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A FAMILY OF K-STEP TRIGONOMETRICALLY-FITTED BLOCK FALKNER METHODS FOR SOLVING SECOND-ORDER INITIAL-VALUE PROBLEMS WITH OSCILLATING SOLUTIONS

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ABSTRACT. A family of K-step Trigonometrically-fitted Block Falkner Methods is considered for the direct solution of second order Oscillatory Initial value problems. As unique to Falkner methods, two main formulas (one for the method and one for the derivative) for each k-step and some additional formulas. This method shall be adapted to general oscillatory second order ordinary differential equations via the multistep collocation technique. The idea employed in this study is the generalized collocation technique based on fitting functions that are combination of trigonometric and algebraic polynomials, which is then implemented in a block mode to get approximations at all the grid points simultaneously. As in other block methods, there is no need of other procedures to provide starting values, and thus the methods are selfstarting (sharing this advantage of Runge-kutta methods). The study of the properties of the proposed adapted block Falkner methods reveals that they are consistent and zero-stable, and thus, convergent. Furthermore, the stability analysis and the algebraic order conditions of the proposed methods are established. As evident from the numerical results, the methods are efficient and accurate when compared with some recent methods in the literature.

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1. Introduction

Numerical methods for the solution of second order differential equation with oscillatory behaviour has gained a lot of attention. This oscillatory problems arise in a wide range of fields such as astronomy, molecular dynamics, classical mechanics, quantum mechanics, chemistry, biology and engineering. These problems can be modeled by initial value problems of second order differential equations with a linear term characterizing the oscillatory characteristic. Many of the numerical methods that have been applied to this problems do not preserve the structure in long term computation. It is possible that these problem can be integrated by reformulating it as a system of first order ODEs and applying one of the methods available for those systems. Nevertheless, a numerical methods that can integrate it directly without transforming it into a first order system will be more accurate and efficient. It also reduces the number of function of evaluation in the implementation by half the number required for reducing to a system of first order equations. This paper focuses on the direct numerical integration of the initial value problem of the form:

$$y''(x) = f(x, y(x), y'(x)) : y(0) = y_\circ, \quad y'(0) = y'_\circ, \tag{1.1}$$

whose solution is assumed to be oscillatory or periodic, and the frequency is approximately known in advance, with $f: \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ a smooth function that satisfies existence and uniqueness of solution's conditions, where m is the system's dimension. One of the most useful procedures for the construction of numerical methods that approximate the solution of second-order initial value problems with oscillatory behaviour is the Adapted methods. Adapted methods are numerical procedures whose coefficients are related to the frequency of the problem, which can be identified in advance. Usually, a combination of polynomial and appropriate non-polynomial functions are used as fitted functions. In the excellent works by Vigo-Aguiar and Ramos [1], Jator *et al.* ([6]- [8]), Awoyemi [9], Liu and Wu [12], Li *et al.* [13] , which adopted the direct integration of the general second order IVPs containing the first derivative and their implementation based on a step-by-step fashion. Some used predictor-corrector modes. Nevertheless, they are computationally expensive, especially, for higher-order methods and large systems of equation . It becomes apparent that, some of these methods do not take advantage of the oscillatory or even periodic behavior of the solutions. If the frequency is known or can be estimated in advance this could be considered in the development of the method in order to improve its performance. One of the numerical integrators for the general second order IVP in which the first derivative appears explicitly is an explicit method due to Falkner [28], while the implicit form is due to Collatz [18]. For some modifications on the Falkner methods(see [1], [2], [3], [4]). The adapted Falkner methods that preserve the osclla-tory characteristics and the structure in long term computation can be found in the works by Li and Wu [12], Li [13], and Ehigie and Okunuga [16] respectively. The use of adapted methods started with the elegant work by Gautchi [27] and later by Lyche [20]. Other adapted methods than have been considered can be found therein Franco ([24], [25]), Ixaru *et al.* [21], Vanden Berghe and Van Daele [22], Jator *et al*. [7], Jator ([6], [8]), Ramos and Vigo-Aguiar [1], Vigo-Aguiar and Ramos ([2], [4]), Coleman and Duxbury [18], Coleman and Ixaru [19], Fang *et al.* [17]. Inspite of that,further research is needed to explore methods that can give better performance. The proposed block Falkner method shall be adapted to general oscillatory second order ordinary differential equations via the multistep collocation techniques. The idea employed in this study is the generalized collocation technique based on fitting functions that are combination of trigonometric and algebraic polynomials. This approach shall be used to develop block methods, whose coefficients are functions of the frequency and stepsize. The rest of this paper is organized as follows: the derivation of (TFBFM) is presented in section 2. The analysis of the characteristics of the (TFBFM) is discussed in section 3, while some numerical experiments are presented in section 4. Finally,some concluding remarks will be given in section 5.

2. Development of The TFBFM

For emphasis, consider the general second order IVP of the form

$$y'' = f\left(x, y, y'\right), \quad y\left(x_0\right) = y_0, \quad y'\left(x_0\right) = y'_0, \tag{2.1}$$

whose solution is periodic with the frequency that can be estimated or known in advance and $f : R \times R^m \to R^m$ is a smooth function that satisfies the conditions of existence and uniqueness of solution, and m is the dimension of the system. To develop a discrete Trigonometrically Fitted Block Falkner Method for each family member, a Continuous Trigonometrically Fitted Block Falkner Method (CTFBFM) on the interval $[x_n, x_{n+k}]$ is first constructed. In order to do this, m = 1, the scalar case, is considered. We define its generalized continuous formulation for the direct integration of IVPs(2.1) related with the methods in Ramos and Rufai [30], that will aid the derivation of the TFBFM.

Definition 2.1. *The continuous formulation of the* $k-$*step Trigonometrically Fitted Block Falkner Method for approximating the solution of equation (2.1) is defined by*

$$\begin{aligned}\bar{y}\left(x\right) &= \alpha_{k0}\left(x,u\right)y_{n+1} + h\alpha_{k1}\left(x,u\right)y'_{n+1} \\ &\quad + h^2\sum_{j=0}^{k}\beta_{kj}\left(x,u\right)f_{n+j} + h^3\gamma_k\left(x,u\right)g_{n+k},\end{aligned} \tag{2.2}$$

where $\alpha_{k0}\left(x,u\right)$, $\alpha_{k1}\left(x,u\right)$, $\beta_{kj}\left(x,u\right)$ *and* $\gamma_k\left(x,u\right)$ *are functions of* x *and* $u=\omega h$.

Definition 2.2. *The primary formulas of the adapted* $k-$*step Trigonometrically Fitted Block Falkner Method for the numerical solution of equation (2.1) are given by*

$$\begin{cases} y_{n+k} = y_{n+1} + (k-1)hy'_{n+1} + h^2\sum\limits_{j=0}^{k}\beta_{kj}\left(u\right)f_{n+j} + h^3\gamma_k\left(u\right)g_{n+k} \\ hy'_{n+k} = hy'_{n+1} + h^2\sum\limits_{j=0}^{k}\bar{\beta}_{kj}\left(u\right)f_{n+j} + h^3\bar{\gamma}_{ki}\left(u\right)g_{n+k} \end{cases}, \tag{2.3}$$

where y_{n+j}, y'_{n+j}, f_{n+j} and g_{n+k} are the numerical approximation to the exact values $y(x_{n+j})$, $y'(x_{n+j})$, $f(x_{n+j}, y(x_{n+j}), y'(x_{n+j}))$ and $g(x_{n+j}, y(x_{n+j}), y'(x_{n+j}))$ respectively, where

$$g(x,y,y') = y'''(x) = f_x(x,y,y') + f_y(x,y,y')y'(x) + f_{y'}(x,y,y')f(x,y,y').$$

Definition 2.3. *The* $(2k-2)$ *secondary formulas of the adapted* $k-$*step Trigonometrically Fitted Block Falkner Method for the numerical solution of equation (2.1) are given by*

$$\begin{cases} y_{n+\mu} = y_{n+1} + (\mu-1)hy'_{n+1} + h^2\sum\limits_{j=0}^{k}\beta^{\mu}_{kj}\left(u\right)f_{n+j} + h^3\gamma^{\mu}_k\left(u\right)g_{n+k} \\ hy'_{n+\mu} = hy'_{n+1} + h^2\sum\limits_{j=0}^{k}\bar{\beta}^{\mu}_{kj}\left(u\right)f_{n+j} + h^3\bar{\gamma}^{\mu}_k\left(u\right)g_{n+k}, \end{cases}, \tag{2.4}$$

where $\mu = 0, 2(1)(k-1)$.

