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LARGE DEVIATIONS FOR STOCHASTIC VOLTERRA EQUATIONS WITH REFLECTION IN HÖLDERIAN NORM

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ABSTRACT. In this paper, we study the large deviations principle **(LDP)** of the Volterra process with reflection in Hölderian norm by using the Azencott method. As an application, we obtain the large deviations principle **(LDP)** of a perturbed reflected diffusion process driven by the Fractional Brownian Motion with Hurst parameter $H \in [\frac{1}{2}, 1)$.

1. INTRODUCTION

The purpose of this paper is to study small perturbations of the solution of the following Volterra-type stochastic differential equations (SDE):

(1.1)
$$X_t = x_0 + \int_0^t b(t,s,X_s)\,ds + \int_0^t \sigma(t,s,X_s)\,dB_s + \beta \sup_{0\le s\le t} X_s, \quad t \in [0,1],$$

and let $T = (T_t), t \ge 0$ be the solution of the stochastic differential equation

$$T_t = y + \int_0^t \varsigma(t,s,T_s)\,dB_s + \beta \sup_{0\le s\le t} T_s + L_t, \quad t \in [0,1], \tag{1.2}$$

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with $\beta \in [0,1], x \in \mathbb{R}, y \in \mathbb{R}_+$ are deterministic. Here $\{L_t, t \in [0,1]\}$ is a continuous increasing process with L_0 and

$$\int_0^t \chi_{\{T_s=0\}}\, dL_s = L_t. \tag{1.3}$$

We can assume $\{L_t, t \in [0,1]\}$ as the local time of the semimartingale $\{T_t, t \in [0,1]\}$ at the origin. $\{B_t, t \in [0,1]\}$ the standard one-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P})$.

Let $b, \sigma : [0,1] \times [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\varsigma : [0,1] \times [0,1] \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ are measurable functions satisfying the following conditions:

(1) The functions b and σ are lipschitzian with respect to the second variable. In other words, there exist a constant L such that, for all $x, y \in \mathbb{R}$, $s, t \in [0,1]$, we have:

$$|b(t,s,x) - b(t,s,y| \leq L|x-y|,$$

$$|\sigma(t,s,x) - \sigma(t,s,y)| \leq L|x-y|.$$

(2) There exists a constant L such that, for all $x, y \in \mathbb{R}$, $s, t \in [0,1]$, we have

$$|\zeta(t,s,x) - \zeta(t,s,y)| \leq L|x-y|.$$

(3) The functions $b(t,s,x)$, $\sigma(t,s,x)$ and $\varsigma(t,s,x)$ are bounded.

For $\alpha \in]0, \frac{1}{2}[$, let $C^\alpha([0,1], \mathbb{R})$ be the space of continuous functions from $[0,1]$ to $\mathbb{R}$ equipped with the α-Hölderian norm. Let us consider the small perturbations solutions of the SDE (1.4)

$$X_t^\varepsilon = x_0 + \int_0^t b(t,s,X_s^\varepsilon)\, ds + \sqrt{\varepsilon} \int_0^t \sigma(t,s,X_s^\varepsilon)\, dB_s + \beta \sup_{0\leq s\leq t} X_s^\varepsilon, \tag{1.4}$$

$H \in (0,1)$, $t \in [0,1]$, and let $T^\varepsilon = (T_t^\varepsilon), t \geq 0$ be the solution of the stochastic differential equation

$$T_t^\varepsilon = y + \sqrt{\varepsilon} \int_0^t \varsigma(t,s,T_s^\varepsilon)\, dB_s + \beta \sup_{0\leq s\leq t} T_s^\varepsilon + L_t^\varepsilon \tag{1.5}$$

with $t \in [0,1]$. Here $\{L_t^\varepsilon, t \in [0,1]\}$ is a continuous growing process with L_0^ε and

$$\int_0^t \chi_{\{T_s^\varepsilon=0\}}\, dL_s^\varepsilon = L_t^\varepsilon. \tag{1.6}$$

In this work, we propose to show for $\alpha \in]0, \frac{1}{2}[$ and $\beta \in]0, 1[$ (resp. $\alpha \in]0, \frac{1}{2}[$ and $\beta \in]0, \frac{1}{2}[$) the large deviations principle (**LDP**) for the laws of X^ε (resp. T^ε) solution of the equation (1.4) (resp. (1.5)) in $C^\alpha([0,1], \mathbb{R})$using the Azencott method.

In [6] Ventzell and Freidlin (1970) considered the **LDP** for the diffusion processes of stochastic differential equations driven by the standard Brownian motion. Azencott (1980) [1], later Priouret (1982) [12] have extended this estimations to the general class of diffusions where the coefficients are lipschitzian functions. In [7], Priouret-Doss established the **LDP** for perturbed and reflected stochastic differential equations driven by a standard Brownian motion. The stochastic Volterra differential equation in the plane was studied by Rovira and Sanz-Solé (1997), then David Nualart and Carles Rovira [10] studied the **LDP** of the Voltera equation. Later, Boualem Djehiche and M'hamed Eddahbi [2] extended their results to the Besov-Orlicz norm. In [9], El Hassan LAKHEL extended the results of Boualem Djehiche and M'hamed Eddahbi in $\mathbb{R}^d$.

As an application of the **LDP** for Volterra stochastic differential equations, we study the stochastic differential equation with reflexion driven by a Fractional Brownian motion.

This paper is organized as follows. In Section 2, we announce the first preliminary results. In section 3, we give the rigorous formulation of the problem and we present the main theorem, the theorem 3.1 , the theorem 3.3 and 3.4. The proof of these theorems depends on the reflection principle and the Lemma 2.2 given in section 2. In section 4, we prove the **LDP** of the solution of the equation (1.4) and (1.5). Finally, section 5 gives an example where we can apply this method.

2. Preliminary results

Let E be a Polish metric space (complete metric space), $\mathcal{B}(E)$ its borelian tribute, $(\mathbb{P}^\varepsilon)_{\varepsilon>0}$ a family of probability measures on $\mathcal{B}(E)$.

We suppose that $(P^\varepsilon)_{\varepsilon>0}$ converges tightly to δ_{x_0} and we want to quantify this convergence.

Definition 2.1. *A family of probabilities measures $(\mathbb{P}^\varepsilon)_{\varepsilon>0}$ is satisfies the large deviations principle (or shorter LDP) with the rate function I if there exists a function I defined on E such that:*

- $0 \leq I(x) \leq +\infty$ *for any* $x \in E$,
- I *is semicontinuous in lower terms, in other words, for all* $l < \infty, \{x : I(x) \leq l\}$ *is a closed of E or for all* $x_n \longrightarrow x$, *then* $I(x) \leq \liminf I(x_n)$
- *For every closed subset* $F \subset E$,

$$\limsup_{\varepsilon \longrightarrow 0} \varepsilon \log \mathbb{P}^{\varepsilon}(F) \leq - \inf_{x \in F} I(x).$$

- *For every open subset* $O \subset E$,

$$\liminf_{\varepsilon \longrightarrow 0} \varepsilon \log \mathbb{P}^{\varepsilon}(O) \geq - \inf_{x \in O} I(x).$$

If additionally, for any $l < \infty, \{x : I(x) \leq l\}$ *is a compact of* E *then we say that I is a good rate function*

Theorem 2.1. ***(Contraction principle)*** *Let* $(\mathbb{P}^{\varepsilon})_{\varepsilon>0}$ *be a family of probabilities satisfying a principle of large deviations of good rate function* I. *Let, for all* $\longrightarrow$, $f_{\longrightarrow} : E \longrightarrow \mathbb{R}$ *a continuous function of E in a separable metric space F. Assume that there exist* $f : E \longrightarrow F$ *such that*

$$\limsup_{\varepsilon \longrightarrow 0} \sup_{x: I(x) \leq l} d_F(f_{\varepsilon}(x), f(x)) = 0.$$

Then $(Q^{\varepsilon} = \mathbb{P}^{\varepsilon} \circ f_{\varepsilon}^{-1})$ *satisfies a large deviation principle with the rate function J where*

$$J(y) = \inf_{x: f(x) = y} I(x).$$

Theorem 2.2. ***(Schilder's theorem)*** *The family* $(\mathcal{W}^{\varepsilon})_{\varepsilon>0}$ *satisfy a large deviation principle**(LDP)*** *on* E *of rate function* λ *given by :*

$$\lambda(h) = \begin{cases} \frac{1}{2} \int_0^T |\dot{h}(s)|^2 \, ds & \text{if } h \in \mathcal{H} \\ +\infty & \text{otherwise} \end{cases},$$

where $\mathcal{H} = \left\{ \int_0^t \dot{h}(s) ds : \dot{h} \in L^2([0,1]) \right\}$.

Proposition 2.1. ***(Azencott's method)*** *Let* (E_i, d_i), $i = 1, 2$ *be two Polish spaces and* $X_{\varepsilon}^i \rightarrow E_i$, $\varepsilon > 0, i = 1, 2$ *two families of random variables. Suppose that* $\{X_1^{\varepsilon}, \varepsilon > 0$ *satisfy a* ***LDP*** *with the rate function* $I_1 : E_1 \rightarrow [0, +\infty]$.

Let $\Phi : \{I_1 < \infty\} \rightarrow E_2$ an application such that its restriction to compact sets $\{I_1 \leq a\}$ is continuous in the topology of E_1. For all $g \in E_2$ we pose $I(g) =$

$\inf\Big\{ I_1(f), \Phi(f)=g\Big\}$. Suppose that for all $R,\ \rho,\ \ a>0$ there exist α and $\varepsilon_0>0$ such that for all $h\in E_1$ satisfying $I_1(h)\le a$ and $\varepsilon\le\varepsilon_0$ we have

Lemma 2.1. *[11](**Gronwall's Lemma**) Soit $T>0$ and let f and h be two positive functions measurable for any $y\in[0,T]$ satisfying $\frac{\partial h}{\partial t}(.,y)$ and $h(y,y)$ exists. Supposed that there exist a constant $a\ge 0$ such that for any $t\in[0,T]$, we have:*

$$f(t)\le a+\int_0^t h(t,s)f(s)ds. \tag{2.1}$$

Then, we have $f(t)\le a\exp\Big(\int_0^t h(t,u)du\Big)$ for any $t\in[0,T]$.

