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NUMERICAL SIMULATION OF A PHENOMENON OF SILTING UP OF RIVER BANKS

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ABSTRACT. In this paper we have placed a particular emphasis on the construction of the algorithmic scheme leading to the codes to identify the parameters of the silting of the banks of the rivers. To overcome the lack of real field data, we generated experimental data by solving a carefully chosen partial differential equation . All the codes obtained were executed on the Matlab 7.14(R2012 a) interface and the results of the simulation were satisfactory.

1. INTRODUCTION

The model describing the process of silting or sedimentation of a river bank is described by the following equations:

$$\frac{\partial S}{\partial t} + k_x(u,v)\frac{\partial S}{\partial x} + k_y\frac{\partial S}{\partial y} = \Delta(\Phi(S,u,v)) + f(t,x,y) \tag{1.1}$$

$$\frac{\partial S}{\partial n} = g_i, \text{on } \Gamma_i^r \tag{1.2}$$

where $S(t,x,y)$ is the height of the sediments at time t; $k_x(u,v)$ and $k_y(u,v))$ are convective terms of sedimentation respectively in directions x and y. $\Delta(\Phi(S,u,v))$

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is a sediment dispersion term, $f(t,x,y)$ is the source term and $g_i; i = 1,2$ are supposed known functions. Furthermore, the velocity field components u and ν must satisfy Navier Stokes equations describing the flow of water.

The model (1.1) - (1.2) is an incomplete data problem. In fact due to the variability of flows on a cross section of a river, the system (1.1) - (1.2) is thus a boundary value problem for which the boundary conditions on a part of the boundary of Ω is not provided or is poorly known. Such a system belongs to a class of ill-posed problems for which their resolutions require inverse methods. However, since our problem is the description of a silting process in the vicinity of river banks we shall adopt an asymptotic approach.

2. The direct model

Equations of type (1.1)) - (1.2) are widely studied in the literature. Notably it has been shown that there exists at each time t a shock line where the solution admits a C^1 discontinuity (2.3). This shock line can be interpreted as a line where silting wrinkles are formed. So, let us assume that we are given of a shock line near a river bank described by the equation

$$y = \eta(t,x), \tag{2.1}$$

where η is a sufficiently smooth function. Then we consider a domain $\Omega^{\epsilon,t}$ round the shock line (2.1) such that (2.1) as $\Omega^{\epsilon,t} \to \Omega^0$ as $\epsilon \to 0$, Ω^0 being a sub-domain of Ω. For an asymptotic study in the domain $\Omega^{\epsilon,t}$ we introduce inner variables by setting

$$x^* = x, \quad y^* = \frac{y - \eta(t,x)}{\delta(\epsilon}, \quad t^* = t, \tag{2.2}$$

where $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. We denote by $u^\epsilon(t,x,y)$ and $v^\epsilon(t,x,y)$ the components of velocity fields and $S^\epsilon(t,x,y)$ the solution of the equation system (1.1) - (1.2) valid in the shock zone $\Omega^{\epsilon,t}$. Then we look for an asymptotic expansion of these solutions in the form:

$$S^\epsilon(t,x,y) = \epsilon^{-\alpha} S^0(t^*,x^*,y^*) + o(\epsilon^{-\alpha}), \tag{2.3}$$

$$u^\epsilon(t,x,y) = \epsilon^{-\beta} u^0(t^*,x^*,y^*) + o(\epsilon^{-\beta}), \tag{2.4}$$

$$v^\epsilon(t,x,y) = \epsilon^{-\gamma} v^0(t^*,x^*,y^*) + o(\epsilon^{-\alpha}), \tag{2.5}$$

where α, β and γ are appropriate parameters and where we have set

$$\lim_{\epsilon\to 0} \frac{o(\epsilon^M)}{\epsilon^M} = 0. \tag{2.6}$$

for $M \in \mathbb{R}^*$. One obtains, after calculations, in the main order, as $\epsilon \to 0$ where t^*, x^*, y^* are fixed, the equation

$$k_y^0 v^0 \frac{\partial S}{\partial y^*} - \mu^0 v^0 \frac{\partial^2 S}{\partial y^* 2} - \mu^0 v^0 \left(\frac{\partial \eta}{\partial x}\right)^2 \frac{\partial^2 S}{\partial y^* 2} = \lim_{\epsilon\to 0} \delta(\epsilon)\epsilon^{\alpha+\gamma} f \tag{2.7}$$

Setting

$$f = f^0(t^*,x^*,y^*)\frac{\epsilon^{-\alpha+\gamma}}{\delta(\epsilon})^2 + \cdots , \tag{2.8}$$

one obtains

$$-\mu^0 v^0 \left[1 + \left(\frac{\partial \eta(t,x)}{\partial x}\right)^2\right] \frac{\partial^2 S^0}{\partial y^* 2} + k_y^0 v_0 \frac{\partial S^0}{\partial y^*} + k_y^0 v^0 \frac{\partial S^0}{\partial y^*} = f^0. \tag{2.9}$$

Furthermore, according to asymptotic expansions theory, the so-called matching properties lead to the following boundary condition,

$$\lim_{|y^*|\to+\infty} S^0(t^*,x^*,y^*) = 0. \tag{2.10}$$

It is easy to show that for given non null parameters and sufficiently regular functions μ^0, k_y^0, η, v^0 and f^0 the problem (2.9) - (2.10) admits a unique solution.