Definition 2.4. *The $k-$step Trigonometrically Fitted Block Falkner Method consists of the primary formulas in equation (2.3) and the secondary formulas in equation (2.4).*

Derivation of $TFBFM$

Let $\Delta = \{\Delta_0, \Delta_1, \Delta_2, \cdots, \Delta_{k+3}\}$ be a set of $k+4$ linearly independent functions. We seek an approximate solution $I(x) \in span\ \Delta$ called a fitted function associated to the Trigonometrically Fitted Falkner method which satisfies the IVP in equation (2.1) at some specified points.

For the construction of the adapted Falkner methods, Δ is taken as

$$\Delta = \{1, x, \cdots, x^{k+1}\} \cup \{\sin(\omega x), \cos(\omega x)\}. \tag{2.5}$$

To get the coefficients of the fitting function associated to the set Δ in (2.5), $I(x)$ is interpolated at the point $x = x_{n+1}$, and the following collocating conditions are considered: $I'(x)$ at $x = x_{n+1}$, $I''(x)$ at the points $x = x_{n+j}, j = 0, 1 \cdots, k$, and $I'''(x)$ at $x = x_{n+k}$. This leads to the following system of $k+4$ equations

$$\begin{cases} I(x_{n+1}) = y_{n+1}, \\ I'(x_{n+1}) = y'_{n+1}, \\ I''(x_{n+j}) = f_{n+j},\ j = 0, 1, \cdots, k, \\ I'''(x_{n+k}) = g_{n+k}. \end{cases} \tag{2.6}$$

Theorem 2.1. *Let $I(x)$ be the fitting function associated to the set Δ in (2.5), $\{\Delta_i(x)\}_{i=0}^{k+3} = \{1, x, \cdots, x^{k+1}, \sin(\omega x), \cos(\omega x)\}$ and the vector $\Lambda =(y_{n+1}, y'_{n+1}, f_n, f_{n+1}, \cdots f_{n+k}, g_{n+k})^t)$, where t is the transpose. Consider the following square matrix of dimension $k+4$ which is the matrix of coefficients of the system in (2.6),*

$$\Omega = \begin{pmatrix} \Delta_0(x_{n+1}) & \Delta_1(x_{n+1}) & \cdots & \Delta_{k+3}(x_{n+1}) \\ \Delta'_0(x_{n+1}) & \Delta'_1(x_{n+1}) & \cdots & \Delta'_{k+3}(x_{n+1}) \\ \Delta''_0(x_n) & \Delta''_1(x_n) & \cdots & \Delta''_{k+3}(x_n) \\ \vdots & \ddots & \ddots & \vdots \\ \Delta''_0(x_{n+k}) & \Delta''_1(x_{n+k}) & \cdots & \Delta''_{k+3}(x_{n+k}) \\ \Delta'''_0(x_{n+k}) & \Delta'''_1(x_{n+k}) & \cdots & \Delta'''_{k+3}(x_{n+k}) \end{pmatrix}$$

and Ω_i obtained by replacing the ith column of Ω by the vector Λ. If we impose that $I(x)$ satisfies the system of $k+4$ equations in (2.6) then it can be written as

$$I\left(x\right)=\sum_{i=0}^{k+3}\frac{det\left(\Omega_{i}\right)}{det\left(\Omega\right)}\Delta_{i}(x). \tag{2.7}$$

Proof. Let the fitting function $I(x)$ associated to the set Δ be defined as follows

$$\begin{aligned}I\left(x\right)=&\alpha_{k0}\left(x,u\right)y_{n+1}+h\alpha_{k1}\left(x,u\right)y_{n+1}'+h^{2}\sum_{j=0}^{k}\beta_{kj}\left(x,u\right)f_{n+j}\\&+h^{3}\gamma_{k}\left(x,u\right)g_{n+k}.\end{aligned} \tag{2.8}$$

To solve the system of equations in (2.6), it is required that the coefficients in (2.8) are expressed in terms of the assumed basis functions as follow

$$\begin{cases}\alpha_{j0}\left(x,u\right)=\sum_{i=0}^{\mathrm{k}+3}\alpha_{i,j}\left(x,u\right)\Delta_{i}\left(x\right), & j=0,1,\\ \alpha_{j1}\left(x,u\right)=\sum_{i=0}^{\mathrm{k}+3}\alpha_{i,j}\left(x,u\right)\Delta_{i}\left(x\right), & j=0,1,\end{cases} \tag{2.9}$$

$$\beta_{j}\left(x,u\right)=\sum_{i=0}^{\mathrm{k}+3}\beta_{i,j}\left(x,u\right)\Delta_{i}\left(x\right), \quad j=0,1,\cdots,k, \tag{2.10}$$

$$\gamma_{j}\left(x,u\right)=\sum_{i=0}^{\mathrm{k}+3}\gamma_{i,j}\left(x,u\right)\Delta_{i}\left(x\right), \quad j=k. \tag{2.11}$$

Substituting equations (2.9),(2.10)and (2.11) into the equation (2.8) yields

$$\begin{aligned}I\left(x\right)=\sum_{i=0}^{\mathrm{k}+3}\Bigg\{&\sum_{j=0}^{1}\alpha_{i,j0}(x,u)y_{n+j}+\sum_{j=0}^{1}\alpha_{i,j1}(x,u)y_{n+j}'+h^{2}\sum_{j=0}^{\mathrm{k}}\beta_{i,j}(x,u)f_{n+j}\\&+h^{3}\sum_{j=\mathrm{k}}^{\mathrm{k}}\gamma_{i,j}(x,u)g_{n+j}\Bigg\}\Delta_{i}(x).\end{aligned} \tag{2.12}$$

Let

$$\xi_{i}=\sum_{j=0}^{1}\alpha_{i,j0}(x,u)y_{n+j}+\sum_{j=0}^{1}\alpha_{i,j1}(x,u)y_{n+j}'+h^{2}\sum_{j=0}^{\mathrm{k}}\beta_{i,j}(x,u)f_{n+j}++h^{3}\sum_{j=\mathrm{k}}^{\mathrm{k}}\gamma_{i,j}(x,u)g_{n+j}.$$

Then equation (2.12) becomes

$$I\left(x\right)=\sum_{i=0}^{\mathrm{k}+3}\xi_{i}\Delta_{i}(x), \tag{2.13}$$

where ξ is an undetermined vector written as $(\xi = \xi_0, \xi_1, \cdots, \xi_{k+3})^t$, t is the transpose. By imposing conditions in (2.6) on equation (2.13), a system of $k+4$ is obtained and can be expressed in matrix form as

$$\Omega\xi = \Lambda. \tag{2.14}$$

□

Specification of The TFBFM

We emphasize that for each k, there are two primary formulas of the form in equation (2.3) and $(2k-2)$ secondary formulas as those in equation (2.4) (which are obtained by evaluating the fitting function in (2.7) at the corresponding points) that combined together form the proposed TFBFM. Hence the TFBFM has $2k$ formulas.

As an illustration, we specified how to obtain the TFBFM for $k = 2$.

For $k = 2$, we evaluate the fitting function in (2.7) and its first derivative at $x = \{x_{n+2}, x_n\}$ to obtain the two primary formulas and the two secondary formulas as Evaluate the fitting function in (2.7) and its first derivative at $x = \{x_{n+2}, x_n\}$ to obtain the two primary methods and the two secondary methods as

$$\begin{cases} y_{n+2} = y_{n+1} + hy'_{n+1} + h^2 \sum\limits_{j=0}^{2} \beta_{2j}(u) f_{n+j} + h^3 \gamma_2(u) g_{n+2} \\ hy'_{n+2} = hy'_{n+1} + h^2 \sum\limits_{j=0}^{2} \bar{\beta}_{2j} f_{n+j}(u) + h^3 \bar{\gamma}_2(u) g_{n+2} \\ y_n = y_{n+1} - hy'_{n+1} + h^2 \sum\limits_{j=0}^{2} \beta^0_{2j}(u) f_{n+j} + h^3 \gamma^0_2(u) g_{n+2} \\ hy'_n = hy'_{n+1} + h^2 \sum\limits_{j=0}^{2} \bar{\beta}^0_{2j}(u) f_{n+j} + h^3 \bar{\gamma}^0_2(u) g_{n+2}. \end{cases} \tag{2.15}$$

Remark 2.1. *For small values of u, the coefficients of the TFBFM may be subject to heavy cancellations. In that case the Taylor series expansion of the coefficients is preferable (see Lambert, [33]). Specific coefficients of the two primary formulas and their corresponding series expansion up to $O(u^{16})$ for $k = 2$ are provided in Appendix B.*

3. Analysis of the TFBFM

We discuss the basic analysis of the proposed $TFBFM$ in this section. The analysis includes the Algebraic Order, Local Truncation Error, Consistency, Zero-Stability, Convergence and Linear Stability of the $TFBFM$.