Theorem 2.3. *(**Reflection principle**) Let $(U_t)_{t\ge0}$ et $(D_t)_{t\ge0}$ two continuous processes on $[0,T]$, with $T>0$, and $\beta\in(0,1)$. So the equation*

$$U_t=D_t+\beta\sup_{0\le s\le t}U_s \tag{2.2}$$

admits a unique solution of the form

$$U_t=D_t+\frac{\beta}{1-\beta}\sup_{0\le s\le t}D_s. \tag{2.3}$$

Proof. Let us note by : $U_t^*=\sup_{s\le t}U_s$ and by $D_t^*=\sup_{s\le t}D_s$. From the formula (2.2), we have $U_t^*\le D_t^*+\beta U_t^*$. So, $(1-\beta)U_t^*\le D_t^*$. Let $s_0(t)$ the point where D_t^* is attained, then by virtue of equation (2.2), we have : $U_{s_0(t)}=D_t^*+\beta U^*_{s_0(t)}$, then it follows that $D_t^*\le(1-\beta)U^*_{s_0(t)}\le(1-\beta)U_t^*$ because $U_{s_0(t)}\le U^*_{s_0(t)}$ and $s_0(t)\le t$, hence $D_t^*=(1-\beta)U_t^*$ and formula (2.2) is rewritten:

$$U_t=D_t+\frac{\beta}{1-\beta}D_t^*. \tag{2.4}$$

□

Theorem 2.4. *(**Corollary of the Reflection Principle**) Let $(U_t)_{t\ge0}$ be a process of the form*

$$U_t=U_0+\int_0^t b(t,s,x)ds+\int_0^t\sigma(t,s,x)dB_s+\beta\sup_{s\le t}U_s.$$

Then,

$$\sup_{s\le t}U_s\le\frac{1}{1-\beta}\Big(U_0+\sup_{s\le t}\int_0^s b(s,u,x)du+\sup_{s\le t}\int_0^s\sigma(s,u,x)dB_u\Big).$$

Lemma 2.2. *Let* $\{Z(t,s)\}_{t\in T}$ *be a real process satisfying the following conditions:*

(1) $Z : [0,T]\times[0,T]\times\Omega \to \mathbb{R}$ *est* $\mathcal{B}([0,T])\otimes\mathcal{B}([0,T])\otimes\mathcal{F}-$*mesurable.*
(2) $Z(t,s) = 0$ *if* $s > t$.
(3) $Z(t,s)$ *is* $\mathcal{F}-adapted$.
(4) *There exists a random variable* ζ *and a real* $\delta \in]0,2]$ *such that for all* $t, r \in [0,T]$*, we have:*

$$\int_0^{min(r,t)} |Z(t,s) - Z(r,s)|^2 ds \leq \zeta |t-r|^\delta. \tag{2.5}$$

Then, for any β*,* $0 < \beta \leq min(1,\delta)$*, there exist positive constants* $K_1(\beta), K_2, K_3$ *such that, we have:*

$$\begin{aligned} &\mathbb{P}\left(\| \int_0^t Z(t,s)dB_s \|_\alpha > a, \| Z \|_\infty \leq K_Z, \zeta \leq C_Z\right) \\ &\leq K_1 exp\{-\frac{L^2}{(TK_Z^2 + T^\alpha C_Z)}K_3\}. \end{aligned} \tag{2.6}$$

Proof. Readers are referred in [10] for more detail of the proof of Lemma 2.2. □

3. Statement of the main results

3.1. Statement of LDP results for the solution of (1.4).

Theorem 3.1. *For every* $\alpha \in]0,\frac{1}{2}[, \beta \in]0,1[$*. By denoting* $\{\eta_\varepsilon, \varepsilon > 0\}$ *the family of probability measures associated to* X^ε *solution of the SDE (1.4), considered as a random variable in* $C^\alpha([0,1],\mathbb{R})$ *equipped with the Hölderian norm* $\| \,.\, \|_\alpha$*, then the family* η_ε *satisfy a* ***LDP*** *with the good rate function* $I(.)$ *defined by:*

$$I(g) = \inf_{h\in\mathcal{H}; g=\Phi^x(h)} \left\{\frac{1}{2} \| \dot{h} \|^2_{L^2(T)}, g = \Phi^x(h)\right\} \text{ for any } g \in C^\alpha([0,1],\mathbb{R})$$

where $\mathcal{H}$ *is the Cameron Martin space associated with the Brownian motion* B*:*

$$\mathcal{H} = \left\{ \begin{array}{c} h : [0,1] \to \mathbb{R}, \ \ h(t) = \int_0^t \dot{h}(s)ds \text{ such that} \\ h(0) = 0 \text{ and } \int_0^1 |\dot{h}_s|^2 \, ds < +\infty \end{array} \right\},$$

$\Phi^x(h)(t)$ *is the unique solution of the following solution of the deterministic differential perturbed equation, called the skeleton of* (1.4)*:*

$$\begin{aligned}\Phi^x(h)(t) = x_0 + \int_0^t b\Big(t, s, \Phi^x(h)(s)\Big)ds + \int_0^t \sigma\Big(t, s, \Phi^x(h)(s)\Big)\dot{h}(s)ds \\ + \beta \sup_{0\le s\le t} \Phi^x(h)(s)\end{aligned} \tag{3.1}$$

First, we need to study the existence and uniqueness of the solution of the equation (1.4) and (3.1).

3.2. **Existence and uniqueness of the solution of the equation** (1.4)**.**

Theorem 3.2. *For every* $\alpha \in]0, \frac{1}{2}[,\ \beta \in]0, 1[$*, for every* $x_0 > 0$*, the equation* (1.4) *admits a unique solution in* $C^\alpha([0, 1], \mathbb{R})$*.*

Proof. Consider the approximating sequence $(X_t^{\varepsilon,n})_{n\ge 0}$ defined by:

$$X_t^{\varepsilon,n+1} = x_0 + \int_0^t b(t, s, X_s^{\varepsilon,n})\, ds + \sqrt{\varepsilon}\ \int_0^t \Big[\ \sigma(t, s, X_s^{\varepsilon,n})\Big]\, dB_s + \beta \sup_{0\le s\le t} X_s^{\varepsilon,n}$$

$H \in (0, 1)$, $t \in [0, 1]$. It follows that,

$$\begin{aligned} X_t^{\varepsilon,n+1} - X_t^{\varepsilon,n} \ &= \ \int_0^t \sigma(t, s, X_s^{\varepsilon,n})\ - \sigma(t, s, X_s^{\varepsilon,n-1})dB_s \\ &+ \ \int_0^t b(t, s, X_s^{\varepsilon,n})\ - b(t, s, X_s^{\varepsilon,n-1})\ ds \\ &+ \ \beta\Big(\sup_{0\le s\le t} X_s^{\varepsilon,n} - \sup_{0\le s\le t} X_s^{\varepsilon,n-1}\Big).\end{aligned}$$

By **the Reflection principle**, and by virtue of the fact that for two continuous functions u and v on $\mathbb{R}_+$.

$$\Big|\sup_{0\le s\le t} u(s) - \sup_{0\le s\le t} v(s)\Big| \le \sup_{0\le s\le t} \Big|u(s) - v(s)\Big|.$$

Thus, **by the Reflection principle**,it follows that, for $t \in [0, 1]$

$$\begin{aligned} |X_t^{\varepsilon,n+1} - X_t^{\varepsilon,n}| \ &\le \ \int_0^t |\sigma(t, s, X_s^{\varepsilon,n})\ - \sigma(t, s, X_s^{\varepsilon,n-1})dB_s| \\ &+ \ \int_0^t |b(t, s, X_s^{\varepsilon,n})\ - b(t, s, X_s^{\varepsilon,n-1})|\ ds\end{aligned}$$

$$\begin{aligned} &+ \quad \tfrac{\beta}{1-\beta}\Big(\sup_{0\le s\le t} \int_0^s |\sigma(s,u,X_u^{\varepsilon,n}) \ - \sigma(s,u,X_u^{\varepsilon,n-1}) dB_u| \\ &+ \quad \sup_{0\le s\le t} \int_0^s |b(s,u,X_u^{\varepsilon,n}) \\ &- \quad b(s,u,X_u^{\varepsilon,n-1})| \ ds\Big). \end{aligned}$$

By Using the following estimate, for any $u \in \mathbb{R}$ and $v \in \mathbb{R}$, we have:

$$(u+v)^2 \le 2(u^2+v^2),$$

and by applying **Burkhölder's inequality** on the martingale M_t defined by

$$M_t^n = \int_0^t |\sigma(t,u,X_u^{\varepsilon,n}) \ - \sigma(t,u,X_u^{\varepsilon,n-1}) dB_u| (*),$$

and considering $p=2$, we have

$$\mathbb{E}[\sup_{0\le s\le t} |X_s^{\varepsilon,n+1} - X_s^{\varepsilon,n}|^2] \le C_p \frac{L^n}{n!} \mathbb{E}[\sup_{0\le s\le t} |X_s^{\varepsilon,1} - X_s^{\varepsilon,0}|^2].$$

Again, by applying **Burkhölder inequality** on the martingale M_t' which appears in the expression of the $|X_s^{\varepsilon,1} - X_s^{\varepsilon,0}|^2$, it follows that

$$\mathbb{E}[\sup_{0\le s\le t} |X_s^{\varepsilon,1} - X_s^{\varepsilon,0}|^2] < \infty.$$

It should be noted that

$$\mathbb{P}[\sup_{0\le s\le T} |X_s^{\varepsilon,n+1} - X_s^{\varepsilon,n}| \ge \frac{1}{2^n}] = \mathbb{P}[\sup_{0\le s\le T} |X_s^{\varepsilon,n+1} - X_s^{\varepsilon,n}|^2 \ge \frac{1}{4^n}].$$

By **Markov inequality** and the relation (*), we have

$$\mathbb{P}[\sup_{0\le s\le T} |X_s^{\varepsilon,n+1} - X_s^{\varepsilon,n}|^2 \ge \frac{1}{4^n}] \le 4^n \mathbb{E}[\sup_{0\le s\le T} |X_s^{\varepsilon,n+1} - X_s^{\varepsilon,n}|^2] \le C\frac{(4L)^n}{n!}.$$

It is easy to see that the general term series $U_n = \frac{(4L)^n}{n!}$ is convergent.

By using the fact that if $(X^{\varepsilon,n})_{n\ge 0}$ a sequence of random variables such that for a convergent positive term series $(U_n)_{n\ge 0}$, we have $\sum_{n\ge 0} \mathbb{P}[|X_s^{\varepsilon,n+1} - X_s^{\varepsilon,n}| > U_n] < \infty$, then $(X^{\varepsilon,n})_{n\ge 0}$ is almost surely convergent. Hence, the exitence of the solution of the equation (1.4).

The uniqueness of the solution of the equation (1.4) will be obtained by successively applying **Burkhölder's inequality** the Fubini-Tonelli theorem and the Gronwall Lemma. Indeed, if U_t and V_t are two solutions of the equation (1.4) we have:

$$|U_t - V_t|^2 \leq K_1 \sup_{0\leq s\leq t} \int_0^s |\sigma(s,u,U_u) \ - \sigma(t,u,V_u)dB_u|^2$$
$$+ K_2 \sup_{0\leq s\leq t} \int_0^s |b(s,u,U_u) \ - b(t,u,V_u)|^2 du.$$

By **the Burkholder inequality**, we have

$$\mathbb{E}[|U_t - V_t|^2] \leq K_2 \mathbb{E}[\int_0^t |U_s - V_s|^2 ds].$$

By using the **Fubini-Tonelli Theorem**

$$\mathbb{E}[\int_0^t |U_s - V_s|^2 ds] = \int_0^t (\mathbb{E}[|U_s - V_s|^2])ds.$$

From **Gronwall's Lemma**, we have:

$$\mathbb{E}[|U_t - V_t|^2] \leq 0 \exp t = 0.$$

Consequently, we have the uniqueness of the solution. □

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3.3. **Existence and uniqueness of the solution of the equation** (3.1).

Lemma 3.1. *Suppose that the functions σ and b are bounded and lipschitzian, then the equation defined in the formula* (3.1) *admits a unique solution.*

Proof. Let us note by $\Phi^{(1)}(h)$ and $\Phi^{(2)}(h)$ two solutions of the equation (3.1).