3. Discrete formulation of the direct problem

To construct the approached direct problem , we consider the finite difference method. Let $Y^*_{\max}$ be a sufficiently large positive real value. Then we divide the interval $[-Y^*_{\max}; Y^*_{\max}]$ in $2N+1$ values for a given parameter N and, let us put $h = \frac{2Y^*_{\max}}{2N+1}$ the step size of this subdivision.

Let S_i^0 be the value of the solution of the equation 2.9 at points $y_j^* = jh$ with $j = -N, \ldots, N$ and where we have assumed that t^* and x^* are fixed. Following centered finite differences formulas are considered:

$$\frac{\partial^2 S^0(t^*,x^*,y^*)}{\partial y^{*2}} = \frac{1}{h^2}(S^0_{i+1} - 2S^0_i + S^0_{i-1}) + o(1), \tag{3.1}$$

and

$$\frac{\partial S^0(t^*,x^*,y^*)}{\partial y^*} = \frac{1}{2h}(S^0_{i+1} - S^0_{i-1}) + o(h). \tag{3.2}$$

Setting

$$a(t^*,x^*,y^*) = \mu_0 v^0 \left[1 + \left(\frac{\partial \eta}{\partial x}\right)^2\right], \tag{3.3}$$

and

$$b(t^*,x^*,y^*) = k^0_y v^0, \tag{3.4}$$

it follows from (2.9) that

$$\left\{ \begin{array}{r} \left(-\frac{a_j}{h^2} + \frac{b_j}{2h}\right) S^0_{-N+j+1} + \frac{2a_j}{h^2} S^0_{-N+j} - \left(\frac{a_j}{h^2} + \frac{b_j}{2h}\right) S^0_{-N+j-1} = f^0_j \\ S^0_{-N} = s^0_N = 0 j = 1,2,\ldots,2N-1. \end{array} \right. , \tag{3.5}$$

with $a_i = a(t^*,x^*,y^*_i)$ and $b_i = b(t^*,x^*,y^*_i)$.

Next, if we set

$$\Lambda = (a_{N+1},\ldots,a_{N-1},b_{-N+1},\ldots,b_{N-1})^T \in \mathbb{R}^{4N-2} \tag{3.6}$$

the vector parameters, then the system (3.5) is written by

$$A(\Lambda)\underline{S}^0 = \underline{f}^0, \tag{3.7}$$

where we have set:

$$A(\Lambda)i,j = \frac{1}{h^2} \begin{cases} 2a_i & \text{if } i = j \\ -a_i - b_i & \text{if } i = j-1 \\ -a_i + b_i & \text{if } i = j+1 \\ 0 & \text{otherwise} \end{cases} \tag{3.8}$$

a $(2N-1)$ tridiagonal matrix and where $\underline{S}^0 = \left[S^0_{-N+1}, S^0_{-N+2},\ldots,S^0_{N-1}\right]^T$ and $\underline{f}^0 = \left[f^0_{-N+1},\ldots,f^0_{N-1}\right]^T$ are, respectively, the discrete state vector and the source vector with respect to inner variables. Parameters a_j and b_j being all non null, one can easily prove that the matrix $A(\Lambda)$ is invertible.

If we denote by $\underline{S}^{\epsilon,k}$ the sedimentation state vector calculated at a time t^*_k on the grid points (x^*_i, y^*_i) in the area of shock wave, then according to the development

(2.3), we can write:

$$A(\Lambda)\underline{S}^{\varepsilon,k} \approx \epsilon^{-\alpha}\underline{f}^{0,k}. \tag{3.9}$$

It should be noted that within the framework of the problematic of this work, the source vectors $\underline{f}^{0,k}$ are indeterminate. We agree to approach them via a spectral Chebyshev approximation by considering the basic $(T_j)_j$ defined by

$$T_j(y^*) = \cos\left(j\cos^{-1}(y^*)\right), \quad j = 0, 1, 2, \dots, \tag{3.10}$$

and allows us to write to a given order of approximation m,

$$f^0(t_k^*, x^*, y^*) \simeq \sum_{j=0}^{m} F_j^k(x^*)T_j(y^*), \tag{3.11}$$

which can be written again in the following matrix form

$$\underline{f}^{0,k} \simeq \mathbb{T} \times \underline{F}^k, \tag{3.12}$$

where T is the matrix of order $(m+1)\times(2N-1)$ whose the $(i,j)^{th}$ component is $T_j(y_i^*)$ and where we have set $\underline{F}^k = \left[F_1^k, \dots, F_m^k\right]^T$.

The expression (3.9) is written by equations

$$A(\Lambda)\underline{S}^{\varepsilon,k} \approx \varepsilon^{-\alpha}\mathbb{T} \times \underline{F}^k, \quad k = 0, \dots, n_{obs}, \tag{3.13}$$

which defines the direct problem where to each couple (Λ, F^k) is associated the state of the system $\underline{S}^{\epsilon,k}$ at time t_k.