Algebraic Order, Local Truncation Errors and Consistency of the TFBFM

The purpose here is to establish that the TFBFM is of uniform order for the individual formula that makes up the $2k$ formulas of the TFBFM and their equivalent local truncation errors with the aid of the theory of linear operator (Lambert, [34]).

Local Truncation Error of TFBFM

Proposition 3.1. *The local truncation error of the of the $k-$step TFBFM is form* $C_{k+4}h^{k+4}(y^{(k+4)+\omega^2 y^{k+2}(x_n)}(x_n)) + O(h^{k+5})$.

Proof. Since the Falkner Methods in equations (2.3) and (2.4) are made up of generalized linear multistep methods, we associate the Falkner methods with linear difference operators $\mathcal{L}\left[y\left(x_n\right);h\right]$, $\mathcal{L}'\left[y\left(x_n\right);h\right]$ for the primary methods and $\mathcal{L}_\mu\left[y\left(x_n\right);h\right]$,${\mathcal{L}'}_\mu\left[y\left(x_n\right);h\right]$ for the secondary methods defined respectively by

$$(3.1)\quad \begin{cases} \mathcal{L}\left[y\left(x_n\right);h\right] = y\left(x_n+kh\right) - \Big(y(x_n+h) + (k-1)hy'\left(x_n+h\right) \\ \qquad\qquad +h^2\displaystyle\sum_{j=0}^{k}\beta_{kj}(u)y''\left(x_n+jh\right) + h^3\gamma_k(u)y'''\left(x_n+kh\right)\Big) \\ \mathcal{L}'\left[y\left(x_n\right);h\right] = hy'\left(x_n+kh\right) - \Big(hy'\left(x_n+h\right) \\ \qquad\qquad +h^2\displaystyle\sum_{j=0}^{k}\bar{\beta}_{kj}(u)y''\left(x_n+jh\right) + h^2\bar{\gamma}_k(u)y'''\left(x_n+kh\right)\Big) \\ \mathcal{L}_\mu\left[y\left(x_n\right);h\right] = y\left(x_n+\mu h\right) - \Big(y(x_n+h) + (\mu-1)hy'\left(x_n+h\right) \\ \qquad\qquad +h^2\displaystyle\sum_{j=0}^{k}\beta^{\mu}_{kj}(u)y''\left(x_n+jh\right) + h^3\gamma^{\mu}_k(u)y'''\left(x_n+kh\right)\Big) \\ {\mathcal{L}'}_\mu\left[y\left(x_n\right);h\right] = hy'\left(x_n+\mu h\right) - \Big(hy'\left(x_n+h\right) \\ \qquad\qquad +h^2\displaystyle\sum_{j=0}^{k}\bar{\beta}^{\mu}_{kj}(u)y''\left(x_n+jh\right) + h^2\bar{\gamma}^{\mu}_k(u)y'''\left(x_n+kh\right)\Big) \end{cases}$$

Consider the Taylor series expansions of $y(x_n+kh)$, $y(x_n+h)$, $y(x_n+\mu h)$, $y'(x_n+kh)$, $y'(x_n+\mu h)$, $y''(x_n+kh)$ and $y'''(x_n+kh)$ about the point x_n and the coefficients of TFBFM specified by $\beta_{kj}(u)$, $\gamma_k(u)$, $\bar{\beta}_{kj}(u)$, $\bar{\gamma}_k(u)$, $\beta^{\mu}_{kj}(u)$, $\gamma^{\mu}_k(u)$,$\bar{\beta}^{\mu}_{kj}(u)$ and $\bar{\gamma}^{\mu}_k(u)$ respectively. By collecting the coefficients of the same power of h, we observe that the Taylor series expansion in equation (3.1) vanishes up to $p+1$. The remaining non zero terms of the Taylor series whose coefficients are $C_{p+2}, C_{p+3}, \dots$ can be equivalently written as $C_{k+4}h^{k+4}(y^{(k+4)+\omega^2 y^{k+2}(x_n)}(x_n)) + O(h^{k+5})$ which is the local truncation error of the $k-$step TFBFM. □ □

Corollary 3.1. *The order of the $k-$step TFBFM is at least $p = k+2$.*

Theorem 3.1. *When the solution of the problem in equation (2.1) is a linear combination of the basis functions $\{I(x)\}_{j=0(1)(k+3)}$, then the local truncation errors vanish.*

Proof. Solving the differential equation $y^{(k+4)} + \omega^2 y^{k+2} = 0$ provides the following solution set $\left\{1, x, \cdots, x^{k+1}, \sin(\omega x), \cos(\omega x)\right\}$, which contains the basis function of the TFBFM, from which the statement follows immediately. □ □

Corollary 3.2. *The order p of the $k-$step TFBFFM is $p = k+2$. Hence the order of BFFM for $k = 2$ is $p = 4$.*

Theorem 3.2. *When the solution of the problem in equation (2.1) is a linear combination of the basis functions $\{I(x)\}_{j=0(1)(k+3)}$, then the local truncation errors vanish.*

Proof. Solving the differential equation $y^{(k+4)} + \omega^2 y^{k+2} = 0$ provides the following solution set $\left\{1, x, \cdots, x^{k+1}, \sin(\omega x), \cos(\omega x)\right\}$, which contains the basis function of the TFBFM, from which the statement follows immediately. □ □

Consistency of The TFBFM

Since the order of the $k-$step TFBFM is at least $p = k+2$, we therefore conclude that it is consistent (Lambert, [33] and Fatunla, [29]).

Stability of The TFBFM

Stability is a key term in numerical analysis. It refers to the degree to which a numerical system is suitable for solving an initial value problem in the context of ordinary differential equations, if little changes in the data cause a modest alteration in the solution a method gives, it is considered to be stable. The proposed method is typically written as a one-step recurrence difference system, after which the requisite definition is applied to the matrices that arise, as in the cases of zero stability and linear stability. Thus the TFBFM specified by equations (2.3) and (2.4) may be written in the form of difference system defined by

$$A_1 Y_{n+1} = A_0 Y_n + h^2 B_0 F_n + h^2 B_1 F_{n+1} + h^3 D_1 G_{n+1} \tag{3.2}$$

where

$$\begin{aligned} Y_{n+1} &= \left(y_{n+1}, y_{n+2}, \cdots, y_{n+k}, hy'_{n+1}, hy'_{n+2}, \cdots, hy'_{n+k}\right)^T, \\ Y_n &= \left(y_{n-k+1}, \cdots, y_{n-1}, y_n, hy'_{n-k+1}, \cdots, hy'_n\right)^T, \\ F_{n+1} &= \left(f_{n+1}, f_{n+2}, \cdots, f_{n+k},\ hf'_{n+1}, \cdots,\ hf'_{n+k}\right)^T, \\ F_n &= (f_{n-k+1}, \cdots, f_{n-1}, f_n, hf'_{n-k+1}, \cdots,\ hf'_n)^T, \\ G_{n+1} &= (g_{n+1}, g_{n+2}, \cdots, g_{n+k})^T. \end{aligned}$$

Here A_0, A_1, B_0, B_1, and D_1 are $2k \times 2k$ matrices defined in canonical form respectively for $k = 2$ and $k = 3$ as follows.