Now, let us denote by $\mathbf{D}\Phi(h)(t) = \Phi^{(1)}(h)(t) - \Phi^{(2)}(h)(t)$. So, $\| \Phi^{(1)}(h) - \Phi^{(2)}(h) \|_\alpha$ can be rewritten as $\| \mathbf{D}\Phi(h) \|_\alpha$,

$$|\mathbf{D}\Phi(h)(t)| \leq L \int_0^t \left[\sigma(t,s,X_s^{\varepsilon,n})\right] \mathbf{D}\Phi(h)(s) \ (1+\dot{h}_s) \ ds$$
$$+ \beta \sup_{0\leq s\leq t} |\mathbf{D}\Phi(h)(s)|. \tag{3.2}$$

Thus,

$$\| \mathbf{D}\Phi(h) \|_\alpha \leq L \frac{1}{1-\beta} \int_0^t \left[\ \sigma(t,s,X_s^{\varepsilon,n})\right] \ \| \mathbf{D}\Phi(h) \|_\alpha \ \ (1+|\dot{h}_s|) \ ds. \tag{3.3}$$

Set,

$$\phi_s = \left|\sigma(t,s,X_s^{\varepsilon,n})\right|(|\dot{h}_s|) \ 1_{[0,t]}(s) \in L^1([0,1]). \tag{3.4}$$

It follows, by **the Cauchy-Schwarz inequality** that

$$\int_0^1 \phi_s \, ds \leq \left(\int_0^1 \left|\sigma(t,s,X_s^{\varepsilon,n})\right|^2 ds \right)^{\frac{1}{2}} \left(\int_0^1 (1+|\dot{h}_s|)^2 ds \right)^{\frac{1}{2}}$$
$$= C(H)(\| h \|_{\mathcal{H}}) < \infty.$$

We can conclude from **Gronwall's Lemma** that $\| \Phi^{(1)}(h) - \Phi^{(2)}(h) \|_\alpha = 0$, hence $\Phi^{(1)}(h) = \Phi^{(2)}(h)$.

To study the existence, we use the following approximating. Let's define

$$\Phi_t^n = x_0 + \int_0^t b(t,s,\Phi_t^{n-1}) \, ds + \int_0^t \sigma(t,s,\Phi_t^{n-1}) \, \dot{h}(s) \, ds + \beta \sup_{0 \leq s \leq t} \Phi_s^{n-1}.$$

We note by $\Phi_n(t) = \| \Phi_t^{n+1} - \Phi_t^n \|_\alpha$, then we have:

$$\Phi_0(t) \leq \frac{L}{1-\beta} \int_0^t \left| \sigma(t,s,X_s^{\varepsilon,n}) \right| (1+\dot{h}_s) \, ds < \infty$$
$$\Phi_n(t) \leq \frac{L}{1-\beta} \int_0^t \left| \sigma(t,s,X_s^{\varepsilon,n}) \right| \Phi_{n-1}(s) \, (1+\dot{h}_s) \, ds < \infty.$$

Then, by iteration

$$\Phi_n(t) \leq \left(\frac{L}{1-\beta} \right)^n D \, K_{n-1}^{(1)}(t).$$

We can then deduce that $\Phi_n(t) \longrightarrow \Phi(t)$ converges uniformly in t and Φ is the solution of the equation (3.1) □

3.4. **The results on the LDP solution of** (1.5)**.** In this paragraph, we will prove the **PGD** for the solution of the reflected diffusion equation (1.5).

For $y \geq 0$ and $f \in C^\alpha([0,1],\mathbb{R})$ (the space of continuous functions from $[0,1]$ to $\mathbb{R}$) with $f(0) = y$, let's define two functionals Γ and K as follows:

$$\Gamma : C^\alpha([0,1],\mathbb{R}) \longrightarrow C^\alpha([0,1],\mathbb{R}_+)$$
$$f \longrightarrow \Gamma f = f + \tilde{f},$$

and

$$K : C^\alpha([0,1],\mathbb{R}) \longrightarrow C^\alpha([0,1],\mathbb{R}_+)$$
$$f \longrightarrow K f = \tilde{f},$$

with $\tilde{f} = -\inf_{s \leq t}(f(s) \wedge 0)$, $t \in [0,1]$.

Let us note that the solution T^ε of equation (1.5) is given by:

$$T_t^\varepsilon = (\Gamma Z^\varepsilon)(t) \text{ and } L_t^\varepsilon = (KZ^\varepsilon)(t), \quad t \in [0,1], \tag{3.5}$$

where Z^ε is a solution of the following differential equation:

$$Z^\varepsilon(t) = y + \sqrt{\varepsilon}\int_0^t \varsigma(t,s,\Gamma Z^\varepsilon)(s))\, dB_s + \beta \sup_{0\le s\le t}(\Gamma Z_s^\varepsilon), \quad t \in [0,1]. \tag{3.6}$$

Indeed,

$$\begin{aligned}
\Gamma Z^\varepsilon(t) &= Z^\varepsilon(t) + KZ^\varepsilon(t) \\
&= y + \sqrt{\varepsilon}\int_0^t \varsigma(t,s,\Gamma Z^\varepsilon)(s))\, dB_s + \beta \sup_{0\le s\le t}(\Gamma Z_s^\varepsilon) + KZ^\varepsilon(t) \\
&= y + \sqrt{\varepsilon}\int_0^t \varsigma(t,s,\Gamma Z^\varepsilon)(s))\, dB_s + \beta \sup_{0\le s\le t}(\Gamma Z_s^\varepsilon) + L_t^\varepsilon \\
&= T_t^\varepsilon.
\end{aligned}$$

For $h \in \mathcal{H}$, let $\tilde{\Phi}^y(h)$ be the unique solution of the following equation:

$$\begin{aligned}
\tilde{\Phi}^y(h)(t) = y + \int_0^t \varsigma(t,s,\tilde{\Phi}^y(h))(s))\, \dot{h}s\, ds \\
+ \beta \sup_{0\le s\le t}(\tilde{\Phi}^y(h)(s)) + \eta(t),
\end{aligned} \tag{3.7}$$

$t \in [0,1]$, where $\tilde{\Phi}^y(h)$ is a continuous, non-negative function, and η is a continuous increasing function satisfying $\eta(t) = \displaystyle\int_0^t \chi_{\tilde{\Phi}^y(h)=0}\, d\eta(s)$. Similarly to the formula (3.5), $\tilde{\Phi}^y(h)$ can also be written as:

$$\tilde{\Phi}^y(h)(t) = (\Gamma V(h))(t) \text{ and } \eta(t) = (KV(h))(t), \quad t \in [0,1], \tag{3.8}$$

where $V(h)$ is a solution of the following deterministic equation:

$$V(h)(t) = y + \int_0^t \varsigma(t,s,\Gamma V(h)(s))\, \dot{h}s\, ds + \beta \sup_{0\le s\le t}(\Gamma V(h)(s)), \quad t \in [0,1]. \tag{3.9}$$

Let ν_ε^1 be the law of Z^ε on $C^\alpha([0,1],\mathbb{R}+)$ equipped with the Hölder norm $\| \,.\, \| \alpha$. We have the following main results:

Theorem 3.3. *For every $\alpha \in]0,\frac{1}{2}[,\ \beta \in]0,\frac{1}{2}[$, by noting $\{\nu_\varepsilon^1, \varepsilon > 0\}$ the family of probability measures associated to Z^ε considered as a random variable in $C^\alpha([0,1],\mathbb{R}_+)$ equipped with the Höderian norm $\|.\|_\alpha$, then ν_ε^1 satisfy the* ***LDP*** *with the rate function*

$\tilde{I}_y(.)$ *defined by:*

$$\tilde{I}_y(g) = \inf_{\{h\in\mathcal{H}; g=\tilde{\Phi}^x(h)\}} \lambda(h), \tag{3.10}$$

where λ *is defined in* (2.2). *We take the convention* $\inf\emptyset = \infty$.

Theorem 3.4. *For every* $\alpha \in]0, \frac{1}{2}[,\ \beta \in]0, \frac{1}{2}[.$ *by noting* $\{\nu_\varepsilon^2, \varepsilon > 0\}$ *the family of probability measures related to* T^ε *considered as a random variable in* $C^\alpha([0,1], \mathbb{R}_+)$ *equipped with the Höderian norm* $\|.\|_\alpha$ *, then* ν_ε^2 *satisfy the* ***LDP*** *with the rate function* $\bar{I}_y(.)$ *defined by:*

$$\bar{I}_y(g) = \inf_{\{\bar{g}=\Gamma g\}} \tilde{I}_y(g)$$

where $\tilde{I}$ *is defined in the theorem* (3.4). *We take the convention* $\inf\emptyset = \infty$.

3.5. Existence and uniqueness of the solution of the equation (1.5).

Theorem 3.5. *For every* $\alpha \in]0, \frac{1}{2}[$*, for all* $y > 0$*, the equation* (1.5) *admits a unique solution.*

Proof. Let's consider the sequence defined by:

$$T_t^{\varepsilon,(n+1)} = y + \sqrt{\varepsilon}\int_0^t \varsigma(t, s, T_s^{\varepsilon,(n)})\, dB_s + \beta \sup_{0\le s\le t} T_s^{\varepsilon,(n)} + L_t^{\varepsilon,(n)}. \tag{3.11}$$

Let $T_t^{\varepsilon,(0)}$ be the unique solution of

$$T_t^{\varepsilon,(0)} = y + \sqrt{\varepsilon}\int_0^t \varsigma(t, s, T_s^{\varepsilon,(0)})\, dB_s + \beta \sup_{0\le s\le t} T_s^{\varepsilon,(0)} + L_t^{\varepsilon,(0)}.$$

Let's construct the first stopping time $\tau_1 = \inf_{\{t\ge 0\}}\{T_t^{\varepsilon,(0)} = 0\}$ and the process $B_t^{(1)} = B_t - B_0$. By invariance, $B_t^{(1)}$ is a Brownian motion. The reflection principle ensures the existence and uniqueness of the solution to the SDE $Z_t^{\varepsilon,(1)} = \int_0^t \varsigma(t, s, Z_t^{\varepsilon,(1)})\, dB_s^{(1)} + L_t^{\varepsilon,(1)}$, with $L_0^{\varepsilon,(1)} = 0$ et $L_t^{\varepsilon,(1)} = \int_0^t 1_{\{Z_s^{\varepsilon,(1)}=0\}} dL_s^{\varepsilon,(1)}$.

Let's now construct the second stopping time $\tau_2 = \inf_{\{t>\tau_1\}}\{Z_{t-\tau_1}^{\varepsilon,(1)} = \sup_{0\le s\le \tau_1} T_s^\varepsilon\}$ and the process $B_t^{(2)} = B_{t+\tau_1} - B_{\tau_1}$. By invariance, $B_t^{(2)}$ is a Brownian motion. Similarly, the reflection principle ensures the existence and uniqueness of the solution of the SDE $Z_t^{\varepsilon,(2)} = (1-\beta)Z_{\tau_2-\tau_1}^{\varepsilon,(1)} + \int_0^t \varsigma(t, s, Z_t^{\varepsilon,(2)})\, dB_s^{(2)} + \beta \sup_{0\le s\le t} Z_s^{\varepsilon,(2)}$.

