \Omega, \tag{2.2}$$

$$u|_\Gamma = \alpha|_\Gamma = 0 \quad \text{on} \quad \partial\Omega, \tag{2.3}$$

$$u|_{t=0} = u_0, \frac{\partial u}{\partial t}|_{t=0} = u_1, \alpha|_{t=0} = \alpha_0, \frac{\partial\alpha}{\partial t}|_{t=0} = \alpha_1. \tag{2.4}$$

We consider the regular potential $f(s) = s^3 - s$ which satisfies the following properties:

(2.5) $$f \quad \text{is of class} \quad C^2, \quad f(0) = 0;$$

(2.6) $$-c_0 \leq f'(s), \quad c_0 \geq 0, \quad \forall s \in R;$$

$$-c_1 \leq F(s) \leq f(s)s + c_2, \quad c_1, c_2 \geq 0, \quad \forall s \in R$$

where

(2.7) $$F(s) = \int_0^s f(\tau) d\tau.$$

3. NOTATIONS

We note $\|.\|$ the usual L^2 norm (with associated scalar product $(.,.)$) and $\|.\|_{-1} = \|(- \ Delta)^{\frac{-1}{2}}.\|$, where $-\Delta$ denotes the Laplace minus operator with Dirichlet boundary conditions. More generally, $\|.\|_X$ denotes the Banach space norm X and $\star|\Omega|$ is a measure of Ω. Throughout the article, c, c' and c" represent the constants which can vary from one line to another or even within the same line. Similarly, the symbol c_p represents the strictly positive constant which can vary from one line to another or even within the same line.

4. A PRIORITI ESTIMATES

The estimates derived in this section are formal, but they can easily be justified within a Galerkin scheme.

In what follows, the Poincaré, Holder and Young inequalities are extensively used, Without further referring to them. We multiply (2.1) by $\frac{\partial u}{\partial t}$ and have, integrating over and by parts,

(4.1) $$\frac{d}{dt}\left(\|\nabla u\|^2 + 2\int_\Omega F(u)dx\right) + 2\|\frac{\partial u}{\partial t}\|^2 = 2\left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t}\right).$$

Multiplying then (2.2) by $\frac{\partial \alpha}{\partial t}$, we obtain

(4.2) $$\frac{d}{dt}\left(\|\nabla \alpha\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2\right) + 2\|\frac{\partial \alpha}{\partial t}\|^2 = -2\left(u, \frac{\partial \alpha}{\partial t}\right) - 2\left(\frac{\partial u}{\partial t}, \frac{\partial \alpha}{\partial t}\right).$$

Summing (4.1) and (4.2), we find the differential egality

$$\begin{aligned}\frac{d}{dt}\left(\|\nabla u\|^2+2\int_\Omega F(u)dx+\|\nabla\alpha\|^2+\|\frac{\partial\alpha}{\partial t}\|^2\right)+2\|\frac{\partial u}{\partial t}\|^2+2\|\frac{\partial\alpha}{\partial t}\|^2 \\ =-2\left(u,\frac{\partial\alpha}{\partial t}\right),\end{aligned} \tag{4.3}$$

where

$$E=\|\nabla u\|^2+2\int_\Omega F(u)dx+\|\nabla\alpha\|^2+\|\frac{\partial\alpha}{\partial t}\|^2. \tag{4.4}$$

Thanks to the estimate (2.7), we obtain

$$(F(u)+c_0,1)\geq 0. \tag{4.5}$$

We multiply (2.1) by u, integrate our Ω and find

$$\frac{d}{dt}\|u\|^2+2\|\nabla u\|^2+2\int_\Omega F(u)dx=2\left(\frac{\partial\alpha}{\partial t},u\right) \tag{4.6}$$

which yields, owing to (2.7). Summing (4.3) and (4.6), we easily find

$$\frac{d}{dt}E_1+2\frac{\partial u}{\partial t}\|^2+2\frac{\partial\alpha}{\partial t}\|^2+2\|\nabla u\|^2+2\int_\Omega F(u)dx=0, \tag{4.7}$$

where

$$E_1=\|\nabla u\|^2+\|u\|^2+2\int_\Omega F(u)dx+\|\alpha\|^2+\|\frac{\partial\alpha}{\partial t}\|^2. \tag{4.8}$$