For $k = 2$:

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & -1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & -\bar{\beta}_{20}^0 & 0 & 0 \\ 0 & -\beta_{20}^0 & 0 & 0 \\ 0 & -\bar{\beta}_{20} & 0 & 0 \\ 0 & -\beta_{20} & 0 & 0 \end{bmatrix}$$

,

$$B_1 = \begin{bmatrix} -\bar{\beta}_{22}^0 & -\bar{\beta}_{21}^0 & 0 & 0 \\ -\beta_{22}^0 & -\beta_{21}^0 & 0 & 0 \\ -\bar{\beta}_{22} & -\bar{\beta}_{21} & 0 & 0 \\ -\beta_{22} & -\beta_{21} & 0 & 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} -\bar{\gamma}_2^0 & 0 & 0 & 0 \\ -\gamma_2^0 & 0 & 0 & 0 \\ -\bar{\gamma}_2 & 0 & 0 & 0 \\ -\gamma_2 & 0 & 0 & 0 \end{bmatrix}.$$

For $k = 3$:

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 0 & -2 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0 & 0 & -\bar{\beta}^0_{30} & 0 & 0 & 0 \\ 0 & 0 & -\beta^0_{30} & 0 & 0 & 0 \\ 0 & 0 & -\bar{\beta}^2_{30} & 0 & 0 & 0 \\ 0 & 0 & -\beta^2_{30} & 0 & 0 & 0 \\ 0 & 0 & -\bar{\beta}_{30} & 0 & 0 & 0 \\ 0 & 0 & -\beta_{30} & 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -\bar{\beta}^0_{33} & -\bar{\beta}^0_{32} & -\bar{\beta}^0_{31} & 0 & 0 & 0 \\ -\beta^0_{33} & -\beta^0_{32} & -\beta^0_{31} & 0 & 0 & 0 \\ -\bar{\beta}^2_{33} & -\bar{\beta}^2_{32} & -\bar{\beta}^2_{31} & 0 & 0 & 0 \\ -\beta^2_{33} & -\beta^2_{32} & -\beta^2_{31} & 0 & 0 & 0 \\ -\bar{\beta}_{3} & -\bar{\beta}_{32} & -\bar{\beta}_{31} & 0 & 0 & 0 \\ -\beta_{33} & -\beta_{32} & -\beta_{31} & 0 & 0 & 0 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} -\bar{\gamma}^0_3 & 0 & 0 & 0 & 0 & 0 \\ -\gamma^0_3 & 0 & 0 & 0 & 0 & 0 \\ -\bar{\gamma}^2_3 & 0 & 0 & 0 & 0 & 0 \\ -\gamma^2_3 & 0 & 0 & 0 & 0 & 0 \\ -\bar{\gamma}_3 & 0 & 0 & 0 & 0 & 0 \\ -\gamma_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Worthy of note is that the difference system in (3.2) can be written in the form

$$Y_{n+1} - Y_n = h\phi_\Xi(Y_n, Y_{n+1}; u, h),$$

where the subscript indicates that the dependence of ϕ on Y_n, Y_{n+1} is through the function Ξ. Thus, the numerical solution of the problem in equation (2.1) according to Abdulganiy *et al.* [35] is the one given by

$$\begin{cases} Y_{n+1} - Y_n = h\phi_{\Xi}(Y_n, Y_{n+1}; u, h), \\ Y_0 = Y(x_0), \quad n = 1, 2, \ldots, N-1. \end{cases} \tag{3.3}$$

Definition 3.1. ***Fatunla [29]*** *A block method is zero stable if the roots of the first characteristic polynomial have modulus less than or equal to one and those of modulus one do not have multiplicity greater than 2. i.e.* $\rho(R) = \det[RA_1 - A_0] = 0$ *satisfies* $|R_i| \leq 1$ *and for those roots with* $|R_i| = 1$*, the multiplicity does not exceed 2.*

Proposition 3.2. *The TFBFM is zero stable.*

Proof. Normalize equation (3.1) to obtain the first characteristic polynomial of BFFM given by $\rho_k(R) = det[RA_1 - A_0]$. So that $\rho_k(R) = 0 \Longrightarrow -R^{k-2}(1+R)^2 = 0$. Consequently, the roots $R_i,\ i = 1, 2, \ldots k$ of $\rho_k(R)$ satisfy$|R_i| = 1$, the roots are simple. Hence for each k, the TFBFM is Zero stable. □

Linear Stability and Region of Stability of TFBFM

Proposition 3.3. *The TFBFM, when applied to the Lambert-Watson test equations* $y'' = \lambda^2 y$ *and* $y''' = \lambda^3 y$ *gives*

$$Y_{n+1} = M(z,v)Y_n,$$

where

$$M(z,v) = \left(A_1 - B_1 z - C_1 z^2 - D_1 z^3\right)^{-1}(A_0 + B_0 z). \tag{3.4}$$

Proof. First, apply the TFBFM to the test equations $y'' = \lambda^2$y and $y''' = \lambda^3$y which are expressed as $f(x,y) = \lambda^2 y$ and g$(x,y) = \lambda^3$y. Since $Y_{n+1}, Y_n, F_{n+1}, F_n$ and G_{n+1} are in vectors form, then the test equations can now be written as $F = \lambda^2 Y \ and \ G = \lambda^3 Y$. Substituting for F and G to obtain a linear difference equation given by

$$A_1 Y_{n+1} = A_0 Y_n + (\lambda h)^2 B_1 Y_{n+1} + (\lambda h)^2 B_0 Y_n + (\lambda h)^3 D_1 Y_{n+1}.$$

Letting $z = \lambda$h and $v = \omega h$ to have

$$A_1 Y_{n+1} = A_0 Y_n + z^2 B_1 Y_{n+1} + z^2 B_0 Y_n + z^3 D_1 Y_{n+1},$$

it follows that

$$Y_{n+1} = M(z,v)Y_n,$$

where

$$M\left(z,v\right)=\left(A_1-B_1z-C_1z^2-D_1z^3\right)^{-1}\left(A_0+B_0z\right).$$

□

The rational function $M\left(z,u\right)$ is called the amplification matrix which determines the stability of the method.

Definition 3.2. *(Coleman and Ixaru, [19]): A region of stability is a region in the $zu-$plane throughout which $|\rho\left(z,v\right)|\leq 1$, where $\rho\left(z,u\right)$ is the spectral radius of $M\left(z,u\right)$.*

Linear Stability and Region of Stability of TFBFM

To analyze the linear stability of TFBFM, the block method in equation (3.2) is applied to the Lambert-Watson test equation $y^{''}=\lambda^2y$. After simple algebraic calculations and letting $z=\lambda h$, we obtain

$$Y_{n+1}=M(z,u)Y_n,$$

where

$$M\left(z,u\right)=\left(A_1-B_1z^2\right)^{-1}\left(A_0+B_0z^2\right). \tag{3.5}$$

The rational function $M\left(z,u\right)$ is called the amplification matrix and determines the stability of the method.

Definition 3.3. *(Coleman and Ixaru, [19]): A region of stability is a region in the $zu-$plane throughout which $|\rho\left(z,u\right)|\leq 1$, where $\rho\left(z,u\right)$ is the spectral radius of $M\left(z,u\right)$.*

Here the colored region (blue) is the stability region corresponding to the test problem $y''=\lambda^2y$. Since the Lambert-Watson test does not contain the first derivative, another usual test equation to analyze linear stability is the one given by

$$y''=-2\lambda y'-\lambda^2y \tag{3.6}$$

which has bounded solutions for $\lambda\geq 0$ that tend to zero when $x\to\infty$. The corresponding stability region for the TFBFM $k=2$ is plotted in Figure 2, where the colored region (green) is the stability region corresponding to the test problem $y''=-2\lambda y'-\lambda^2y$.