4. Formulation of the identification problem

To construct the objective function, we consider observation vectors $\underline{S}^{obs,k}$ at times t_k calculated at the discretization points. The objective function is then defined by the following residual operator

$$J(\Lambda, \underline{F}^1, ..., \underline{F}^{n_{obs}}) = \sum_{k=1}^{n_{obs}} \parallel \underline{S}^{\varepsilon,k}\left(\Lambda, \underline{F}^k\right) - \underline{S}^{obs,k} \parallel^2 + \lambda_{opt} \parallel \Lambda - \Lambda_0 \parallel^2, \tag{4.1}$$

where $\|\|$ denotes the Euclidian norm, $\underline{S}^{\epsilon,k}(\Lambda,^k$ is defined by (3.13), Λ_0 is the a priori information vector and where λ_{opt} is an optimal regularization parameter. The identification problem is then: Find $(\Lambda, F^1, \dots, F^{n_{obs}}) \in \mathbb{R}^{4N-2} \times \mathbb{R}^{m\times n_{obs}}$ which minimizes the operator J defined by (4.1). Solving this problem requires the calculation of the gradient of the residual operator J relative to each of its arguments. However it should be noted that this calculation is not obvious in many cases.

4.1. **Computing the gradient of the residual operator J.** We first notice that the residual operator J defined in (4.1) can be written as

$$\begin{aligned} J(\Lambda, \underline{F}^1, \ldots, \underline{F}^{n_{obs}}) = \sum_{k=0}^{n_{obs}} \langle \underline{S}^{\varepsilon,k}(\Lambda, \underline{F}^k) - \underline{S}^{obs,k}, \underline{S}^{\varepsilon,k}(\Lambda, \underline{F}^k) - \underline{S}^{obs,k} \rangle \\ + \lambda_{opt} \langle \Lambda - \Lambda^0, \quad \Lambda - \Lambda^0 \rangle \end{aligned} \tag{4.2}$$

Here $\langle\rangle$ denotes the Cartesian scalar product. Then, to determine its gradient we have to define associated tangent and adjoint equations.

4.2. **Tangent equations.** To determine the equations of the tangent model we consider the direct model described by equation (3.13). At first a perturbation $\xi\Lambda$ with respect to the control variable Λ leads to

$$A(\Lambda + \xi\delta\Lambda)\underline{S}^{\varepsilon,k}(\Lambda + \xi\delta\Lambda, \underline{F}^k) = \varepsilon^{-\alpha}\mathbb{T}\underline{F}^k. \tag{4.3}$$

Subtracting this last equation with the equation (4.1) gives

$$A(\Lambda + \xi\delta\Lambda)\underline{S}^{\varepsilon,k}(\Lambda + \xi\delta\Lambda, \underline{F}^k) - A(\Lambda)\underline{S}^{\varepsilon,k}(\Lambda, \underline{F}^k) = 0, \tag{4.4}$$

which can be written further

$$\begin{aligned} (A(\Lambda + \xi\delta\Lambda) - A(\Lambda))\, \underline{S}^{\varepsilon,k}(\Lambda + \xi\delta\Lambda, \underline{F}^k) \\ + A(\Lambda) \left(\underline{S}^{\varepsilon,k}(\Lambda + \xi\delta\Lambda, \underline{F}^k) - \underline{S}^{\varepsilon,k}(\Lambda, \underline{F}^k) \right) = 0 \end{aligned} \tag{4.5}$$

If we divide both sides of this equation by ξ and one passes to the limit as ξ tends to 0 then it follows the tangent equation with respect to the variable Λ that is

$$\widehat{A}(\underline{S}^{\varepsilon,k})\delta\Lambda + A\widehat{\underline{S}}^k_\Lambda \delta\Lambda = 0, \qquad k = 0, \ldots, n_{obs}, \tag{4.6}$$

where we have set

$$\begin{aligned} \widehat{A}(\underline{S}^{\varepsilon,k})\delta\Lambda &= \lim_{\xi \searrow 0} \frac{A(\Lambda + \xi\delta\Lambda) - A(\Lambda)}{\xi} \delta\Lambda, \\ \widehat{\underline{S}}^k_\Lambda \delta\Lambda &= \lim_{\xi \searrow 0} \frac{(\underline{S}^{\varepsilon,k}(\Lambda + \xi\delta\Lambda, \underline{F}^k) - \underline{S}^{\varepsilon,k}(\Lambda, \underline{F}^k)}{\xi}. \end{aligned} \tag{4.7}$$

Similarly with respect to variables $\underline{F}^k$ we can write

$$A(\Lambda)\underline{S}^{\varepsilon,k}(\Lambda,\underline{F}^k+\xi\delta\underline{F}^k)=\varepsilon^{-\alpha}\mathbb{T}\times(\underline{F}^k+\xi\delta\underline{F}^k), \tag{4.8}$$

and by subtracting equation (36) to (32) one obtains

$$A(\Lambda)\underline{S}^{\varepsilon,k}(\Lambda,\underline{F}^k+\xi\delta\underline{F}^k)-A(\Lambda)\underline{S}^{\varepsilon,k}(\Lambda,\underline{F}^k)=\varepsilon^{-\alpha}\mathbb{T}\times(\underline{F}^k+\xi\delta\underline{F}^k)-\varepsilon^{-\alpha}\mathbb{T}\underline{F}^k, \tag{4.9}$$

where by carrying as above we obtain the tangent equation with respect to the control variable $\underline{F}^k$,

$$A\widehat{\underline{S}}^k_{\underline{F}^k}\delta\underline{F}^k=\varepsilon^{-\alpha}\mathbb{T}\delta\underline{F}^k,\qquad k=0,\dots,n_{obs}, \tag{4.10}$$

where we have set

$$\widehat{\underline{S}}^k_{\underline{F}^k}\delta\underline{F}^k \;=\; \lim_{\xi\searrow 0}\frac{\underline{S}^{\varepsilon,k}(\Lambda,\underline{F}^k+\xi\delta\underline{F}^k)-\underline{S}^{\varepsilon,k}(\Lambda,\underline{F}^k)}{\xi}. \tag{4.11}$$