By proceeding by recurrence, let us construct the following approximating sequences:

$$(I):\begin{cases} B_t^{(k)} = B_{t+\tau_k} - B_{\tau_k} \\ B_0 = 0 \end{cases}.$$

The sequence $(B_k)_{k\in\mathbb{N}}$ are brownians motions **from the Markov property**.

$$(II):\begin{cases} \tau_{2n} &= \inf_{\{t>\tau_{2n-1}\}}\{Z^{\varepsilon,(2n-1)}_{t-\tau_{2n-1}} = \sup_{0\le s\le t-\tau_{2n-1}} T^{\varepsilon}_s\} \\ \tau_{2n+1} &= \inf_{\{t>\tau_{2n}\}}\{Z^{\varepsilon,(2n)}_{t-\tau_{2n}} = 0\} \end{cases}$$

$$(III):\begin{cases} Z_t^{\varepsilon,(2n-1)} &= \sqrt{\varepsilon}\int_0^t \varsigma(t,s,Z_s^{\varepsilon,(2n-1)})\, dB_s^{(2n-1)} + L_t^{\varepsilon,(2n-1)} \\ Z_t^{\varepsilon,(2n)} &= (1-\beta)T^{\varepsilon}_{\tau_{2n}} + \sqrt{\varepsilon}\int_0^t \varsigma(t,s,Z_s^{\varepsilon,(2n)})\, dB_s^{(2n)} + \beta\sup_{0\le s\le t} T_s^{\varepsilon,(2n)} \end{cases}$$

$$(IV):\begin{cases} L_0^{\varepsilon,(2n-1)} &= 0 \\ L_t^{\varepsilon,(2n-1)} &= \int_0^t 1_{\{Z_s^{\varepsilon,(2n-1)}=0\}} dL_s^{\varepsilon,(2n-1)} \end{cases}$$

$$(V):\begin{cases} L_t^{\varepsilon} &= L^{\varepsilon}_{\tau_{2n-1}} + L^{\varepsilon,(2n-1)}_{\tau_{2n-1}}, \text{if } \tau_{2n-1}\le t\le \tau_{2n} \\ L_t^{\varepsilon} &= L^{\varepsilon}_{\tau_{2n}}, \text{if } \tau_{2n}\le t\le\tau_{2n+1} \end{cases}$$

$$(VI):\begin{cases} T_t^{\varepsilon} &= Z^{\varepsilon,(2n-1)}_{t-\tau_{2n-1}}, \text{if } \tau_{2n-1}\le t\le\tau_{2n} \\ T_t^{\varepsilon} &= T^{\varepsilon,(2n)}_{t-\tau_{2n}}, \text{if } \tau_{2n}\le t\le\tau_{2n+1} \end{cases}$$

Let us show that (Y_t, L_t) satisfying the equation (1.5).

Case 1: if $\tau_{2n}\le t\le\tau_{2n+1}$

$$\begin{aligned} T_t^{\varepsilon} &= T^{\varepsilon,(2n)}_{t-\tau_{2n}} = (1-\beta)T^{\varepsilon}_{\tau_{2n}} + \sqrt{\varepsilon}\int_0^{t-\tau_{(2n)}} \varsigma(t-(\tau_{2n}),s,Z_s^{\varepsilon,(2n)})\, dB_s^{(2n)} \\ &\quad +\beta\sup\nolimits_{0\le s\le t-\tau_{2n}} T_s^{\varepsilon,(2n)} \\ &= (1-\beta)T^{\varepsilon}_{\tau_{2n}} + \sqrt{\varepsilon}\int_0^{t-\tau_{2n}} \varsigma(t-\tau_{2n},s,Z_s^{\varepsilon,(2n)})\,(dB_{s+\tau_{2n}} - dB_{\tau_{2n}}) \\ &\quad +\beta\sup\nolimits_{0\le s\le t-\tau_{2n}} T_s^{\varepsilon,(2n)} \\ &= y + \sqrt{\varepsilon}\int_0^{\tau_{2n}} \varsigma(\tau_{2n},s,T_s^{\varepsilon})\, dB_s + \beta\sup_{0\le s\le\tau_{2n}} T_s^{\varepsilon} + L^{\varepsilon}_{\tau_{2n}} \\ &\quad +\sqrt{\varepsilon}\int_0^{t-\tau_{2n}} \varsigma(t-\tau_{2n},s,Z_s^{\varepsilon,(2n)})\,(dB_{s+\tau_{2n}} - dB_{\tau_{2n}}) + \beta\sup_{0\le s\le t-\tau_{2n}} T_s^{\varepsilon,(2n)} - \beta T^{\varepsilon}_{\tau_{2n}}. \end{aligned}$$

By using a new variable $u = s + \tau_{2n}$ we have

$$\int_0^{t-\tau_{2n}} \varsigma(t-\tau_{2n}, s, Z_s^{\varepsilon,(2n)})\, dB_{s+\tau_{2n}} = \int_{\tau_{2n}}^t \varsigma(t-\tau_{2n}, u, Z_{u-\tau_{2n}}^{\varepsilon,(2n)})\, dB_u$$

Noticing that,

$$T_{\tau_{2n}}^\varepsilon = \sup_{0\le s\le \tau_{2n}} T_s^\varepsilon$$

As a result, it follows that

$$T_t^\varepsilon = y + \sqrt{\varepsilon}\int_0^t \varsigma(t,s,T_s^\varepsilon)dB_s + \beta \sup_{0\le s\le t} T_s + L_t^\varepsilon.$$

<u>**Case 2: if** $\tau_{2n+1} \le t \le \tau_{2n+2}$</u>

$$\begin{aligned} T_t^\varepsilon &= T_{\tau_{2n+1}}^\varepsilon + \sqrt{\varepsilon}\int_0^{t-\tau_{2n+1}} \varsigma(t-(\tau_{2n+1}), s, Z_s^{\varepsilon,(2n+1)})\, dB_s^{(2n+1)} + L_{\tau_{2n+1}}^\varepsilon \\ &= y + \sqrt{\varepsilon}\int_0^{\tau_{2n+1}} \varsigma(t,s,T_s^\varepsilon)\, dB_s + \sqrt{\varepsilon}\int_{\tau_{2n+1}}^t \varsigma(t,s,T_s^\varepsilon)\, dB_s \\ &\quad +\beta \sup\nolimits_{0\le s\le \tau_{2n+1}} T_s^\varepsilon + L_t^\varepsilon \\ &= y + \sqrt{\varepsilon}\int_0^t \varsigma(t,s,T_s^\varepsilon)dB_s + \beta \sup_{0\le s\le t} T_s^\varepsilon + L_t^\varepsilon. \end{aligned}$$

We remark that the sequence $(\tau_n)_{n\ge 0}$ is increasing and its limit $\infty = \tau = \lim_{n\to\infty} \tau_n$ □

We will now prove the Theorem 3.1, so, **it is a direct consequence of the following Theorem 4.1**

4. Proof of main results

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4.1. Main results on the LDP solution of (1.4).

Theorem 4.1. *For every $\alpha \in]0, \frac{1}{2}[,\ \beta \in]0,1[$ and $h \in \mathcal{H}$. For any $R, \delta > 0$ there exist $\rho > 0$ such that*

$$\mathbb{P}\Big(\| X^\varepsilon - \Phi^x(h) \|_\alpha > \rho, \| \sqrt{\varepsilon}B - h \|_\infty < \delta \Big) \le \exp(-\frac{R}{\varepsilon}).$$

4.1.1. *Proof of the Theorem* 4.1.

Proof. By the formulas (1.4) and (3.1), we can then write

$$\begin{aligned} X_t^\varepsilon - \Phi^x(h)(t) &= \int_0^t \sigma(t,s,X_s^\varepsilon)\Big(\sqrt{\varepsilon}\,dB_s - \dot h(s)\Big)\,ds \\ &+ \int_0^t \sigma(t,s,X_s^\varepsilon) - \sigma\Big(t,s,\Phi^x(h)(s)\Big)\,\dot h(s)\,ds \\ &+ \int_0^t b(t,s,X_s^\varepsilon) - b\Big(t,s,\Phi^x(h)(s)\Big)\,ds \\ &+ \beta\Big(\sup_{0\le s\le t} X_s^\varepsilon - \sup_{0\le s\le t}\Phi^x(h)(s)\Big). \end{aligned}$$

Therefore, **by the Reflection principle** , it follows that:

$$\begin{aligned} |X_t^\varepsilon - \Phi^x(h)(t)| &\le \int_0^t |\sigma(t,s,X_s^\varepsilon)\,(\sqrt{\varepsilon}\,dB_s - \dot h(s)\,ds)| \\ &+ L\int_0^t \Big(|X_s^\varepsilon - \Phi^x(h)(s)|\,(1+|\dot h(s)|)\Big)\,ds \\ &+ \beta\sup_{0\le s\le t}|X_s^\varepsilon - \Phi^x(h)(s)|, \end{aligned}$$

where $L>0$ is the Lipschitz coefficient of b and σ, and noting that

$$\Big|\sup_{0\le s\le t} u(s) - \sup_{0\le s\le t} v(s)\Big| \le \sup_{0\le s\le t}\Big|u(s)-v(s)\Big| \tag{4.1}$$

for two continuous functions u and v on $\mathbb{R}_+$. Thus, it follows that, for $t\in[0,1]$,

$$\begin{aligned} \sup_{0\le u\le t}|X_u^\varepsilon - \Phi^x(h)(u)| &\le \tfrac{1}{1-\beta}\sup_{0\le u\le t}\int_0^u |\sigma(t,s,X_s^\varepsilon)\,(\sqrt{\varepsilon}\,dB_s - \dot h(s)ds)| \\ &+ \tfrac{L}{1-\beta}\sup_{0\le u\le t}\int_0^u |X_s^\varepsilon - \Phi^x(h)(s)|\,(1+|\dot h(s)|)\,ds. \end{aligned}$$

By **the Gronwall Lemma and the Cauchy-Schwarz inequality**, we have

$$\begin{aligned} \| X_t^\varepsilon - \Phi^x(h)(t)\|_\infty &\le \tfrac{1}{1-\beta}\sup_{0\le t\le 1}\int_0^t |\sigma(t,s,X_s^\varepsilon)\,(\sqrt{\varepsilon}\,dB_s - \dot h(s))|\,ds \\ &\times \exp\Big(\int_0^t \frac{L}{1-\beta}\,(1+|\dot h(s)|\,ds\Big) \\ &\le C_1(h)\sup_{0\le t\le 1}\int_0^t |\sigma(t,s,X_s^\varepsilon)\Big(\sqrt{\varepsilon}\,dB_s - \dot h(s)\Big)|\,ds \\ &\le C_1(h)\ \|\int_0^t \sigma(t,s,X_s^\varepsilon)\Big(\sqrt{\varepsilon}\,dB_s - \dot h(s)\Big)\,ds\,\|_\infty, \end{aligned}$$

where $C_1(h) = \frac{1}{1-\beta} \exp\Big(C(H)\, L\, (1+ \parallel h \parallel_{\mathcal{H}})/(1-\beta)\Big)$ with $\parallel h \parallel_{\mathcal{H}}= \Big(\int_0^1 |\dot{h}_s|^2\, ds\Big)^{\frac{1}{2}}$ for $h \in \mathcal{H}$, since $t \leq 1$.

Let us first examine the case where $h = 0$.