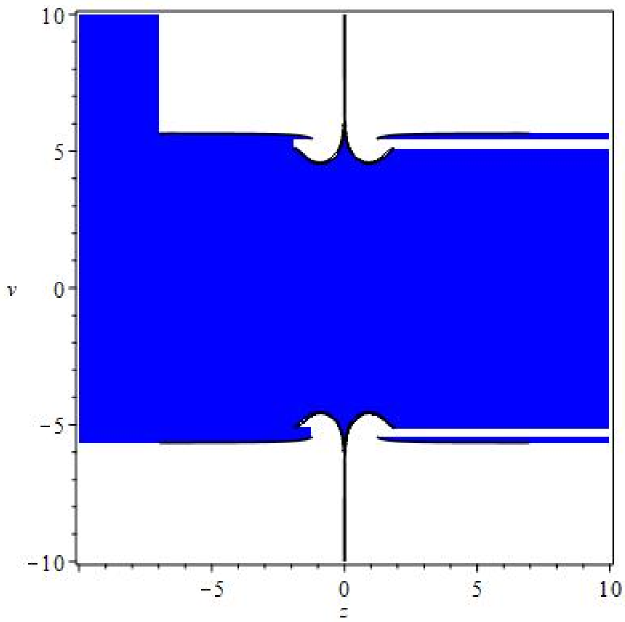

FIGURE 1. $z - v$ stability region of TFBFM for $k = 2$

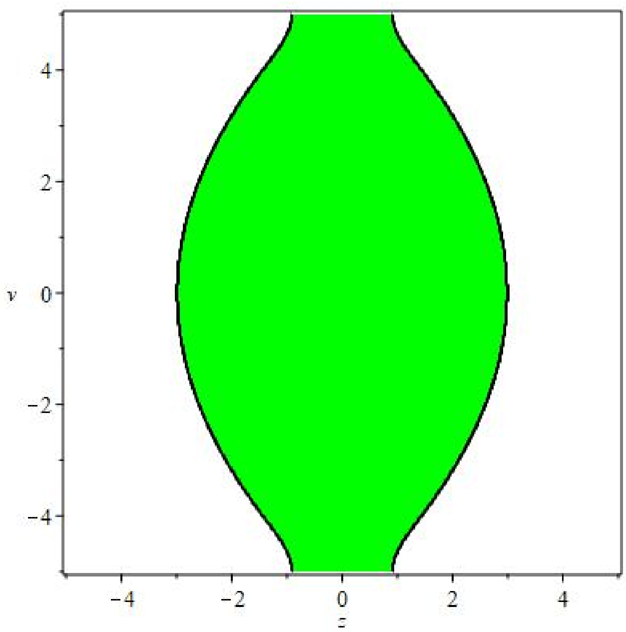

FIGURE 2. $z - v$ stability region of TFBFM for $k = 2$

4. Implementation and Numerical Experiments

Implementation of TFBFM

The TFBFM is implemented using a written code in Maple 2016.1 enhanced by the feature $fsolve$ for both linear nonlinear problems respectively. All numerical experiments are conducted on a Laptop with the following features:

(1) 64 bit Windows 10 Pro Operating System,
(2) Intel (R) Celeron CPU N3060 @ 1.60GHz processor, and
(3) 4.00GB RAM memory.

The summary of how TFBFM is applied to solve initial value problems (IVPs) with oscillatory solutions in a block by-block fashion is as follows:

Step 1: Choose N, $h = (x_N - x_0)/N$ to form the grid $\Gamma_N = \{x_0, x_1, \ldots, x_N\}$ with $x_i = x_0 + ih$. Note that N must be a multiple of $k, N = mk$.

Step 2: Using the difference equation (3.2), $n = 0$, solve for the values of $(y_1, y_2, \ldots, y_k)^T$ and $(y'_1, y'_2, \cdots, y'_k)^T$ simultaneously on the block sub-interval $[x_0, x_k]$, as y_0 and y'_0 are known from the IVP (2.1). We outline the procedure with $k = 2$ for the two first block intervals, when $n = 0$ and $n = 2$,

Step 3: Next, for $n = k$, the values of $(y_{k+1}, y_{k+2}, \ldots, y_{2k})^T$ and $(y'_{k+1}, y'_{k+2}, \ldots, y'_{2k})^T$ are simultaneously obtained over the block sub-interval $[x_k, x_{2k}]$, as y_k and y'_k are known from the previous block.

Step 4: The process is continued for $2k, 3k, \ldots, (N-1)k$ to obtain the numerical solution to (2) on the sub-intervals $[x_0, x_k]$, $[x_k, x_{2k}]$, ... , $[x_{N-k}, x_N]$.

Numerical Examples

To examine the numerical effectiveness of the newly constructed block Falkner methods adapted to general oscillatory initial value problem, we carry out experiments with the TFBFM for $k = 2$ on some well known oscillatory problems that were solved in the recent literature. The accuracy is investigated using the maximum error of the approximate solution defined as $Error = \max_{1 \le n \le N} \|y(x) - y_n\|$, where $y(x)$ is the exact solution and y_n is the numerical solution obtained using TFBFM, while the computational efficiency is shown through the plots of the maximum errors versus the number of function evaluations, NFE, required by each integrator. We emphasize that the fitting frequencies used in the numerical experiments have been obtained from the problems referenced from the literature. For both linear and nonlinear problems, the $TFBFM$ is implemented using written code in Maple 2016.1, which has been improved by the function $fsolve$.

4.1. **Example 1.** Problems involving general second order IVPs:

$$y'' = f(x, y, y'),\, y(x_0) = y_0,\, y'(x_0) = y'_0.$$

Consider the popular Van der Pol equation given by: $y'' + y = \delta(1 - y^2)\, y'$, with initial values $y(0) = 2 + \frac{1}{96}\delta^2 + \frac{1033}{552960}\delta^4 + \frac{1019689}{55738368000}\delta^2,\, y'(0) = 0.$

This is a nonlinear scalar equation. In the numerical experiment, the parameter δ is selected as $\delta = 10^{-3}$ and the principal frequency is chosen as $\omega = 1$. The problem is integrated in the interval $[0, 100]$. For the comparison of error of different methods, the step lengths $h = \frac{1}{2^i}, i = 1, 2, 3, 4$ are considered. It is emphasized that the analytic solution of this problem does not exists, thus, a reference numerical solution which is obtained via special perturbation approach is used. The TFBFM for $k = 2$ results in comparison with the Block Falkner methods (BFM) of order 5 in Ramos *et al.* [30], Modified Block Falkner methods (MBFM) of order 5 in Ehigie and Okunuga [15], and The two-stage and three-stage Two-derivative Runge-Kutta-Nystrom Methods (TDRKN2 and TDRKN3) of orders 4 and 5 respectively are displayed in Table 1, while the efficiency curves are displayed in Figure (3) respectively. It is evident from the Table (1) and Figure (3) that the family of TFBFM performs better than some of the existing methods in the literature.

TABLE 1. Data for Example 1 with $\omega = 1, \delta = 10^{-3}$

h	TFBFM2		BFM		MBFM		TDRKN2		TDRKN3	
	Error	NFE	Error	NFE	Error	NFE	Error	NFE	Error	NFE
$\frac{1}{2}$	5.21×10^{-6}	101	1.38×10^{-2}	101	1.23×10^{-4}	101	1.00×10^{-2}	603	0.75×10^{-4}	631
$\frac{1}{4}$	1.13×10^{-7}	201	2.45×10^{-4}	201	9.55×10^{-7}	201	1.00×10^{-3}	1202	3.98×10^{-6}	1230
$\frac{1}{8}$	1.00×10^{-8}	401	3.98×10^{-6}	401	9.12×10^{-9}	401	1.00×10^{-4}	2344	1.00×10^{-7}	2455
$\frac{1}{16}$	2.80×10^{-10}	801	6.31×10^{-8}	801	5.25×10^{-10}	801	1.00×10^{-5}	4786	1.00×10^{-9}	5012

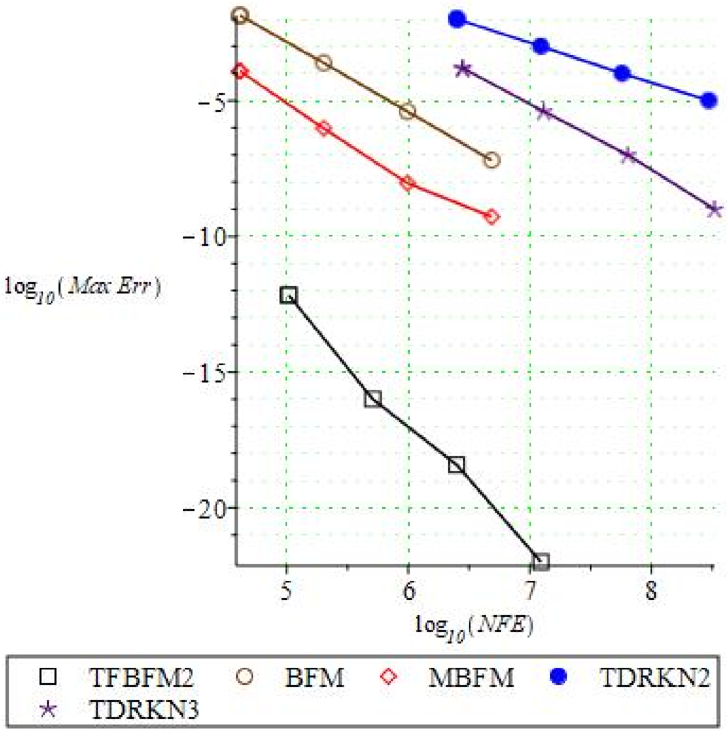

FIGURE 3. Efficiency Curves for Example 1

4.2. **Example 2.** Consider the following general second order IVP

$$y'' + \omega^2 y = -\delta y'$$

with initial conditions $y(0) = 1$ and $y'(0) = -\frac{\delta}{2}$, with analytical solution $y(x) = e^{-\left(\frac{\delta}{2}\right)x} \cos\left(x\sqrt{\omega^2 - \frac{\delta^2}{4}}\right)$.