4.3. **Computation of the component $\nabla_\Lambda J$ of the gradient with respect to Λ.** By perturbing relatively to the residual function (4.2) one obtains

$$\begin{aligned}
&J(\Lambda-\xi\delta\Lambda,\underline{F}^1,\dots,\underline{F}^k)-J(\Lambda,\underline{F}^1,\dots,\underline{F}^k)\\
&=\sum_{k=0}^{n_{obs}}\langle\underline{S}^{\varepsilon,k}(\Lambda-\xi\delta\Lambda,\underline{F}^k)-\underline{S}^{obs,k},\underline{S}^{\varepsilon,k}(\Lambda-\xi\delta\Lambda,\underline{F}^k)-\underline{S}^{obs,k}\rangle\\
&-\sum_{k=0}^{n_{obs}}\langle\underline{S}^{\varepsilon,k}(\Lambda,\underline{F}^k)-\underline{S}^{obs,k},\quad\underline{S}^{\varepsilon,k}(\Lambda,\underline{F}^k)-\underline{S}^{obs,k}\rangle\\
&+\lambda_{opt}\langle\Lambda-\Lambda^0-\xi\delta\Lambda,\Lambda-\Lambda^0-\xi\delta\Lambda\rangle-\lambda_{opt}\langle\Lambda-\Lambda^0,\Lambda-\Lambda^0\rangle
\end{aligned} \tag{4.12}$$

After some calculations it follows

$$\langle\nabla_\Lambda J,\delta\Lambda\rangle=2\sum_{k=0}^{n_{obs}}\langle\widehat{\underline{S}}^k_\Lambda\delta\Lambda,\quad\underline{S}^{\varepsilon,k}-\underline{S}^{obs,k}\rangle+2\lambda_{opt}\langle\Lambda-\Lambda^0,\quad\delta\Lambda\rangle, \tag{4.13}$$

where

$$\langle\nabla_\Lambda J,\delta\Lambda\rangle=\lim_{\xi\searrow 0}\frac{J(\Lambda-\xi\delta\Lambda,\underline{F}^1,\dots,\underline{F}^k)-J(\Lambda,\underline{F}^1,\dots,\underline{F}^k)}{\xi}. \tag{4.14}$$

To give an explicit expression of $\nabla_\Lambda J$ the adjoint method can be performed using inner product to the tangent equation (4.6) with a suitable vectors sequence $\underline{q}^k$.

We then obtain

$$\sum_{k=0}^{n_{obs}}\langle \widehat{A}(\underline{S}^{\varepsilon,k})\delta\Lambda, \quad \underline{q}^k\rangle + \sum_{k=0}^{n_{obs}}\langle A\widehat{\underline{S}}_{\Lambda}^{k}\delta\Lambda, \quad \underline{q}^k\rangle = 0, \tag{4.15}$$

which can be written further

$$\sum_{k=0}^{n_{obs}}\langle \widehat{A}(\underline{S}^{\varepsilon,k})\delta\Lambda, \quad \underline{q}^k\rangle + \sum_{k=0}^{n_{obs}}\langle \widehat{\underline{S}}_{\Lambda}^{k}\delta\Lambda, \quad A^T\underline{q}^k\rangle = 0. \tag{4.16}$$

If we consider the sequence $\underline{q}^k$ satisfying

$$A(\Lambda)^T\underline{q}^k = \underline{S}^{\varepsilon,k} - \underline{S}^{obs,k}, \quad k = 0, \dots, n_{obs}, \tag{4.17}$$

then from the equation(39) we deduce

$$\sum_{k=0}^{n_{obs}}\langle \widehat{\underline{S}}_{\Lambda}^{k}\delta\Lambda, \quad \underline{S}^{\varepsilon,k} - \underline{S}^{obs,k}\rangle = \sum_{k=0}^{n_{obs}}\langle \widehat{A}(\underline{S}^{\varepsilon,k})\delta\Lambda, \quad \underline{q}^k\rangle. \tag{4.18}$$

Therefore, considering the expression (38), one obtains

$$\langle \nabla_\Lambda J, \delta\Lambda\rangle = 2\sum_{k=0}^{n_{obs}}\langle \widehat{A}(\underline{S}^{\varepsilon,k})\delta\Lambda, \quad \underline{q}^k\rangle + 2\lambda_{opt}\langle \Lambda - \Lambda^0, \quad \delta\Lambda\rangle, \tag{4.19}$$

which still writing

$$\langle \nabla_\Lambda J, \delta\Lambda\rangle = \langle 2\sum_{k=0}^{n_{obs}}\left(\widehat{A}(\underline{S}^{\varepsilon,k})\right)^* \underline{q}^k + 2\lambda_{opt}(\Lambda - \Lambda^0), \quad \delta\Lambda\rangle, \tag{4.20}$$

where $\left(\widehat{A}(\underline{S}^{\varepsilon,k})\right)^*$ denotes the adjoint of $\widehat{A}(\underline{S}^{\varepsilon,k})$. Finally this equation yields

$$\nabla_\Lambda J = 2\sum_{k=0}^{n_{obs}}\left(\widehat{A}(\underline{S}^{\varepsilon,k})\right)^* \underline{q}^k + 2\lambda_{opt}(\Lambda - \Lambda^0). \tag{4.21}$$