We multiply (2.2) by α, integrate our Ω and find

$$\frac{d}{dt}\left[2\left(\frac{\partial\alpha}{dt},\alpha\right)+\|\alpha\|^2\right]+\|\nabla\alpha\|^2\leq\|u\|^2+\|\frac{\partial u}{\partial t}\|^2+2\|\alpha\|^2+2\|\frac{\partial\alpha}{\partial t}\|^2, \tag{4.9}$$

where

$$E_2=2\left(\frac{\partial\alpha}{dt},\alpha\right)+\|\alpha\|^2 \quad and \quad E_3=E_1+\gamma E_2.$$

Summing (4.7) and $\gamma(4.9)$, we easily find

$$\begin{aligned}\frac{d}{dt}E_3+(2-\gamma)\|\nabla u\|^2+(2-\gamma)\|\frac{\partial u}{\partial t}\|^2+2(1-\gamma)\|\frac{\partial\alpha}{\partial t}\|^2 \\ +(2-c\gamma)\|\nabla\alpha\|^2+2\int_\Omega F(u)dx\leq c_0|\Omega|,\end{aligned} \tag{4.10}$$

with $\gamma > 0$, $1-\gamma > 0$, $2-\gamma > 0$, $2-c\gamma > 0$, where

$$\begin{aligned} E_3 &= \|\nabla u\|^2 + \|u\|^2 + 2\int_\Omega F(u)dx + \|\alpha\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 \\ &+ 2\gamma\left(\frac{\partial \alpha}{dt}, \alpha\right) + \gamma\|\alpha\|^2 \end{aligned} \tag{4.11}$$

satisfies

$$E_3 \geq C^{-1}\left(\|u\|_{H^1}^2 + \|\alpha\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 + \int_\Omega F(u)dx\right), \tag{4.12}$$

and

$$E_3 \leq C\left(\|u\|_{H^1}^2 + \|\alpha\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 + \int_\Omega F(u)dx\right), \tag{4.13}$$

$$\begin{aligned} &C^{-1}\left(\|u\|_{H^1}^2 + \|\alpha\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 + \int_\Omega F(u)dx\right) \leq E_3 \\ &\leq C\left(\|u\|_{H^1}^2 + \|\alpha\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2 + \int_\Omega F(u)dx\right). \end{aligned} \tag{4.14}$$

We have

$$\frac{d}{dt}E_3 + c(E_3 + \|\frac{\partial u}{\partial t}\|^2) \leq c', \quad c > 0. \tag{4.15}$$

Finally, we conclude that

$$\begin{aligned} u &\in L^\infty(R^+; H_0^1(\Omega)) \cap L^2(0,T; H_0^1(\Omega)), \\ \alpha &\in L^\infty(R^+; H_0^1(\Omega)) \cap L^2(0,T; H_0^1(\Omega)), \\ \frac{\partial \alpha}{\partial t} &\in L^\infty(R^+; L^2(\Omega)) \cap L^2(0,T; L^2(\Omega)), \end{aligned}$$

and

$$\frac{\partial u}{\partial t} \in L^2(0,T; L^2(\Omega)),$$

for all $T > 0$. Multiply (2.2) by $\dfrac{\partial^2 \alpha}{\partial t^2}$ and integrate over Ω. We get

$$\|\frac{\partial^2 \alpha}{\partial t^2}\|^2 + \frac{1}{2}\frac{d}{dt}\|\frac{\partial \alpha}{\partial t}\|^2 + (\nabla\frac{\partial^2 \alpha}{\partial t^2}, \nabla\alpha) = -(\frac{\partial u}{\partial t}, \frac{\partial^2 \alpha}{\partial t^2}) - (u, \frac{\partial^2 \alpha}{\partial t^2}),$$

where

$$(\nabla\frac{\partial^2 \alpha}{\partial t^2}, \nabla\alpha) = \frac{d}{dt}\left(\nabla\frac{\partial \alpha}{\partial t}, \nabla\alpha\right) - \|\nabla\frac{\partial \alpha}{\partial t}\|^2.$$