This problem is solved in the interval [0,100] with $\omega = 1, \delta = 10^{-3}$ and compare the result of TFBFM with the BNM of order 5 in Jator and Oladejo [7], BHT of order 5 and BHTRKNM of order 3 in Ngwane and Jator ([31], [32]) . Table (2) shows the Maximum errors and the Number of Function Evaluations while the efficiency curves of the family of TFBFM are presented in the Figure 4 showing the superiority of the methods over some of the existing methods in the scientific literature.

TABLE 2. Data for Example 2 with $\omega = 1, \delta = 10^{-3}$

h	TFBFM2		BHT		BHTRKNM		BNM	
	Error	NFE	Error	NFE	Error	NFE	Error	NFE
2	2.31×10^{-10}	51	2.74×10^{-4}	26	6.48×10^{-4}	26	6.46×10^{-3}	26
1	2.82×10^{-11}	101	6.34×10^{-6}	51	4.39×10^{-5}	51	1.17×10^{-4}	51
$\frac{1}{2}$	1.79×10^{-12}	101	1.16×10^{-7}	101	2.99×10^{-6}	101	1.88×10^{-6}	101
$\frac{1}{4}$	1.06×10^{-13}	201	1.85×10^{-9}	201	1.88×10^{-7}	201	2.96×10^{-8}	201
$\frac{1}{8}$	6.62×10^{-15}	401	2.92×10^{-11}	401	1.18×10^{-8}	401	4.46×10^{-10}	401

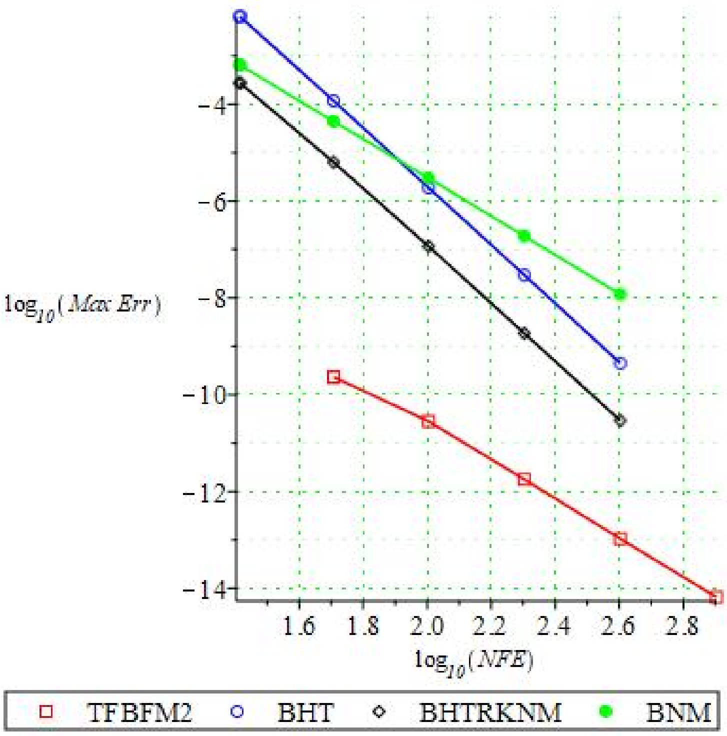

FIGURE 4. Efficiency Curves for Example 2

4.3. **Example 3.** Problems involving special second order IVP $y'' = f(x, y)$, $y(x_0) = y_0$, $y'(x_0) = y'_0$.

The following well known mildly stiff Kepler problem

$$
\begin{aligned}
y_1''(x) &= -\frac{y_1}{r^3},\ y_1(0) = 1,\ y_1'(0) = 0 \\
y_2''(x) &= -\frac{y_2}{r^3},\ y_2(0) = 0,\ y_2'(0) = 0
\end{aligned} \tag{4.1}
$$

where $r = \sqrt{y_1^2 + y_2^2}$ and whose analytic solution is given by $y_1(x) = \cos(x)$,$y_2(x) = \sin(x)$ is considered in the integration interval $0 \leq x \leq 30$. The fitting frequency ω is chosen as $\omega = 1$ and the step size h is chosen as $h = 1/2^i$, where $i = 2, 3, 4, 5, 6$. Whereas the accuracy of the family of TFBFM in comparison with the MBFM presented by Ehigie and Okunuga [14] and the TBNM of Jator et al [6]. is presented in Table 3, the efficiency is presented visually in Figure 5 showing the advantage of the Family of TFBFM.

TABLE 3. Data for Example 3 with $\omega = 1, \delta = 10^{-3}$

h	TFBFM2		MBFM		TBNM	
	Error	NFE	Error	NFE	Error	NFE
$\frac{1}{4}$	1.18×10^{-18}	26	9.00×10^{-13}	31	1.90×10^{-12}	31
$\frac{1}{8}$	4.54×10^{-21}	51	5.80×10^{-14}	61	3.40×10^{-14}	61
$\frac{1}{16}$	6.09×10^{-22}	101	1.20×10^{-14}	121	9.20×10^{-14}	121
$\frac{1}{32}$	1.82×10^{-24}	201	1.90×10^{-16}	241	1.10×10^{-15}	241
$\frac{1}{64}$	4.37×10^{-25}	401	1.00×10^{-17}	481	1.00×10^{-17}	481

4.3.1. *Example 4.* Consider the following Undamped Duffing Equation

$$\begin{cases} y'' + y^3 + y = (\cos(x) + \epsilon \sin(10x))^3 - 99\epsilon \sin(10x),\ 0 \leq x \leq 1000 \\ y(0) = 1,\ y'(0) = 10\epsilon \end{cases},$$

whose analytic solution is $y(x) = \cos(x) + \epsilon \sin(10x)$. For this problem, $\omega = 1$ is selected as principal frequency with parameter $\epsilon = 10^{-10}$. Table 4 shows the performnce of the family of TFBFM in comparison with the TFARKN by Fang *et al.* [23], the EFRK by Franco [24] and the EFRKN by Franco [25] respectively. The efficiency curves of the BFFM and the other methods used for comparisons are displayed in Figure 6.

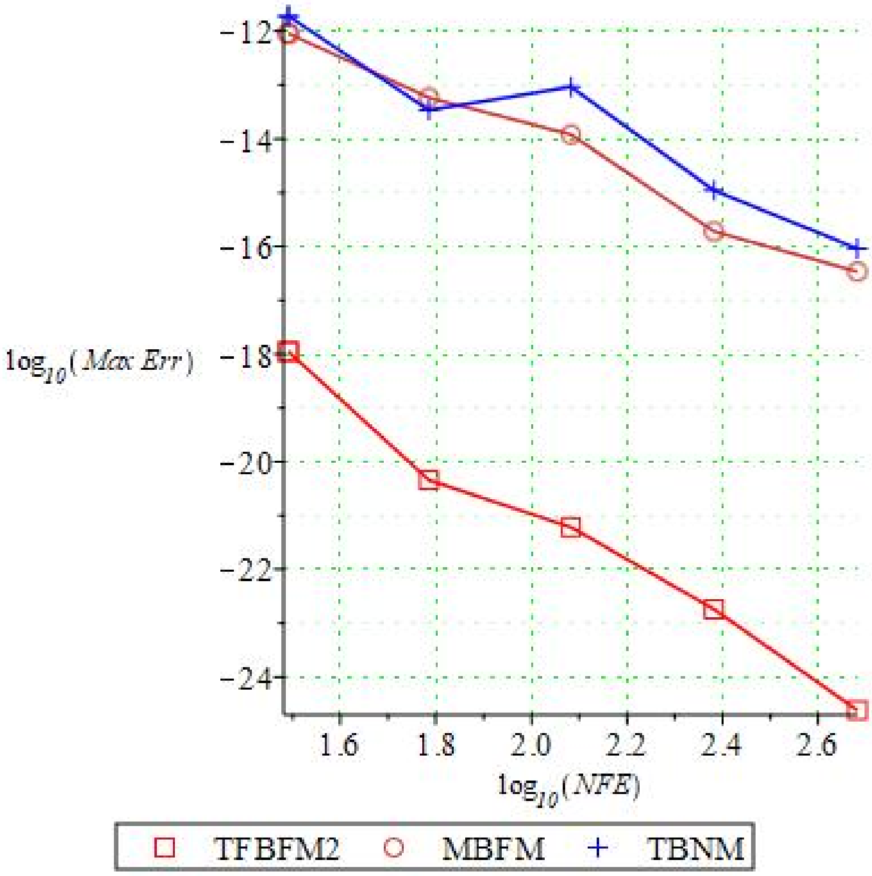

FIGURE 5. Efficiency Curves for Example 3

TABLE 4. Data for Example 4 with $\omega = 1$ and $\epsilon = 10^{-10}$

TFBFM2		TFARKN		EFRK		EFRKN	
Error	NFE	Error	NFE	Error	NFE	Error	NFE
1.55×10^{-9}	301	2.63×10^{-2}	300	1.26×10^{-6}	8000	7.94×10^{-6}	2000
1.20×10^{-11}	601	4.47×10^{-6}	400	0.75×10^{-7}	14000	0.75×10^{-7}	5000
6.02×10^{-13}	1201	3.72×10^{-8}	600	0.75×10^{-8}	22000	1.26×10^{-8}	9000
3.62×10^{-14}	2401	1.17×10^{-13}	4200	6.31×10^{-9}	38000	1.00×10^{-9}	19000