To compute $\nabla_\Lambda J$ we will need to determine the adjoint $\left(\widehat{A}(\underline{S}^{\varepsilon,k})\right)^*$. Firstly it is easy to see that $\left(\widehat{A}(\underline{S}^{\varepsilon,k})\right)^* = \widehat{A}(\underline{S}^{\varepsilon,k})^T$. So we need to calculate $\widehat{A}(\underline{S}^{\varepsilon,k})$. Since the application which associates to Λ associe $A(\Lambda)$ is linear, then

$$\lim_{\xi\searrow 0}\frac{A(\Lambda + \xi\delta\Lambda) - A(\Lambda)}{\xi}\delta\Lambda = A(\delta\Lambda). \tag{4.22}$$

Hence

$$\widehat{A}(\underline{S}^{\varepsilon,k})\delta\Lambda = \lim_{\xi\searrow 0} \frac{A(\Lambda+\xi\delta\Lambda)-A(\Lambda)}{\xi}\delta\Lambda\underline{S}^{\varepsilon,k} = A(\delta\Lambda)\underline{S}^{\varepsilon,k}. \tag{4.23}$$

Let us now set

$$\delta\Lambda = \begin{bmatrix} \delta\Lambda^1 \\ \delta\Lambda^2 \end{bmatrix}, \tag{4.24}$$

with

$$\delta\Lambda^1 = \left[\delta\Lambda^1_1,\ldots,\delta\Lambda^1_{2N-1}\right]^T \text{ et } \delta\Lambda^2 = \left[\delta\Lambda^2_1,\ldots,\delta\Lambda^2_{2N-1}\right]^T. \tag{4.25}$$

Denoting $\underline{S} = [S_1,\ldots,S_{2N-1}]^T$ yields the following.

Lemma 4.1. *We have*

$$A(\delta\Lambda)\underline{S} = \mathit{diag}\left(A_1\underline{S}\right)\delta\Lambda^1 + \mathit{diag}\left(A_2\underline{S}\right)\delta\Lambda^2, \tag{4.26}$$

where

$$A_1 = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & -1 & & & \\ -1 & 0 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & -1 \\ & & & -1 & 0 \end{pmatrix}. \tag{4.27}$$

Next, by setting the following block matrix

$$D(\underline{S}) = \left[diag(A_1\underline{S}) \mid diag(A_2\underline{S})\right]. \tag{4.28}$$

Lemma 4.2. *One has*

$$A(\delta\Lambda)\underline{S} = D(\underline{S})\delta\Lambda. \tag{4.29}$$

Based on these two lemmas we can state the following result

Proposition 4.1. *The adjoint of $\widehat{A}(\underline{S}^{\varepsilon,k})$ and the gradient of the operator J with respect to Λ are respectively given by*

$$\left(\widehat{A}(\underline{S}^{\varepsilon,k})\right)^* = D(\underline{S}^{\varepsilon,k})^T \tag{4.30}$$

and

$$\nabla_{\Lambda} J = 2 \sum_{k=0}^{n_{obs}} D(\underline{S}^{\varepsilon,k})^T \underline{q}^k + 2\lambda_{opt}(\Lambda - \Lambda^0). \tag{4.31}$$

4.4. **Computation of the component $\nabla_{\underline{F}^k} J$ of the gradient with respect to $\underline{F}^k$.** As in the previous section, it is easily shown

$$\langle \nabla_{\underline{F}^k} J, \delta \underline{F}^k \rangle = 2 \langle \widehat{\underline{S}}^k_{\underline{F}^k} \delta \underline{F}^k, \quad \underline{S}^{\varepsilon,k} - \underline{S}^{obs,k} \rangle, \quad k = 0, \dots, n_{obs}. \tag{4.32}$$

According to the tangent equation (4.10) we deduce

$$\widehat{\underline{S}}^k_{\underline{F}^k} \delta \underline{F}^k = \varepsilon^{-\alpha} A(\Lambda)^{-1} \mathbb{T} \delta \underline{F}^k. \tag{4.33}$$

Hence,

$$\begin{aligned} \langle \nabla_{\underline{F}^k} J, \delta \underline{F}^k \rangle &= 2\varepsilon^{-\alpha} \langle A(\Lambda)^{-1} \mathbb{T} \delta \underline{F}^k, \underline{S}^{\varepsilon,k} - \underline{S}^{obs,k} \rangle \\ &= 2\varepsilon^{-\alpha} \langle \mathbb{T}^T (A(\Lambda)^{-1})^T (\underline{S}^{\varepsilon,k} - \underline{S}^{obs,k}), \delta \underline{F}^k \rangle. \end{aligned} \tag{4.34}$$

We finally state

$$\nabla_{\underline{F}^k} J = 2\varepsilon^{-\alpha} \mathbb{T}^T (A(\Lambda)^{-1})^T (\underline{S}^{\varepsilon,k} - \underline{S}^{obs,k}), \quad k = 0, \dots, n_{obs}. \tag{4.35}$$

5. Algorithmic scheme

The algorithmic scheme is based on the calculation of the gradient of the function objective. To solve the problem, we use Newton's method.