By using the fact that for any $f \in C([0,1],\mathbb{R})$, $\parallel f \parallel_\infty \leq \parallel f \parallel_\alpha$ and $h = 0$, it can be deduced that

$$\parallel X_t^\varepsilon - \Phi^x(0) \parallel_\infty \leq C_1(0) \parallel \sqrt{\varepsilon} \int_0^t \sigma(t,s,X_s^\varepsilon)\; dB_s \parallel_\alpha . \tag{4.2}$$

For any $t \in [0,1]$, for any continuous function $f : [0,1] \to \mathbb{R}$, let us denote by

$$\parallel f \parallel_{\alpha,t} = \sup_{0\leq u\neq v\leq t} \frac{|f(u)-f(v)|}{|v-u|^\alpha} < \infty. \tag{4.3}$$

Set

$$\mathbf{D}_{\Phi^x(0)}^{X^\varepsilon}(u) = X_u^\varepsilon - \Phi^x(0)(u).$$

By the Reflection principle, we will get:

$$\begin{aligned} \frac{\left|\mathbf{D}_{\Phi^x(0)}^{X^\varepsilon}(t) - \mathbf{D}_{\Phi^x(0)}^{X^\varepsilon}(s)\right|}{|t-s|^\alpha} \quad &\leq \quad \frac{1}{|t-s|^\alpha}\Bigg(\Bigg|\frac{1}{1-\beta}\int_s^t \sqrt{\varepsilon}\,\sigma(t,v,X_v^\varepsilon)\, dB_v \\ &+ \quad \frac{\beta}{1-\beta}\sup_{s\leq u\leq t}\Big(\sqrt{\varepsilon}\int_s^u \sigma(u,v,X_v^\varepsilon)\, dB_v\Big) \\ &+ \quad \frac{\beta}{1-\beta}\sup_{s\leq u\leq t}\int_s^u b(u,v,X_v^\varepsilon) - b\Big(u,v,\Phi^x(0)(v)\Big)\; dv\Bigg|\Bigg). \end{aligned}$$

Therefore, we obtain:

$$\begin{aligned} \parallel (X^\varepsilon - \Phi^x(0)) \parallel_{\alpha,t} \;\; &\leq \frac{1}{1-\beta} \;\; \parallel \sqrt{\varepsilon}\int_0^t \sigma(t,v,X_v^\varepsilon)\, dB_v \parallel_{\alpha,t} \\ &+ \frac{\beta\, L}{1-\beta} \;\; \parallel X^\varepsilon - \Phi^x(0) \parallel_\infty \\ &+ \frac{\beta\, L}{1-\beta} \;\; \int_0^t \parallel X^\varepsilon - \Phi^x(0) \parallel_{\alpha,t}\;\; dv. \end{aligned}$$

By using the formula (4.2), we have:

$$\begin{aligned} \parallel (X^\varepsilon - \Phi^x(0)) \parallel_\alpha \leq \left(\frac{1}{1-\beta} + \frac{\beta\, C_1(0)\, L}{1-\beta}\right) \parallel \sqrt{\varepsilon}\int_0^t \sigma(t,v,X_v^\varepsilon)\, dB_v \parallel_\alpha \\ + \frac{\beta\, L}{1-\beta}\int_0^t \parallel X^\varepsilon - \Phi^x(0) \parallel_\alpha \;\; dv. \end{aligned}$$

By Gronwall's Lemma , we have:

$$\| (X^{\varepsilon}-\Phi^{x}(0)) \|_{\alpha}\leq \left(\frac{1}{1-\beta}+\frac{\beta\, C_1(0)\, L}{1-\beta}\right) \| \times\sqrt{\varepsilon}\int_0^t \sigma(t,v,X_v^{\varepsilon})\, dB_v \|_{\alpha}\, \Theta(0), \tag{4.4}$$

where $\Theta(0) = \exp\left(\frac{\beta\, L}{1-\beta}\right)$.

Now, let us examine the case where $h \neq 0$.

In the general case, we proceed to a translation on the Wiener space by the Cameron-Martin formula to obtain the case $h = 0$. By an argument similar to that of the proof of the formula (4.2), we obtain

$$\| X_t^{\varepsilon} - \Phi^{x}(h)(t) \|_{\infty}\leq C_1(h) \| \int_0^t \sigma(t,s,X_s^{\varepsilon})\left(\sqrt{\varepsilon}\, dB_s - \dot{h}(s)\right)ds \|_{\alpha} . \tag{4.5}$$

Set

$$\mathbf{D}^{X^{\varepsilon}}_{\Phi^{x}(h)}(u) = X_u^{\varepsilon} - \Phi^{x}(h)(u).$$

Therefore,

$$\begin{aligned}
\frac{\left|\mathbf{D}^{X^{\varepsilon}}_{\Phi^{x}(h)}(t)-\mathbf{D}^{X^{\varepsilon}}_{\Phi^{x}(h)}(s)\right|}{|t-s|^{\alpha}} &\leq \frac{1}{|t-s|^{\alpha}}\left(\left|\frac{1}{1-\beta}\int_s^t \sigma(t,v,X_v^{\varepsilon})\left(\sqrt{\varepsilon}\, dB_v - \dot{h}(v)\right) dv\right.\right.\\
&+ \frac{\beta}{1-\beta}\sup_{s\leq u\leq t}\left(\int_s^u \sqrt{\varepsilon}\,\sigma(u,v,X_v^{\varepsilon})\, dB_v\right)\\
&+ \frac{\beta}{1-\beta}\sup_{s\leq u\leq t}\left(\int_s^u b(u,v,X_v^{\varepsilon}) - b\Big(u,v,\Phi^{x}(h)(v)\Big)\, dv\right.\\
&+ \left.\left.\frac{\beta}{1-\beta}\sup_{s\leq u\leq t}\int_s^u \left[\sigma(u,v,X_v^{\varepsilon})(v) - \sigma\Big(u,v,\Phi^{x}(h)\Big)(v)\right]\dot{h}(v)\, dv\right|\right).
\end{aligned}$$

By applying the formula (4.5), and **by the Reflection principle**, we obtain

$$\begin{aligned}
\| (X^{\varepsilon} - \Phi^{x}(h)) \|_{\alpha,t} &\leq \frac{1}{1-\beta} \| \int_0^t \sigma(t,s,X_s^{\varepsilon})\left[\sqrt{\varepsilon}\, dB_s - \dot{h}(s)\right]ds \|_{\alpha,t}\\
&+ \frac{\beta\, L}{1-\beta}C_1(h) \| \int_0^t \sigma(t,s,X_s^{\varepsilon})\left[\sqrt{\varepsilon}\, dB_s - \dot{h}(s)\right] ds \|_{\alpha}\\
&+ \frac{\beta\, L}{1-\beta}\int_0^t (|\dot{h}_s|) \| X^{\varepsilon} - \Phi^{x}(h) \|_{\alpha,t}\ \ ds.
\end{aligned}$$

By Gronwall's Lemma, it follows that

$$\| (X^{\varepsilon} - \Phi^{x}(h)) \|_{\alpha}\leq \left(\frac{1}{1-\beta} + \frac{\beta\, L\, C_1(h)}{(1-\beta)}\right) \tag{4.6}$$

$$\times \parallel \int_0^t \sigma(t,s,X_s^\varepsilon)\,(\sqrt{\varepsilon}\,dB_s - \dot{h}_s\,ds) \parallel_\alpha \ \Theta(h),$$

where $\Theta(h) = \exp\left(\frac{\beta L(1+\|h\|_{\mathcal{H}})}{1-\beta}\right)$. □

On the basis of the relation (4.6) which we obtained previously, we have to show the following Theorem 4.2 to complete the proof of the Theorem 4.1.

Theorem 4.2. *For every* $\alpha \in]0, \frac{1}{2}[,\ \beta \in]0,1[$. *For any* $R, \delta, \tilde{a} > 0$, *there exist* $\rho > 0$ *and* $\varepsilon_0 > 0$ *such that, for any* $h \in C^\alpha([0,1],\mathbb{R})$ *satisfying* $\lambda(h) \le \tilde{a}$ *and* $\varepsilon \le \varepsilon_0$

$$\mathbb{P}\bigg(\parallel \int_0^t \sigma(t,s,X_s^\varepsilon)\left(\sqrt{\varepsilon}\,dB_s - \dot{h}(s)\,ds\right) \parallel_\alpha > \rho, \parallel \sqrt{\varepsilon}\,B - h \parallel_\infty < \delta \bigg)$$
$$\le \exp\left(-\frac{R}{\varepsilon}\right).$$

Proof. For $\varepsilon > 0$, let us define a probability measure $\mathbb{P}^\varepsilon$ on Ω by

$$\text{(4.7)} \qquad d\mathbb{P}^\varepsilon = M_\varepsilon\,d\mathbb{P} = \exp\left(\frac{1}{\sqrt{\varepsilon}}\int_0^1 \dot{h}_s\,dB_s - \frac{1}{2\varepsilon}\int_0^1 |\dot{h}_s|^2\,ds\right)d\mathbb{P}.$$

Now, **Girsanov Theorem** ensures that $\left\{B_t^\varepsilon = B_t - \frac{1}{\sqrt{\varepsilon}}\,\dot{h}_t,\ t \in [0,1]\right\}$ is a Wiener process with respect to the probability P^ε. Let $\{U_t^\varepsilon, 0 \le t \le 1\}$ the solution to the following deterministic differential equation:

$$\text{(4.8)} \qquad \begin{aligned} U^\varepsilon(t) = x_0 + \int_0^t b(t,s,U^\varepsilon(s))\,ds + \int_0^t \sigma(t,s,U^\varepsilon(s))\,\dot{h}(s)\,ds \\ + \beta \sup_{0\le s\le t} U^\varepsilon(s). \end{aligned}$$

To simplify the notation, let's define for all ρ, α, $\varepsilon > 0$,

$$A^\varepsilon = \bigg\{ \parallel \int_0^t \sigma(t,s,X_s^\varepsilon)\left(\sqrt{\varepsilon}\,dB_s - \dot{h}(s)\,ds\right) \parallel_\alpha > \rho, \parallel \sqrt{\varepsilon}\,B - h \parallel_\infty < \delta \bigg\}.$$

Using the definition of B_t^ε, we have:

$$A^\varepsilon = \bigg\{ \parallel \int_0^t \sigma(t,s,X_s^\varepsilon)\,\sqrt{\varepsilon}\,dB_s^\varepsilon\,ds \parallel_\alpha > \rho, \sqrt{\varepsilon} \parallel\ B^\varepsilon \parallel_\infty < \delta \bigg\}.$$

By the **Cauchy-Schwarz** inequality,

$$\mathbb{P}(A^\varepsilon) = \int_\Omega M_\varepsilon^{-1}\chi_{\{A^\varepsilon(w)\}}\mathbb{P}^\varepsilon(dB) \le \left(\int_\Omega M_\varepsilon^{-2}(w)\,\mathbb{P}^\varepsilon(dB)\right)^{\frac{1}{2}}\left(\mathbb{P}^\varepsilon(A^\varepsilon)\right)^{\frac{1}{2}}$$

An application of formula (4.7) yields

$$\begin{aligned}\int_\Omega M_\varepsilon^{-2}(w)\mathbb{P}^\varepsilon(dB) &= \mathrm{E}_{\mathbb{P}^\varepsilon}\Big[\exp\Big(-\tfrac{2}{\sqrt{\varepsilon}}\int_0^1 \dot h_s\, dB_s + \frac{1}{\varepsilon}\int_0^1 |\dot h_s|^2\, ds\Big)\Big]\\ &= \mathrm{E}_{\mathbb{P}^\varepsilon}\Big[\exp\Big(-\tfrac{2}{\sqrt{\varepsilon}}\int_0^1 \dot h_s\, (dB^\varepsilon + \frac{1}{\sqrt{\varepsilon}}\dot h_s) + \frac{1}{\varepsilon}\int_0^1 |\dot h_s|^2\, ds\Big)\Big]\\ &\le \mathrm{E}_{\mathbb{P}^\varepsilon}\Big[\exp\Big(-\tfrac{2}{\sqrt{\varepsilon}}\int_0^1 \dot h_s\, dB_s - \frac{2}{\varepsilon}\int_0^1 |\dot h_s|^2\, ds\Big)\Big]\\ &\quad \times\exp\Big(\tfrac{1}{\varepsilon}\int_0^1 |\dot h_s|^2\, ds\Big)\end{aligned}$$

By using the **Ito formula**, the process $L_t = \exp\Big(-\tfrac{2}{\sqrt{\varepsilon}}\int_0^1 \dot h_s\, dB_s - \frac{2}{\varepsilon}\int_0^1 |\dot h_s|^2\, ds\Big)$ is a finite martingale. Therefore, we have:

$$\int_\Omega M_\varepsilon^{-2}(w)\, \mathbb{P}^\varepsilon(dB) \le \exp\Big(\frac{1}{\varepsilon}\parallel h \parallel_H^2\Big).$$

Therefore, if $\lambda(h) \le a$, then

$$\mathbb{P}(A^\varepsilon) \le \exp\Big(\tfrac{a}{\varepsilon}\Big)\Big(\mathbb{P}^\varepsilon(A^\varepsilon)\Big)^{\frac{1}{2}}. \tag{4.9}$$

So, we are left to estimate the quantity $\mathbb{P}^\varepsilon(A^\varepsilon)$ to complete the proof of the theorem. 4.2. Notice that:

$$\begin{aligned}\mathbb{P}^\varepsilon(A^\varepsilon) &= \mathbb{P}^\varepsilon\Big(\parallel \int_0^t \sigma(t,s,X_s^\varepsilon)\Big(\sqrt{\varepsilon}\, dB_s - \dot h(s)ds\Big)\parallel_\alpha > \rho, \parallel \sqrt{\varepsilon}\, B_s - h\parallel_\infty < \delta\Big)\\ &= \mathbb{P}^\varepsilon\Big(\parallel \sqrt{\varepsilon}\int_0^t \sigma(t,s,X_s^\varepsilon)\, dB_s^\varepsilon \parallel_\alpha > \rho, \parallel \sqrt{\varepsilon}\, B^\varepsilon \parallel_\infty < \delta\Big)\\ &= \mathbb{P}\Big(\parallel \sqrt{\varepsilon}\int_0^t \sigma(t,s,U_s^\varepsilon)\, dB_s \parallel_\alpha > \rho, \parallel \sqrt{\varepsilon}\, B \parallel_\infty < \delta\Big).\end{aligned}$$

□

We will prove the following Theorem 4.3 to complete the proof of Theorem 4.2.