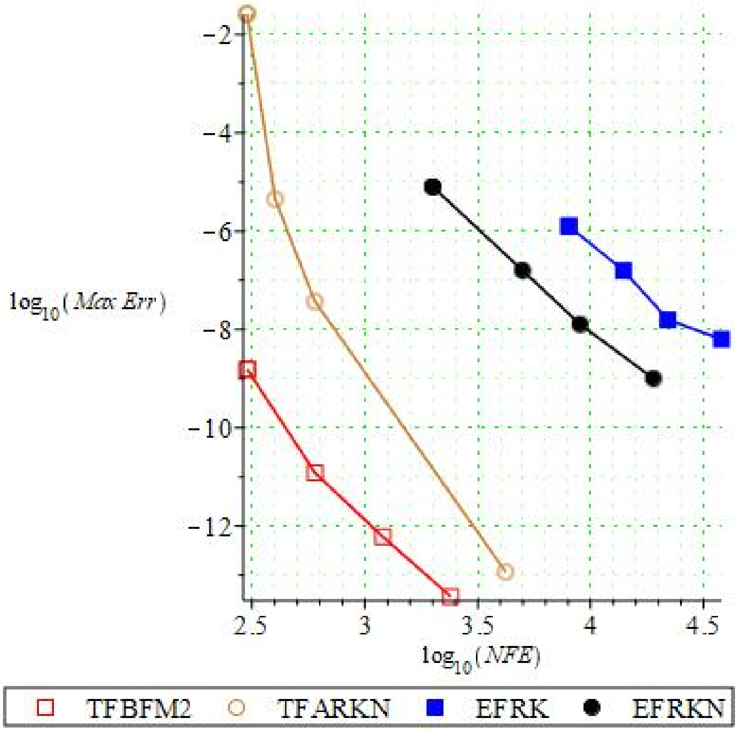

FIGURE 6. Efficiency Curves for Example 4

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APPENDIX A. SPECIFICATION OF ENTRIES OF MATRIX Ω, THE DETERMINANT OF Ω AND DETERMINANTS OF ξ_i

$$\Omega = \begin{pmatrix} 1 & x_{n+1} & {x_{n+1}}^2 & {x_{n+1}}^3 & \cos(\omega\, x_{n+1}) & \sin(\omega\, x_{n+1}) \\ 0 & 1 & 2\,x_{n+1} & 3\,{x_{n+1}}^2 & -\sin(\omega\, x_{n+1})\,\omega & \cos(\omega\, x_{n+1})\,\omega \\ 0 & 0 & 2 & 6\,x_n & -\cos(\omega\, x_n)\,\omega^2 & -\sin(\omega\, x_n)\,\omega^2 \\ 0 & 0 & 2 & 6\,x_{n+1} & -\cos(\omega\, x_{n+1})\,\omega^2 & -\sin(\omega\, x_{n+1})\,\omega^2 \\ 0 & 0 & 2 & 6\,x_{n+2} & -\cos(\omega\, x_{n+2})\,\omega^2 & -\sin(\omega\, x_{n+2})\,\omega^2 \\ 0 & 0 & 0 & 6 & \sin(\omega\, x_{n+2})\,\omega^3 & -\cos(\omega\, x_{n+2})\,\omega^3 \end{pmatrix},$$

$$\xi = \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{pmatrix}, \Lambda = \begin{pmatrix} y_{n+1} \\ y'_{n+1} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ g_{n+2} \end{pmatrix}.$$

$$\begin{aligned}
\xi_0 = {} & [8\,v^3((f_{n+1} - 1/4\,f_{n+2})h^3 + ((9/4\,f_{n+1} - 3/4\,f_{n+2})x_n - 3/2\,\delta_{n+1})h^2 \\
& + ((3/2\,f_{n+1} - 3/4\,f_{n+2}){x_n}^2 - 3/2\,\delta_{n+1}x_n + 3/2\,y_{n+1})h \\
& + 1/4\,{x_n}^3(f_{n+1} - f_{n+2}))(\cos(v))^2 \\
& + 2\,(v(h^3 g_{n+2} + (3\,x_n g_{n+2} - 3\,f_{n+1})h^2 + (3\,g_{n+2}{x_n}^2 - 6\,f_{n+1}x_n + 6\,\delta_{n+1})h \\
& + {x_n}^3 g_{n+2} - 3\,{x_n}^2 f_{n+1} + 6\,\delta_{n+1}x_n - 6\,y_{n+1})h\sin(v) \\
& + ((-2\,f_n - f_{n+2})h^3 + ((-9/2\,f_n - 3/2\,f_{n+2})x_n + 6\,\delta_{n+1})h^2 \\
& + (-3\,f_n{x_n}^2 + 6\,y'_{n+1}x_n - 6\,y_{n+1})h - 1/2\,{x_n}^3(f_n - f_{n+2}))v^2 \\
& - 6\,(h^2 g_{n+2} + (x_n g_{n+2} + f_{n+1} - f_{n+2})h + 1/2\,x_n(f_n - f_{n+2}))h^2)v\cos(v) \\
& - 2\,((h^3 g_{n+2} + (3\,x_n g_{n+2} + 3/2\,f_n - 6\,f_{n+1} + 3/2\,f_{n+2})h^2 \\
& + (3\,g_{n+2}{x_n}^2 - 6\,f_{n+1}x_n + 6\,y'_{n+1})h + {x_n}^3 g_{n+2} \\
& + (-3/2\,f_n - 3/2\,f_{n+2}){x_n}^2 + 6\,y'_{n+1}x_n \\
& - 6\,y_{n+1})v^2 + 3\,h^2(f_n - 2\,f_{n+1} + f_{n+2}))h\sin(v) \\
& + 12\,v(x_n + h)(1/12\,(x_n + h)^2(f_n - 2\,f_{n+1} + f_{n+2})v^2 \\
& + h^2(hg_{n+2} + 1/2\,f_n - 1/2\,f_{n+2}))] \\
& \cdot [12\,v^2 h(\cos(v) - 1)(v\cos(v) - \sin(v))]^{-1}
\end{aligned}$$

$$\begin{aligned}
\xi_1 = {} & [-6\,v^2((f_{n+1} - 1/3\,f_{n+2})h^2 + ((4/3\,f_{n+1} - 2/3\,f_{n+2})x_n - 2/3\,y'_{n+1})h \\
& + 1/3\,{x_n}^2(f_{n+1} - f_{n+2}))(\cos(v))^2 \\
& + (-2\,(h^2 g_{n+2} + (2\,x_n g_{n+2} - 2\,f_{n+1})h + g_{n+2}{x_n}^2 - 2\,f_{n+1}x_n + 2\,y'_{n+1})vh\sin(v) \\
& + ((3\,f_n + f_{n+2})h^2 + (4\,f_n x_n - 4\,y'_{n+1})h + {x_n}^2(f_n - f_{n+2}))v^2 \\
& + 4\,h^2(hg_{n+2} + 1/2\,f_n - 1/2\,f_{n+2}))\cos(v) \\
& + 2\,(h^2 g_{n+2} + (2\,x_n g_{n+2} - 2\,f_{n+1})h + g_{n+2}{x_n}^2 + (-f_n - f_{n+2})x_n + 2\,y'_{n+1})vh\sin(v) \\
& - (x_n + h)^2(f_n - 2\,f_{n+1} + f_{n+2})v^2 - 4\,h^2(hg_{n+2} + 1/2\,f_n - 1/2\,f_{n+2})] \\
& \cdot [4\,vh(\cos(v) - 1)(v\cos(v) - \sin(v))]^{-1}
\end{aligned}$$