Let be the vectors $\Lambda \in \mathbb{R}^{4N-2}$ et $\underline{F}^k \in \mathbb{R}^m; k = 1, 2, \dots, n_{obs}$, where the parameters N, m and n are fixed. Then the gradient calculation procedure summarizes as follows:

(1) Read input arguments
   - $\Lambda \in \mathbb{R}^{4N-2}$: sedimentation parameter vector;
   - $\underline{F}^k \in \mathbb{R}^m$: source function approximation vector.

(2) Read $\underline{S}^{obs,k}$, $k = 1, 2, \dots, n_{obs}$;

(3) Read the parameters Λ^0 and λ_{opt}.

(4) Calculate the matrix $\mathbb{T}$ of order

$$(m+1) \times (2N-1)$$

with the formula (3.10);

(5) Calculate the tridiagonal matrix A_1 et A_2 of order $2N-1$ with the formula (4.29);
(6) Calculate $A(\Lambda)$ with the formula (3.8);
(7) For $k=1$ to $k=n_{obs}$, do
- Calculate $\underline{S}^{\epsilon,k}$ by resolving the equation (3.13);
- Calculate $D(\underline{S}^{\epsilon,k})$ with the formula (4.28);
- calculate $\underline{q}^k$ by resolving the equation (4.17);
- Do $\nabla_\Lambda J \longleftarrow \nabla_\Lambda J + 2D(\underline{S}^{\epsilon,k})^T\underline{q}^k$;
- Do $\nabla_{F^k} J \longleftarrow 2\epsilon^{-\alpha}\mathbb{T}^T(A(\Lambda)^{-1})^T(\underline{S}^{\epsilon,k} - \underline{S}^{obs,k})$.

(8) Do $\nabla_\Lambda \longleftarrow \nabla_\Lambda J + 2\lambda_{opt}(\Lambda - \Lambda_0)$;
(9) Write $\nabla_\Lambda J, \nabla_{\underline{F}^k} J,\ k=1,2,...,n_{obs}$.

6. Implementation

6.1. **Experimental data.** In the process of numerical experimentation, in the absence of real physical measurements, we must generate experimental data. We consider observations represented by $\underline{X}^{obs,k}$ and $\underline{Y}^{obs,k}$, respectively the abscissa and ordonate vectors of the observation points at an instant t_k. We denote by δt the time interval between two consecutive observations and by $\underline{S}^{obs,k}$ the height vector of sediments at time t_k at the observation points. We propose a procedure allowing to generate the observed data S by solving the following partial differential equation analogous to that proposed by Lloyd N Trefethen [7],

$$\frac{\partial S}{\partial x} + c(t,y)\frac{\partial S}{\partial y} = 0 \tag{6.1}$$

on a stretch of the river; with $0 \le x \le 8$ and $0 \le y \le 6$, taking $c(t,y) = 0,2\sqrt{2}sin^2(y-t)$, with a boundary condition $S(t,0,y) = exp(-100(y-t)^2)$.

From these experimental data we deduce the vectors of the observed shock line.

6.2. **The direct scheme.** The direct scheme consists in calculing the sedimentation state vector $\underline{S}^{\epsilon,k}$ from the application:

$$(\Lambda, \underline{F}^k) \mapsto \underline{S}^{\epsilon,k} = \epsilon^{-\alpha}A(\Lambda)^{-1}\mathbb{T}\underline{F}^k. \tag{6.2}$$

Matrix $\mathbb{T}$ can be generated by the simple command:

```
T=cos(acos((-N+1:N-1)*h/N)'*(0:m));
```

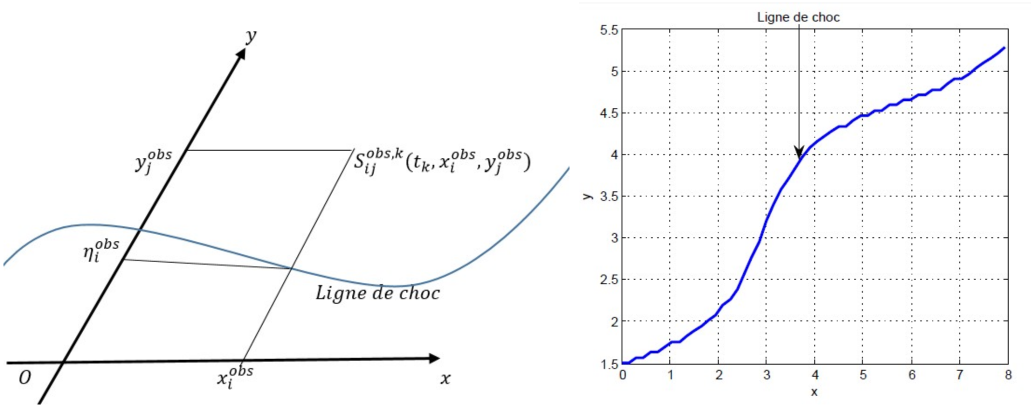

FIGURE 1. *sedimentation state observed(left) and the corresponding shock line(right).*

6.3. **The reverse scheme.** The inverse code consists in calculing at each iteration k, the vector $\underline{q}^k$ solution of the adjoint equation (4.17) and in the calculation of the gradient.