Theorem 4.3. *For every $\alpha \in]0,\frac{1}{2}[$, $\beta \in]0,1[$. For any $R,\delta,\tilde a > 0$, there exist $\rho > 0$ and $\varepsilon_0 > 0$ such that, for all $h \in C^\alpha([0,\tau],\mathbb{R})$ satisfying $\lambda(h) \le \tilde a$ and $\varepsilon \le \varepsilon_0$,*

$$\mathbb{P}\Big(\parallel \sqrt{\varepsilon} \int_0^t \sigma(t,s,U_s^\varepsilon) \ dB_s \parallel_\alpha > \rho, \parallel \sqrt{\varepsilon}\ B \parallel_\infty < \delta \Big) \leq \exp\left(-\frac{R}{\varepsilon}\right).$$

Proof. For all $n \in \mathbb{N}^*$, we consider the sequence of approximating of the process U^ε defined by

$$U_t^{\varepsilon,n} = U^\varepsilon_{\frac{j}{n}}, \text{if } t \in \Big[\frac{j}{n}, \frac{j+1}{n}\Big[\text{ for any } j = 0,1,2,...,n-1.$$

For $\gamma > 0$ and for each $n \in \mathbb{N}$, we have

$$A^\varepsilon = \Big\{ \parallel \sqrt{\varepsilon} \int_0^t \sigma(t,s,U_s^\varepsilon)\ dB_s \parallel_\alpha \geq \rho, \parallel \sqrt{\varepsilon} B \parallel_\infty \leq \delta \Big\} \subset A_1^\varepsilon \cup A_2^\varepsilon \cup A_3^\varepsilon,$$

where

$$\begin{cases} A_1^\varepsilon = \Big\{ \parallel \sqrt{\varepsilon} \int_0^t \Big(\sigma(t,s,U_s^\varepsilon) - \sigma(t,s,U_s^{\varepsilon,n})\Big) dB_s \parallel_\alpha \geq \dfrac{\rho}{2}, \parallel U^\varepsilon - U^{\varepsilon,n} \parallel_\infty \leq \gamma \Big\} \\ A_2^\varepsilon = \Big\{ \parallel U^\varepsilon - U^{\varepsilon,n} \parallel_\infty \geq \gamma \Big\} \\ A_3^\varepsilon = \Big\{ \parallel \sqrt{\varepsilon} \int_0^t \sigma(t,s,U_s^{\varepsilon,n})\ dB_s \parallel_\alpha \geq \dfrac{\rho}{2}, \parallel \sqrt{\varepsilon} B \parallel_\infty \leq \delta \Big\} \end{cases}.$$

In the subsets $\{\parallel U^\varepsilon - U^{\varepsilon,n} \parallel_\infty \leq \gamma\}$, we have the following estimates

$$\parallel \sqrt{\varepsilon}[\sigma(t,s,U_s^\varepsilon) - \sigma(t,s,U_s^{\varepsilon,n})] \parallel_\alpha \leq \sqrt{\varepsilon} L \gamma, \tag{4.10}$$

and by using **the Lemma** 2.2, it follows that

$$\mathbb{P}(A_1^\varepsilon) \leq K_1 \exp\Big\{ -\Big(\frac{\rho}{2\sqrt{\varepsilon} L \gamma} K_2 - 1\Big)^2 \Big\}.$$

To estimate $\mathbb{P}\Big(A_3^\varepsilon\Big)$, in the subsets $\{\parallel \sqrt{\varepsilon} B^\varepsilon \parallel_\infty \leq \delta\}$, if σ is bounded by M, we get

$$\begin{aligned} &\parallel \sqrt{\varepsilon} \int_0^t \sigma(t,s,U_s^{\varepsilon,n})\ dB_s \parallel_\alpha \\ &= \sqrt{\varepsilon} \parallel \sum_{j=0}^{n-1} \sigma(t_j,s,U_{t_j}^{\varepsilon,n}) \Big(B(t_{j+1} \wedge .) - B^\varepsilon(t_j \wedge .)\Big) \parallel_\alpha \\ &\leq M \sum_{j=0}^{n-1} \sqrt{\varepsilon} \parallel \Big(B(t_{j+1}) - B(t_j)\Big) \parallel_\infty \\ &\leq n\ M\ \delta, \end{aligned}$$

where $M > 0$ is a common bound of b and σ. Consequently, if $\delta \leq \frac{\rho}{2\ n\ M}$ then $\mathbb{P}(A_3^\varepsilon) = 0$. By using the formula (2.17) in Bo and Zhang [3], we have:

$$\mathbb{P}(A_2^\varepsilon) \leq n\ \exp\Big\{ -\frac{n\ \gamma^2(1-\beta)^2}{8L^2\varepsilon}\Big\}.$$

□

We will now prove the continuity of the function Φ^x solution of (3.1).

Proposition 4.1. *Let $\alpha \in]0, \frac{1}{2}[$ and $\beta \in]0, 1[$. For any $a \geq 0$, the map*

$$\Phi^x : C^\alpha([0,1], \mathbb{R}) \cap \Bigg(\Big\{h\ \in\ \mathcal{H} :\| \ h\ \|_{\mathcal{H}}^2 \leq \tilde{a}\Big\}\Bigg) \longrightarrow (C^\alpha([0,1], \mathbb{R}), \|\ .\ \|_\alpha)$$

is continuous.

Proof. Let $(h_n)_n$ is a convergent sequence in $C^\alpha([0,1], \mathbb{R}) \cap \Bigg(\Big\{h\ \in\ \mathcal{H} :\|\ h\ \|_{\mathcal{H}}^2 \leq a\Big\}\Bigg)$ and converges to h. By combining the formulas (1.4) and (3.1),

$$\begin{aligned}
\Phi^x(h_n)(t) - \Phi^x(h)(t) &= \int_0^t \sigma(t, s, \Phi^x(h)(s))\ \Big(\dot{h_n}(s) - \dot{h}(s)\Big)\ ds \\
&+ \int_0^t \sigma(t, s, \Phi^x(h_n)(s))\ - \sigma\Big(t, s, \Phi^x(h)(s)\Big)\ \dot{h_n}(s)\ ds \\
&+ \int_0^t b(t, s, \Phi^x(h_n)(s))\ - b\Big(t, s, \Phi^x(h)(s)\Big)\ ds \\
&+ \beta\Big(\sup_{0\leq s\leq t} \Phi^x(h_n)(s) - \sup_{0\leq s\leq t} \Phi^x(h)(s)\Big).
\end{aligned}$$

Consequently,

$$\begin{aligned}
|\Phi^x(h_n)(t) - \Phi^x(h)(t)| &\leq \int_0^t |\sigma(t, s, \Phi^x(h_n)(s))\ (\dot{h_n}(s) - \dot{h}(s)\ ds)| \\
&+ L \int_0^t \Big(|\Phi^x(h_n)(s)\ - \Phi^x(h)(s)|\ (|\dot{h_n}(s)|)\Big)\ ds \\
&+ \beta \sup_{0\leq s\leq t} |\Phi^x(h_n)(s) - \Phi^x(h)(s)|,
\end{aligned}$$

where $L > 0$ is the Lipschitz coefficient of σ, and noting that

(4.11) $$\Big|\sup_{0\leq s\leq t} u(s) - \sup_{0\leq s\leq t} v(s)\Big| \leq \sup_{0\leq s\leq t} \Big|u(s) - v(s)\Big|$$

for two continuous functions u and v on $\mathbb{R}_+$. Thus, **by the Reflection principle**, it follows that for $t \in [0,1]$,

$$\begin{aligned}\sup_{0\le u\le t} |\Phi^x(h_n)(u) - \Phi^x(h)(u)| &\le \tfrac{1}{1-\beta} \sup_{0\le u\le t} \int_0^u |\sigma(t,s,\Phi^x(h_n))\,(\dot{h_n}(s) - \dot{h}(s)ds)| \\ &+ \tfrac{L}{1-\beta} \sup_{0\le u\le t} \int_0^u |\Phi^x(h_n)(s) - \Phi^x(h)(s)|\,(|\dot{h}(s)|)\,ds.\end{aligned}$$

By **the Gronwall Lemma and the Cauchy-Schwarz inequality**, we have

$$\begin{aligned}\| \Phi^x(h_n) - \Phi^x(h) \|_\infty &\le \tfrac{1}{1-\beta} \sup_{0\le t\le 1} \int_0^t |\sigma(t,s,\Phi^x(h_n)(s))\,(\dot{h_n}(s) - \dot{h}(s)|\,ds \\ &\quad \times \exp\Big(\int_0^t \frac{L}{1-\beta}\,(|\dot{h}(s)|\Big)\,ds \\ &\le C_1(h) \sup_{0\le t\le 1} \int_0^t |\sigma(t,s,\Phi^x(h_n))\,\Big(\dot{h_n}(s) - \dot{h}(s)\Big)|\,ds \\ &\le C_1(h)\ \Big\| \int_0^t \sigma(t,s,\Phi^x(h_n)(s))\,\Big(\dot{h_n}(s) - \dot{h}(s)\Big)\,ds \Big\|_\infty,\end{aligned}$$

where $C_1(h) = \frac{1}{1-\beta} \exp\Big(C(H)\ L\ (\| h \|_{\mathcal{H}})/(1-\beta)\Big)$ with $\| h \|_{\mathcal{H}} = \Big(\int_0^1 |\dot{h}_s|^2\ ds\Big)^{\frac{1}{2}}$ for $h \in \mathcal{H}$.