Applying the Gronwall's lemma, we have

$$\frac{d}{dt}\left(\|\frac{\partial\alpha}{\partial t}\|^2\|^2+2(\nabla\frac{\partial\alpha}{\partial t},\nabla\alpha)\right)+\frac{3}{2}\|\frac{\partial^2\alpha}{\partial t^2}\|^2\leq\|u\|^2+\|\frac{\partial u}{\partial t}\|^2+2\|\nabla\frac{\partial\alpha}{\partial t}\|^2,$$

and we deduce that $\dfrac{\partial^2\alpha}{\partial t^2}\in L^2(0,T;L^2(\Omega)),\quad\forall T>0.$

5. Existence and Uniqueness of solutions

Theorem 5.1. *(Existence) We assume* $(u_0,\alpha_0,\alpha_1)\in H_0^1(\Omega)\times H_0^1(\Omega)\times L^2(\Omega)$ *then the system* (2.1) – (2.4) *possesses at least one solution* (u,α) *such that*

$$\begin{aligned}&u,\alpha\in L^\infty(R^+;H_0^1(\Omega))\cap L^2(0,T;H_0^1(\Omega)),\\&\frac{\partial\alpha}{\partial t}\in L^\infty(R^+;L^2(\Omega))\cap L^2(0,T;L^2(\Omega)),\\&\frac{\partial u}{\partial t}\in L^2(0,T;L^2(\Omega)).\end{aligned}\tag{5.1}$$

Theorem 5.2. *(Uniqueness) Let the assumptions of Theorem 3.2. hold. Then, the system* (2.1) – (2.4) *with initial data* $(u_0^{(1)},u_1^{(1)},\alpha_0^{(1)},\alpha_1^{(1)})$ *and* $(u_0^{(2)},u_1^{(2)},\alpha_0^{(2)},\alpha_1^{(2)})\in H_0^1(\Omega)\times H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega)$*, respectively.*

Proof. We set $u=u^{(1)}-u^{(2)}$ and $\alpha=\alpha^{(1)}-\alpha^{(2)}$ so that (u,α) is solution of the following system

$$\frac{\partial u}{\partial t}-\Delta u+f(u^{(1)})-f(u^{(2)})=\frac{\partial\alpha}{\partial t}\tag{5.2}$$

$$\frac{\partial^2\alpha}{\partial t^2}+\frac{\partial\alpha}{\partial t}-\Delta\alpha=-\frac{\partial u}{\partial t}-u\tag{5.3}$$

$$u|_{\partial\Omega}=\alpha|_{\partial\Omega}=0;\quad\Gamma=\partial\Omega\tag{5.4}$$

$$u|_{t=0}=u_0=u_0^{(1)}-u_0^{(2)};\quad\frac{\partial u}{\partial t}|_{t=0}=u_1=u_1^{(1)}-u_1^{(2)}\tag{5.5}$$

$$\alpha|_{t=0}=\alpha_0=\alpha_0^{(1)}-\alpha_0^{(2)};\quad\frac{\partial\alpha}{\partial t}|_{t=0}=\alpha_1=\alpha_1^{(1)}-\alpha_1^{(2)}\tag{5.6}$$

We multiplying (5.2) by $\dfrac{\partial u}{\partial t}$ and integrate over Ω.

$$\frac{d}{dt}\|\nabla u\|^2+2\|\frac{\partial u}{\partial t}+2\left(f(u^{(1)})-f(u^{(2)}),\frac{\partial u}{\partial t}\right)=2(\frac{\partial\alpha}{\partial t},\frac{\partial u}{\partial t}).$$