And to finish, we take the code of the main program which allows us to identify the sedimentation parameter as well as the vector coefficients in the Chebychev basis of the source function according to the discretization parameters h and N and the approximation parameters α, ϵ and n.

6.4. **Numerical results.** In this part we present the resultsof the numerical simulation on the propagation of a sandbank and the convergence of the approximation scheme over six observation times between the initial time t_0 and the time $t_0 + 25$ for the following values of the settings : $n = 10$, $\varepsilon = 10^{-1}$, $h = 10^{-1}$, $m = 3$ and $\alpha = 1$.

In figure 2, we observe the propagation of a sandbank considered as a shock line from t_0 to $t_0 + 25$.

The following figures illustrate the convergence of the approximation scheme: for larger and larger values of n and for smaller and smaller values of h, the convergence is much clearer.

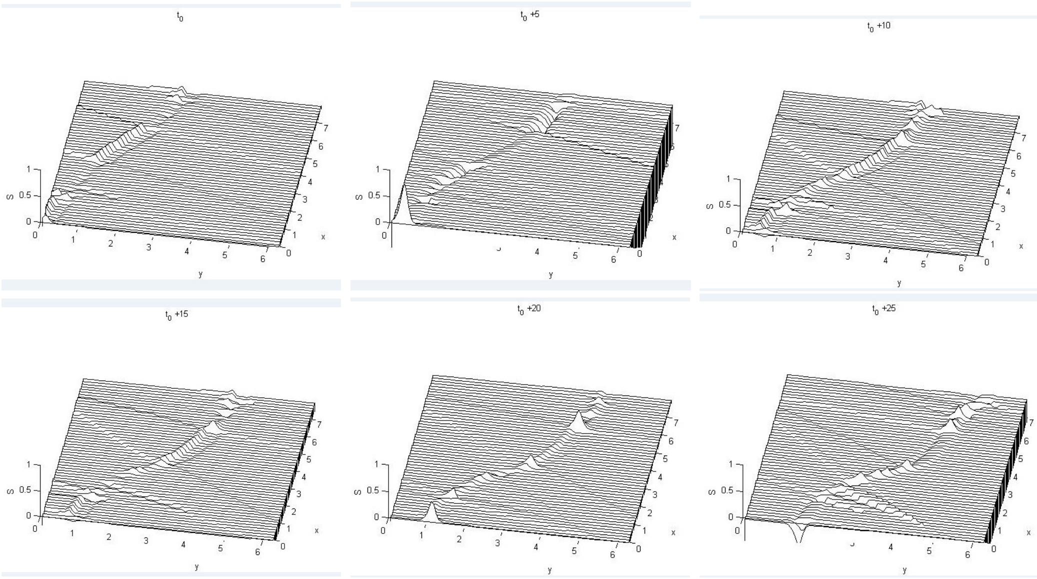

FIGURE 2. *Numerical simulation on the propagation of a sandbank over six observation times between the initial time* t_0 *and the time* $t_0 + 25$ *for* $n = 1$ *and* $L = 0.1$

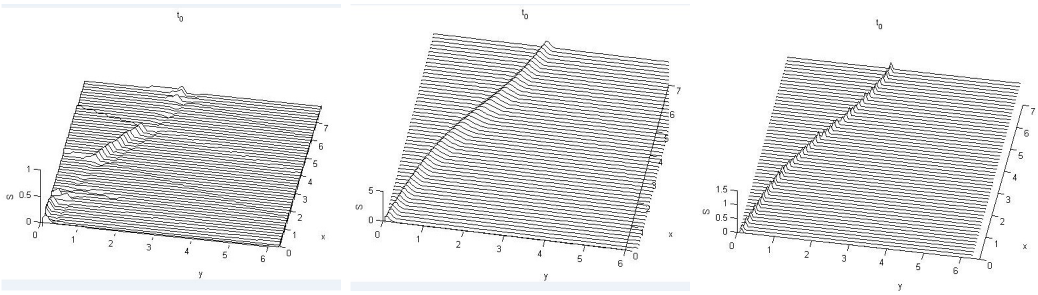

FIGURE 3. *Illustration of the approximation scheme at* t_0*: for the first figure* $n = 10$ *et* $h = 0.1$*, for the second figure* $n = 10$ *and* $h = 0.01$ *and for the third* $n = 100$ *et* $h = 0.01$

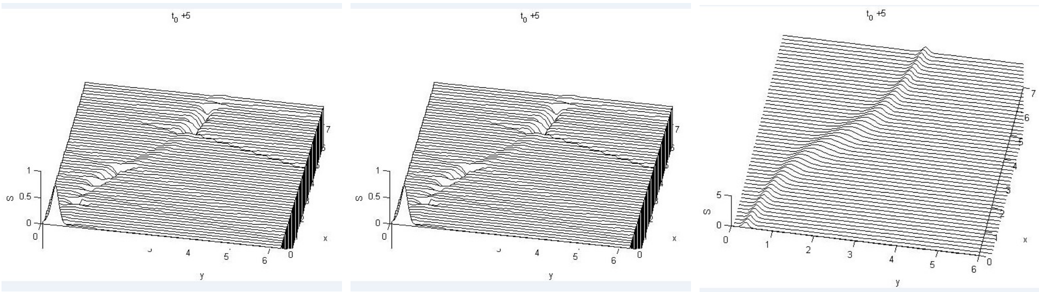

FIGURE 4. *Illustration of the approximation scheme at* $t_0 + 5$*:for the first figure* $n = 10$ *and* $h = 0.1$*, for the second second* $n = 10$ *and* $h = 0.01$ *and for the third* $n = 100$ *and* $h = 0.01$