Now, we set by $\gamma_n(t) = \| \int_0^t \sigma(t,s,\Phi^x(h_n)(s))\,\Big(\dot{h_n}(s) - \dot{h}(s)\Big)\,ds \|_\infty$. We will show that the sequence of functions $(\gamma_n(t))_{n\ge 0}$ is uniformly convergent on $[0,1]$. So,

$$|\gamma_n(t)| \le \left[\int_0^1 \big(1_{[0,t]}\sigma(t,s,\Phi^x(h_n)(s))\big)^2 ds\right]^{\frac{1}{2}} \| h_n - h \|_{\mathcal{H}}\ .$$

Thus, by passing to sup, the sequence $(\gamma_n(t))_{n\ge 0}$ simply converges to 0. As

$$|\gamma_n(t) - \gamma_n(s)| \le \left(\int_0^1 |1_{[s,t]}(s)\sigma(t,s,\Phi^x(h_n)(s)|^2 ds\right)^{\frac{1}{2}} \| h \|_{\mathcal{H}} \le \sqrt{2a}M|t-s|^{\frac{1}{2}}.$$

This ensures that γ_n is uniformly continuous and it follows that Φ^x is continuous for the uniform norm. To study the continuity of Φ^x in the Hölderian norm, let us now note that, for any $\alpha \in]0, \frac{1}{2}[$, we have:

$$\begin{aligned}&\| \Phi^x(h_n) - \Phi^x(h) \|_\alpha \\ &= \sup_{0\le s\le t\le 1} \frac{|(\Phi^x(h_n)(t) - \Phi^x(h)(t)) - (\Phi^x(h_n)(s) - \Phi^x(h)(s))|}{|t-s|^\alpha}\end{aligned}$$

$$\leq \sup_{0\leq s\leq t\leq 1} \frac{|(\Phi^x(h_n)(t)-\Phi^x(h)(t))|}{|t-s|^\alpha} + \sup_{0\leq s\leq t\leq 1} \frac{|(\Phi^x(h_n)(s)-\Phi^x(h)(s))|}{|t-s|^\alpha}$$

$$\leq \tfrac{1}{1-\alpha}\Bigg[\parallel \Phi^x(h_n)-\Phi^x(h) \parallel_\infty + \parallel \Phi^x(h_n)-\Phi^x(h) \parallel_\infty \Bigg]$$

$$\leq \tfrac{2}{1-\alpha}\Bigg[\parallel \Phi^x(h_n)-\Phi^x(h) \parallel_\infty \Bigg].$$

This completes the proof of the Proposition 4.1 □

4.2. **Main results on the LDP solution of** (1.5)**. The Theorem** 3.3 **is a consequence of the following Proposition 4.2:**

Proposition 4.2. *For every* $\alpha \in]0,\frac{1}{2}[,\ \beta \in]0,\frac{1}{2}[$ *and* $h \in \mathcal{H}$*. For any* $R, \rho > 0$ *there exists* $\delta > 0$ *such that for all* $\varepsilon > 0$*,*

$$\mathbb{P}\Bigg(\parallel T^\varepsilon - \tilde{\Phi}^y(h) \parallel_\alpha + \parallel L^\varepsilon - \eta \parallel_\alpha \geq \rho, \parallel \sqrt{\varepsilon}B - h \parallel_\infty < \delta \Bigg) \leq \exp\Bigg(-\frac{R}{\varepsilon}\Bigg).$$

Proof. Note that for f and g two functions with values in $C^\alpha([0,1],\mathbb{R})$, by definition of the function Γ, for $t \in [0,1]$, we have:

$$\parallel \Gamma f - \Gamma g \parallel_\alpha \leq 2 \parallel f-g \parallel_\alpha .$$

From the formulas 3.5 and 3.7, we have:

$$\parallel T^\varepsilon - \tilde{\Phi}^y(h) \parallel_\alpha + \parallel L^\varepsilon - \eta \parallel_\alpha \leq 3 \parallel Z^{\varepsilon,y} - V^y(h) \parallel_\alpha .$$

□

Consequently, the proof of the Proposition 4.2 reduces to showing the following

Theorem 4.4. *For* $\alpha \in]0,\frac{1}{2}[,\ \beta \in]0,\frac{1}{2}[$ *and* $h \in \mathcal{H}$*. For any* $R, \rho > 0$ *there exist* $\delta > 0$ *such that for all* $\varepsilon > 0$*,*

$$\mathbb{P}\Bigg(\parallel Z^{\varepsilon,y} - V^y(h) \parallel_\alpha > \rho, \parallel \sqrt{\varepsilon}B - h \parallel_\infty < \delta \Bigg) \leq \exp\Bigg(-\frac{R}{\varepsilon}\Bigg).$$

Proof. Let us first consider the case where $h = 0$. More precisely, we have the following theorem. □

Theorem 4.5. *For every $\alpha \in]0, \frac{1}{2}[,\ \beta \in]0, \frac{1}{2}[$. For any $R, \rho > 0$ there exist $\delta > 0$ and $\varepsilon_0 > 0$ such that, for any ε small enough,*

$$\mathbb{P}\bigg(\parallel \sqrt{\varepsilon} \int_0^t \varsigma(t, s, \Gamma Z_s^{\varepsilon,y}(s)) \ dB_s \parallel_\alpha > \rho, \parallel \sqrt{\varepsilon}\ B \parallel_\infty < \delta \bigg) \leq \exp\bigg(-\frac{R}{\varepsilon}\bigg).$$

Proof. For all $n \in \mathbb{N}^*$, we consider the sequence of approximating of the process $(Z^\varepsilon)_{\varepsilon>0}$ defined by

$$Z_t^{\varepsilon,n} = Z^\varepsilon_{\frac{j}{n}}, if\ t \in \Big[\frac{j}{n}, \frac{j+1}{n}\Big[\text{ for all } j = 0, 1, 2, \ldots, n-1.$$

For $\alpha > 0$ and for each $n \in \mathbb{N}$, we have

$$\tilde{A}^\varepsilon = \Big\{ \parallel \sqrt{\varepsilon} \int_0^\cdot \varsigma(Y_s^\varepsilon)\ dB_s^\varepsilon \parallel_\alpha \geq \rho, \parallel \sqrt{\varepsilon} B^\varepsilon \parallel_\infty \leq \delta \Big\} \subset \tilde{A}_1^\varepsilon \cup \tilde{A}_2^\varepsilon \cup \tilde{A}_3^\varepsilon,$$

where

$$\begin{cases} \tilde{A}_1^\varepsilon = \Big\{ \parallel \sqrt{\varepsilon} \displaystyle\int_0^t \Big(\varsigma(t, s, \Gamma Z_s^\varepsilon) - \varsigma(t, s, \Gamma Z_s^{\varepsilon,n,y}) \Big)\ dB_s^\varepsilon \parallel_\alpha \\ \qquad \geq \dfrac{\rho}{2}, \parallel Z^\varepsilon - Z^{\varepsilon,n,y} \parallel_\infty \leq \gamma \Big\}, \\ \tilde{A}_2^\varepsilon = \Big\{ \parallel Z^{\varepsilon,y} - Z^{\varepsilon,n,y} \parallel_\infty \geq \gamma \Big\}, \\ \tilde{A}_3^\varepsilon = \Big\{ \parallel \sqrt{\varepsilon} \displaystyle\int_0^t \varsigma(t, s, \Gamma Z_s^{\varepsilon,n,y})\ dB_s^\varepsilon \parallel_\alpha \geq \dfrac{\rho}{2}, \parallel \sqrt{\varepsilon} B^\varepsilon \parallel_\infty \leq \delta \Big\}. \end{cases}$$

By the result of Bo and Zhang [3], we have

$$\mathbb{P}(\tilde{A}_2^\varepsilon) \leq n \ \text{exp}\bigg(-\frac{n\gamma(1-2\beta)^2}{8N^2\varepsilon}\bigg).$$

For any $R, \gamma > 0$ there exist $\tilde{\varepsilon}_0 > 0$ and $\tilde{n}_0 > 0$ such that if $\varepsilon \leq \tilde{\varepsilon}_0$ and $n \geq \tilde{n}_0$,

$$\mathbb{P}(\tilde{A}_2^\varepsilon) \leq \ \text{exp}\bigg(-\frac{R}{\varepsilon}\bigg).$$

By the Lemma 2.2 and the Theorem 4.4,

$$\mathbb{P}(\tilde{A}_1^\varepsilon) \leq C \ \text{exp}\bigg(-\frac{\rho^2}{8L\gamma^2\varepsilon}\bigg).$$

To estimate $\mathbb{P}\Big(\tilde{A}_3^\varepsilon\Big)$, in the subsets $\{\parallel \sqrt{\varepsilon} B^\varepsilon \parallel_\infty \leq \delta\}$, if M is the boundness of coefficient ς, we have

$$\begin{aligned}
&\| \sqrt{\varepsilon} \int_0^t \varsigma(t,s,\Gamma Z_s^{\varepsilon,n,y})\, dB_s^\varepsilon \|_\alpha \\
&= \sqrt{\varepsilon} \,\| \sum_{j=0}^{n-1} \varsigma(t_j,s,\Gamma Z_s^{\varepsilon,n,y})\Big(B^\varepsilon(t_{j+1}\wedge .) - B^\varepsilon(t_j \wedge .)\Big) \|_\alpha \\
&\leq M \sum_{j=0}^{n-1} \sqrt{\varepsilon} \,\| \Big(B^\varepsilon(t_{j+1}) - B^\varepsilon(t_j)\Big) \|_\infty \leq n\, M\, \delta.
\end{aligned}$$

Consequently, if $\delta \leq \frac{\rho}{2\ n\ M}$ then $\mathbb{P}(\tilde{A}_3^\varepsilon) = 0$. □

We will now verify the continuity of the function $\tilde{\Phi}^y$ solution de (3.7).

Proposition 4.3. *Let $\alpha \in]0, \frac{1}{2}[$ and $\beta \in]0,1[$. For any $\tilde{a} \geq 0$, the map $\tilde{\Phi}^y : C^\alpha([0,1],\mathbb{R}) \cap \Big(\Big\{h \in \mathcal{H} : \| h \|_{\mathcal{H}}^2 \leq \tilde{a}\Big\}\Big) \longrightarrow (C^\alpha([0,1],\mathbb{R}), \| \,.\, \|_\alpha)$ and $\eta : C^\alpha([0,1],\mathbb{R}) \cap \Big(\Big\{h \in \mathcal{H} : \| h \|_{\mathcal{H}}^2 \leq \tilde{a}\Big\}\Big) \longrightarrow (C^\alpha([0,1],\mathbb{R}), \| \,.\, \|_\alpha)$ are continuous.*

Proof. The proof of this Proposition is similar to the proof of Proposition 4.1. Indeed,

Case 1: Continuity of the map Φ^y.