Lagrange theorem gives a estimates

$$\begin{aligned} f(u^{(1)})-f(u^{(2)}) &= \int_0^1 f'(u^{(2)}+s(u^{(1)}-u^{(2)}))dsu \\ &= \int_0^1 (3(su^{(1)}+(1-s)\,u^{(2)})^2-1))ds|u|, \end{aligned}$$

which implies

$$\begin{aligned} \|f(u^{(1)}-f(u^{(2)})\|^2 &\leq 36\int_\Omega \left((u^{(2)})^2+(u^{(1)})^2+1\right)^2 |u|^2dx \\ &\leq 36\left(\|u^{(2)}\|_{L^6}^4+\|u^{(1)}\|_{L^6}^4+1\right)\|u\|_{L^6}^2 \\ &\leq C\left(\|\nabla u^{(2)}\|^4+\|\nabla u^{(1)}\|^4+1\right)\|\nabla u\|^2, \end{aligned}$$

and we have

$$\frac{d}{dt}\|\nabla u\|^2+\|\frac{\partial u}{\partial t}=c_1\|\nabla u\|^2+2(\frac{\partial u}{\partial t},\frac{\partial \alpha}{\partial t}). \tag{5.7}$$

Multiplying (5.3) by $\dfrac{\partial \alpha}{\partial t}$ and integrating over Ω, we get.

$$\frac{d}{dt}\left(\|\nabla\alpha\|^2+\|\frac{\partial\alpha}{\partial t}\|^2\right)+2\|\frac{\partial\alpha}{\partial t}\|^2=-2(u,\frac{\partial\alpha}{\partial t})-2(\frac{\partial u}{\partial t},\frac{\partial\alpha}{\partial t}). \tag{5.8}$$

Summing (5.7) and (5.8) then we obtain

$$\frac{d}{dt}\left(\|\nabla u\|^2+\|\nabla\alpha\|^2+\|\frac{\partial\alpha}{\partial t}\|^2\right)+\|\frac{\partial u}{\partial t}\|^2+2\|\frac{\partial\alpha}{\partial t}\|^2=c_2\|\nabla u\|^2-2(u,\frac{\partial\alpha}{\partial t}) \tag{5.9}$$

where

$$E_4=\|\nabla u\|^2+\|\nabla\alpha\|^2+\|\frac{\partial\alpha}{\partial t}\|^2. \tag{5.10}$$

Multiplying (5.2) by u and integrating over Ω, we get

$$\frac{d}{dt}\|u\|^2+2\|\nabla u\|^2=c_3\|u\|^2+2(u,\frac{\partial\alpha}{\partial t}). \tag{5.11}$$

Now summing (5.9) and (5.11) then we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\|u\|^2+\|\nabla u\|^2+\|\nabla\alpha\|^2+\|\frac{\partial\alpha}{\partial t}\|^2\right)+\|\frac{\partial u}{\partial t}\|^2+2\|\frac{\partial\alpha}{\partial t}\|^2+2\|\nabla u\|^2 \\ &\leq c_2\|\nabla u\|^2+c_3\|u\|^2, \end{aligned} \tag{5.12}$$

where

$$E_5 = \|u\|^2 + \|\nabla u\|^2 + \|\nabla \alpha\|^2 + \|\frac{\partial \alpha}{\partial t}\|^2, \tag{5.13}$$

which implies that

$$\frac{d}{dt}E_5 + c(\|u\|^2_{H^1} + \|\frac{\partial u}{\partial t}\|^2 + 2\|\frac{\partial \alpha}{\partial t}) \leq c', \tag{5.14}$$

which yields the uniqueness, owing to Gronwall's lemma. □

6. Conclusion

We have just shown the theorems of existence and uniqueness of solution for Caginalp phase-field system based on the Cattaneo Law.

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