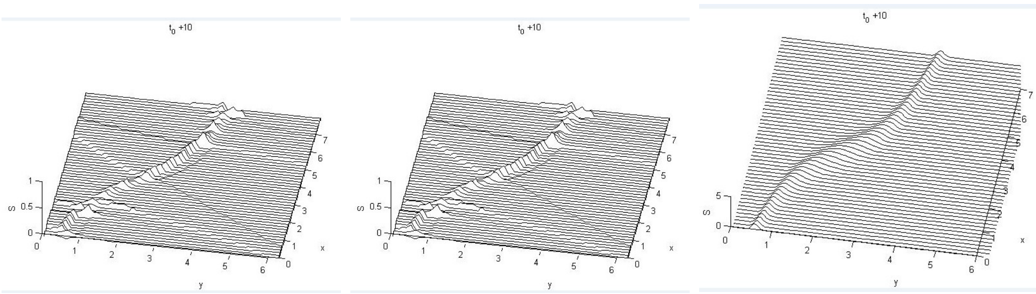

FIGURE 5. *Illustration of the approximation scheme at* $t_0 + 10$*:for the first figure* $n = 10$ *and* $h = 0.1$ *, for the second second* $n = 10$ *and* $h = 0.01$ *and for the third* $n = 100$ *and* $h = 0.01$

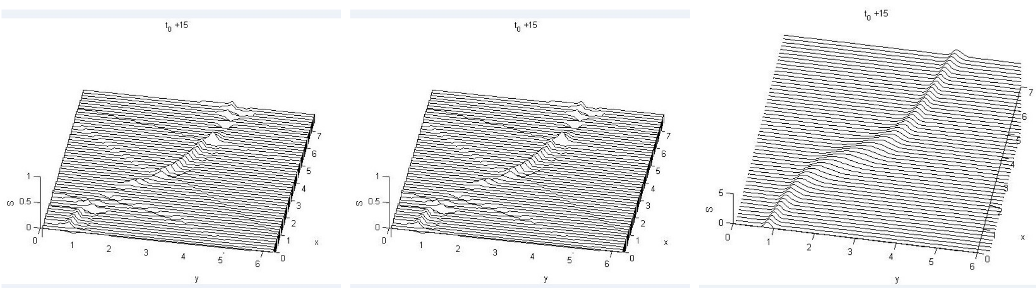

FIGURE 6. *Illustration of the approximation scheme at* $t_0 + 15$*:for the first figure* $n = 10$ *and* $h = 0.1$*, for the second second* $n = 10$ *and* $h = 0.01$ *and for the third* $n = 100$ *and* $h = 0.01$

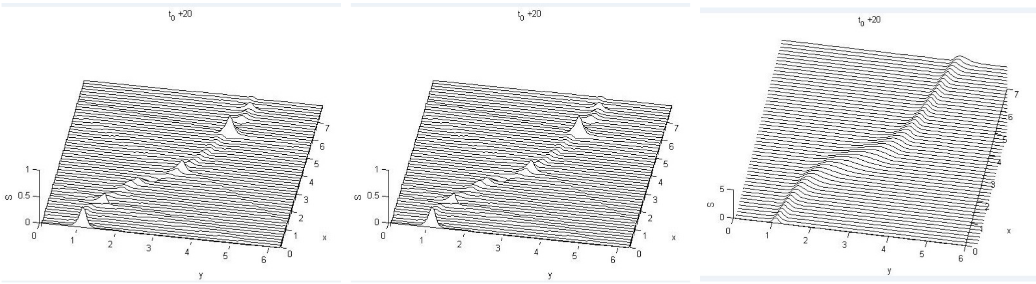

FIGURE 7. *Illustration of the approximation scheme at* $t_0 + 20$*: for the first figure* $n = 10$ *and* $h = 0.1$*, for the second second* $n = 10$ *and* $h = 0.01$ *and for the third* $n = 100$ *and* $h = 0.01$

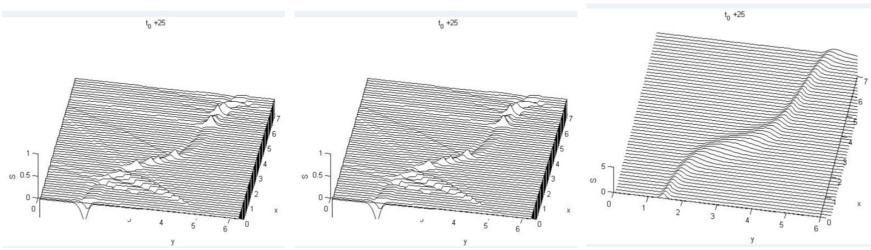

FIGURE 8. *Illustration of the approximation scheme at* $t_0 + 25$*: for the first figure* $n = 10$*,* $h = 0.1$*, for the second* $n = 10$*,* $h = 0.01$*, for the third* $n = 100$*,* $h = 0.01$

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