Let $(h_n)_{(n\geq 0)}$ a convergent sequence of $C^\alpha([0,1],\mathbb{R}) \cap \Big(\Big\{h \in \mathcal{H} : \| h \|_{\mathcal{H}}^2 \leq \tilde{a}\Big\}\Big)$ and converges to h. By the formula (3.7), we have:

$$\begin{aligned}
&\tilde{\Phi}^y(h_n)(t) - \tilde{\Phi}^y(h)(t) \\
=\ & \int_0^t \Big(\varsigma(t,s,\tilde{\Phi}^y(h_n)(s)) \ - \varsigma(t,s,\tilde{\Phi}^y(h)(s))\Big)\, \dot{h_n}(s)\, ds \\
+\ & \beta\Big(\sup_{0\leq s\leq t} \tilde{\Phi}^y(h_n)(s) - \sup_{0\leq s\leq t} \tilde{\Phi}^y(h)(s)\Big) + \Big(\eta(h_n)(t) - \eta(h)(t)\Big).
\end{aligned}$$

By using the formula (3.8), we have:

$$\begin{aligned}
&\tilde{\Phi}^y(h_n)(t) - \tilde{\Phi}^y(h)(t) \\
=\ & \int_0^t \Big(\varsigma(t,s,\Gamma V(h_n)(s)) \ - \varsigma(t,s,\Gamma V(h)(s))\Big)\, \dot{h_n}(s)\, ds \\
+\ & \beta\Big(\sup_{0\leq s\leq t} \Gamma V(h_n)(s) - \sup_{0\leq s\leq t} \Gamma V(h)(s)\Big) + \Big(KV(h_n)(t) - KV(h)(t)\Big).
\end{aligned}$$

Therefore, using the fact that for two continuous functions u and v on $\mathbb{R}_+$, we have:

$$\big|\sup_{0\le s\le t} u(s) - \sup_{0\le s\le t} v(s)\big| \le \sup_{0\le s\le t}\big|u(s)-v(s)\big|.$$

As a result, it follows that

$$\begin{aligned} & |\tilde{\Phi}^y(h_n)(t) - \tilde{\Phi}^y(h)(t)| \\ \le\ & L\int_0^t \Big(|\Gamma V(h_n)(s)\ -\Gamma V(h)(s)|(|\dot{h_n}(s)|)\Big)\,ds \\ +\ & \beta\sup_{0\le s\le t}|\Gamma V(h_n)(s)-\Gamma V(h)(s)| + \Big|KV(h_n)(t)-KV(h)(t)\Big|, \end{aligned}$$

where $L>0$ is the Lipschitz coefficient. By definition of the function Γ, for $t\in[0,1]$, we have:

$$|\Gamma V(h_n)(t)-\Gamma V(h)(t)| \le 2|V(h_n)(t)-V(h)(t)|.$$

Then by definition of the function K, for $t\in[0,1]$, we have :

$$|KV(h_n)(t)-KV(h)(t)| \le |V(h_n)(t)-V(h)(t)|.$$

Thus, it follows that for $t\in[0,1]$,

$$\begin{aligned} & |\tilde{\Phi}^y(h_n)(t) - \tilde{\Phi}^y(h)(t)| \\ \le\ & 2L\int_0^t \Big(|V(h_n)(s)\ -V(h)(s)|(|\dot{h_n}(s)|)\Big)\,ds \\ +\ & 2\beta\sup_{0\le s\le t}|V(h_n)(s)-V(h)(s)| + \Big|V(h_n)(t)-V(h)(t)\Big|. \end{aligned}$$

Therefore, the proof of the continuity of $\tilde{\Phi}^y$ can be reduced to the proof of the continuity of the function V defined in (3.9)

$$\begin{aligned} V(h_n)(t)-V(h)(t) &= \int_0^t \varsigma(t,s,V(h)(s))\Big(\dot{h_n}(s)-\dot{h}(s)\Big)\,ds \\ &+ \int_0^t \Big(\varsigma(t,s,V(h_n)(s))\ -\varsigma(t,s,V(h)(s)\Big)\,\dot{h_n}(s)\,ds \\ &+ \beta\Big(\sup_{0\le s\le t}V(h_n)(s)-\sup_{0\le s\le t}V(s)\Big). \end{aligned}$$

Consequently,

$$|V(h_n)(t)-V(h)(t)| \le \int_0^t |\varsigma(t,s,V(h_n)(s))\,(\dot{h_n}(s)-\dot{h}(s)\,ds)|$$

$$
\begin{aligned}
&+ \quad L \int_0^t \Big(|V(h_n)(s) \; - V(h)(s)| (|\dot{h_n}(s)|) \Big) \, ds \\
&+ \quad \beta \sup_{0 \le s \le t} |V(h_n)(s) - V(h)(s)|,
\end{aligned}
$$

where $L > 0$ is the Lipschitz coefficient of ς. Thus, by the **Reflection principle**, it follows that, for $t \in [0,1]$,

$$
\begin{aligned}
\sup_{0 \le u \le t} |V(h_n)(u) - V(h)(u)| \quad &\le \quad \tfrac{1}{1-\beta} \sup_{0 \le u \le t} \int_0^u |\varsigma(u, s, V(h_n)) \, (\dot{h_n}(s) - \dot{h}(s) ds)| \\
&+ \quad \tfrac{L}{1-\beta} \sup_{0 \le u \le t} \int_0^u |V(h_n)(s) - V(h)(s)| \;\; (|\dot{h}(s)|) \, ds.
\end{aligned}
$$

By **the Gronwall Lemma and the Cauchy-Schwarz inequality**, we have

$$
\begin{aligned}
\| \, V(h_n)(t) - V(h)(t) \, \|_\infty \quad &\le \quad \tfrac{1}{1-\beta} \sup_{0 \le t \le 1} \int_0^t |\varsigma(t, s, V(h_n)(s)) \, (\dot{h_n}(s) - \dot{h}(s)| \, ds \\
&\times \quad \exp\Big(\int_0^t \frac{L}{1-\beta} \, \big(|\dot{h}(s)|\big) \, ds \\
&\le \quad C_1(h) \sup_{0 \le t \le 1} \int_0^t |\varsigma(t, s, V(h_n)) \, \Big(\dot{h_n}(s) - \dot{h}(s)\Big)| \, ds \\
&\le \quad C_1(h) \;\; \| \int_0^t \varsigma(t, s, V(h_n)(s)) \, \Big(\dot{h_n}(s) - \dot{h}(s)\Big) \, ds \, \|_\infty,
\end{aligned}
$$

where $C_1(h) = \frac{1}{1-\beta} \exp\Big(C(H) \; L \; (\| \; h \; \|_{\mathcal{H}})/(1-\beta)\Big)$ with $\| \; h \; \|_{\mathcal{H}} = \Big(\int_0^1 |\dot{h}_s|^2 \; ds \Big)^{\frac{1}{2}}$ for $h \in \mathcal{H}$.

Now let us note by $\gamma_n(t) = \| \int_0^t \varsigma(t, s, V(h_n)(s)) \; \Big(\dot{h_n}(s) - \dot{h}(s)\Big) \; ds \; \|_\infty$. We will show that the functions sequences $\gamma_n(t)$ is uniformly convergent on $[0,1]$. So,

$$
|\gamma_n(t)| \le \Big[\int_0^1 \big(1_{[0,t]}(V(h_n)(s))\big)^2 ds \Big]^{\frac{1}{2}} \; \| \; h_n - h \; \|_{\mathcal{H}} \; .
$$

Therefore, by passing to sup, the sequence γ_n simply converges to 0. As

$$
|\gamma_n(t) - \gamma_n(s)| \le \Big(\int_0^1 |1_{[s,t]}(s) \tilde{\sigma}(t, s, V(h_n)(s)|^2 ds \Big)^{\frac{1}{2}} \; \| \; h \; \|_{\mathcal{H}} \le \sqrt{2a} M |t-s|^{\frac{1}{2}}.
$$

This assures us that γ_n is uniformly continuous and it follows that V is continuous for the uniform norm, then $\tilde{\Phi}^y$ is also continuous for the topology of uniform convergence. Let us note that, for any $\alpha \in]0, \frac{1}{2}[$, we have:

$$
\begin{aligned}
& \| \tilde{\Phi}^y(h_n) - \tilde{\Phi}^y(h) \|_\alpha \\
= & \sup_{0\le s\le t\le 1} \frac{|(\tilde{\Phi}^y(h_n)(t) - \tilde{\Phi}^y(h)(t)) - (\tilde{\Phi}^y(h_n)(s) - \tilde{\Phi}^y(h)(s))|}{|t-s|^\alpha} \\
\le & \sup_{0\le s\le t\le 1} \frac{|(\tilde{\Phi}^y(h_n)(t) - \tilde{\Phi}^y(h)(t))|}{|t-s|^\alpha} + \sup_{0\le s\le t\le 1} \frac{|(\tilde{\Phi}^y(h_n)(s) - \tilde{\Phi}^y(h)(s))|}{|t-s|^\alpha} \\
\le & \tfrac{1}{1-\alpha}\Big[\| \Phi^y(h_n) - \Phi^y(h) \|_\infty + \| \Phi^y(h_n) - \Phi^y(h) \|_\infty \Big] \\
\le & \tfrac{2}{1-\alpha}\Big[\| \Phi^y(h_n) - \Phi^y(h) \|_\infty \Big]
\end{aligned}
$$

This last inequality shows that $\tilde{\Phi}^y$ is continuous with respect to the topology induced by the Hölderian norm of order $\alpha \in]0, \frac{1}{2}[$.

Case 2: Continuity of the map η.

The proof of the continuity of the map η resides in the facts that:

$$
\| \eta(h_n) - \eta(h) \|_\alpha \ \le \ \tfrac{2}{1-\alpha}\Big[\| \eta(h_n) - \eta(h) \|_\infty \Big]
$$

and

$$
\begin{aligned}
\| \eta(h_n) - \eta(h) \|_\infty & \le \| KV(h_n) - KV(h) \|_\infty \\
& \le \| \tilde{V}(h_n) - \tilde{V}(h) \|_\infty \\
& \le \| V(h_n) - V(h) \|_\infty .
\end{aligned}
$$

This completes the proof of the Proposition 4.3 □

5. Application: Large deviations for stochastic Volterra equations driven by Fractional Brownian Motion With Hurst parameter $H \in [\frac{1}{2}, 1)$

Definition 5.1. *A fractional Brownian motion $B^H = \{B_t^H, t \ge 0\}$, of parameter H in $(0,1)$, is a Gaussian process centred, satisfying the following conditions:*

(1) *B^H is a process with stationary increases;*

(2) *$\mathbb{E}(B_t^H)^2 = t^{2H}$;*

(3) *$B_0^H = 0$.*

The parameter H is the Hurst parameter.

Proposition 5.1. *B^H admits as covariance function the function R_H defined for every $(s,t) \in \mathbb{R}^+ \times \mathbb{R}^+$ by:*

$$R_H(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}). \tag{5.1}$$

Proposition 5.2. *For any $0 < H < 1$ and $(s,t) \in [0,1]^2$ we can write:*

$$K^H(t,s) = s^{-|H-\frac{1}{2}|}(t-s)_+^{H-\frac{1}{2}} L^H(t,s), \tag{5.2}$$

where L^H is a continuous function on $[0,1] \times [0,1]$.

In [4, 5], L. Coutin and L. Decreusefond have studied the existence, uniqueness and regularity of a solution of the stochastic differential equation directed by the fractional Brownian motion of Hurst parameter $H \in [\frac{1}{2}, 1)$ and have considered that such an equation is of type Volterra of the form:

$$\begin{aligned} X_t^\varepsilon = x_0 &+ \int_0^t K^H(t,s)\, b(s, X_s^\varepsilon)\, ds + \sqrt{\varepsilon} \int_0^t K^H(t,s)\, \sigma(s, X_s^\varepsilon)\, dB_s \\ &+ \beta \sup_{0 \le s \le t} X_s^\varepsilon, \end{aligned} \tag{5.3}$$

$H \in [\frac{1}{2}, 1)$, $t \in [0,1]$, where b and σ are two continuous bounded lipschitzian functions and B_t is a standard Brownian motion. In this case, we can consider the solution of the SDE driven by fractional Brownian motion to be a Volterra-type equation of the form:

$$X_t^\varepsilon = x_0 + \int_0^t \tilde{b}(t,s,X_s^\varepsilon)\, ds + \sqrt{\varepsilon} \int_0^t \tilde{\sigma}(t,s,X_s^\varepsilon) dB_s + \beta \sup_{0 \le s \le t} X_s^\varepsilon, \tag{5.4}$$

with

$$\tilde{b}(t,s,x) = K^H(t,s) b(s,x) \text{ and } \tilde{\sigma}(t,s,x) = K^H(t,s)\sigma(s,x).$$

and the new coefficients $\tilde{\sigma}$ and $\tilde{b}$ satisfy the conditions of the Theorem 3.1.

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