$$\begin{aligned}\xi_2 = {} & [4\,((f_{n+1} - 1/2\, f_{n+2})h + 1/2\, x_n(f_{n+1} - f_{n+2}))v(\cos(v))^2 \\ & + (2\,h(hg_{n+2} + x_n g_{n+2} - f_{n+1})\sin(v) - 2\,(hf_n + 1/2\, x_n(f_n - f_{n+2}))v)\cos(v) \\ & - 2\,(hg_{n+2} + x_n g_{n+2} - 1/2\, f_n - 1/2\, f_{n+2})h\sin(v) + v(f_n - 2\, f_{n+1} + f_{n+2})(x_n + h)] \\ & \cdot [4\,h(\cos(v) - 1)(v\cos(v) - \sin(v))]^{-1}\end{aligned}$$

$$\xi_3 = \frac{-2\,v(f_{n+1} - f_{n+2})\cos(v) - 2\,h\sin(v)g_{n+2} + v(f_n - 2\, f_{n+1} + f_{n+2})}{12(v\cos(v) - \sin(v))h}$$

$$\begin{aligned}\xi_4 = {} & [-\cos(\frac{v(2\,h + x_n)}{h})h^3 g_{n+2} + 2\,\cos(\frac{v(x_n + h)}{h})h^3 g_{n+2} \\ & - \sin(\frac{v(2\,h + x_n)}{h})h^2 v f_n + 2\,\sin(\frac{v(2\,h + x_n)}{h})h^2 v f_{n+1} \\ & - \sin(\frac{v(2\,h + x_n)}{h})h^2 v f_{n+2} - \cos(\frac{vx_n}{h})h^3 g_{n+2} - \cos(\frac{v(2\,h + x_n)}{h})h^2 f_n \\ & + \cos(\frac{v(2\,h + x_n)}{h})h^2 f_{n+1} + \cos(\frac{v(x_n + h)v}{h})h^2 f_n - \cos(\frac{v(x_n + h)}{h})h^2 f_{n+2} \\ & - \cos(\frac{vx_n}{h})h^2 f_{n+1} + \cos(\frac{vx_n}{h})h^2 f_{n+2}] \\ & \cdot [4\,h(\cos(v) - 1)(v\cos(v) - \sin(v))]^{-1}\end{aligned}$$

$$\begin{aligned}\xi_5 = {} & [-\cos(\frac{v(2\,h + x_n)}{h})h^3 g_{n+2} + 2\,\cos(\frac{v(x_n + h)}{h})h^3 g_{n+2} - \sin(\frac{v(2\,h + x_n)}{h})h^2 v f_n \\ & + 2\,\sin(\frac{v(2\,h + x_n)}{h})h^2 v f_{n+1} - \sin(\frac{v(2\,h + x_n)}{h})h^2 v f_{n+2} - \cos(\frac{vx_n}{h})h^3 g_{n+2} \\ & - \cos(\frac{v(2\,h + x_n)}{h})h^2 f_n + \cos(\frac{v(2\,h + x_n)}{h})h^2 f_{n+1} + \cos(\frac{v(x_n + h)}{h})h^2 f_n \\ & - \cos(\frac{v(x_n + h)}{h})h^2 f_{n+2} - \cos(\frac{vx_n}{h})h^2 f_{n+1} + \cos(\frac{vx_n}{h})h^2 f_{n+2}] \\ & \cdot [2\,v^2(\cos(v) - 1)(v\cos(v) - \sin(v))]^{-1}\end{aligned}$$

Appendix B. Coefficients of the main and complementary methods of the TFBFM for $k = 2$

$$\text{(B.1)}\quad \begin{cases} \beta_{20} = \frac{2\cos(v)v^2 - 9\sin(v)v + v^2 - 12\cos(v) + 12}{6v(2v\cos(v) - v\cos(2v) - 2\sin(v) + \sin(2v) - v)} \\ \beta_{21} = \frac{\left(3v^2+6\right)\sin(2v) - 2\cos(2v)v^3 - 4v^3 + 12v^2\sin(v) + 12v\cos(v) - 12v - 12\sin(v)}{6(2v\cos(v) - v\cos(2v) - 2\sin(v) + \sin(2v) - v)v^2} \\ \beta_{22} = \frac{4v^3\cos(v) - \cos(2v)v^3 - 9v^2\sin(v) + 12\sin(v) - 6\sin(2v)}{6(-v + 2v\cos(v) - v\cos(2v) - 2\sin(v) + \sin(2v))v^2} \\ \gamma_2 = \frac{\left(v^2+6\right)\sin(2v) - 2v^2\sin(v) - 12v\cos(v) + 12v - 12\sin(v)}{6(-v + 2v\cos(v) - v\cos(2v) - 2\sin(v) + \sin(2v))v^2} \end{cases}$$

$$\text{(B.2)}\quad \begin{cases} \bar{\beta}_{20} = \frac{\cos(v)v^2 - 4\sin(v)v + v^2 - 4\cos(v) + 4}{2v(-v + 2v\cos(v) - v\cos(2v) - 2\sin(v) + \sin(2v))} \\ \bar{\beta}_{21} = \frac{-\cos(2v)v^2 + 4\sin(v)v + 2\sin(2v)v - 3v^2 + 2\cos(2v) - 2}{2v(-v + 2v\cos(v) - v\cos(2v) - 2\sin(v) + \sin(2v))} \\ \bar{\beta}_{22} = \frac{3\cos(v)v^2 - \cos(2v)v^2 - 4\sin(v)v + 4\cos(v) - 2\cos(2v) - 2}{2v(-v + 2v\cos(v) - v\cos(2v) - 2\sin(v) + \sin(2v))} \\ \bar{\gamma}_2 = \frac{-2\sin(v)v + \sin(2v)v - 8\cos(v) + 2\cos(2v) + 6}{2v(-v + 2v\cos(v) - v\cos(2v) - 2\sin(v) + \sin(2v))} \end{cases}$$

$$\text{(B.3)}\quad \begin{cases} \beta_{20}^0 = \frac{-12v(\cos(v))^2 + \left(-4v^3 + 12\sin(v)\right)\cos(v) + v^3 - 3v^2\sin(v) + 12v - 12\sin(v)}{12v^2(\cos(v) - 1)(v\cos(v) - \sin(v))} \\ \beta_{21}^0 = \frac{\left(4v^3 + 12v\right)(\cos(v))^2 + \left(-3v^2\sin(v) - 6v - 6\sin(v)\right)\cos(v) - v^3 + 6v^2\sin(v) - 6v + 6\sin(v)}{6v^2(\cos(v) - 1)(v\cos(v) - \sin(v))} \\ \beta_{22}^0 = \frac{-2v^2(\cos(v))^2 - 2\cos(v)v^2 - 12(\cos(v))^2 - 3\sin(v)v + v^2 + 12\cos(v)}{12v(\cos(v) - 1)(v\cos(v) - \sin(v))} \\ \gamma_2^0 = \frac{v^2\sin(v) + 6\sin(v) - 6v}{6v^2(v\cos(v) - \sin(v))} \end{cases}$$

$$\text{(B.4)}\quad \begin{cases} \bar{\beta}_{20}^0 = \frac{-3\cos(v)v^2 + 2\sin(2v)v + v^2 - 4\cos(v) + 2\cos(2v) + 2}{2v(-v + 2v\cos(v) - v\cos(2v) - 2\sin(v) + \sin(2v))} \\ \bar{\beta}_{21}^0 = \frac{3\cos(2v)v^2 + 4\sin(v)v - 6\sin(2v)v + v^2 - 2\cos(2v) + 2}{2v(-v + 2v\cos(v) - v\cos(2v) - 2\sin(v) + \sin(2v))} \\ \bar{\beta}_{22}^0 = \frac{-\cos(v)v^2 - \cos(2v)v^2 + 2\sin(2v)v + 4\cos(v) - 4}{2v(-v + 2v\cos(v) - v\cos(2v) - 2\sin(v) + \sin(2v))} \\ \bar{\gamma}_2^0 = \frac{-2\sin(v)v + \sin(2v)v - 8\cos(v) + 2\cos(2v) + 6}{2v(-v + 2v\cos(v) - v\cos(2v) - 2\sin(v) + \sin(2v))} \end{cases}$$

As $v \to 0$, the Taylor series expansion of coefficients of the TFBFM for $k = 2$ up to the eight order are as follows:

(B.5)
$$\left\{\begin{aligned}
\beta_{20} &= -\frac{1}{80} - \frac{83\,v^2}{50400} - \frac{89\,v^4}{672000} - \frac{39887\,v^6}{4656960000} - \frac{5384573\,v^8}{10897286400000}\\
\beta_{21} &= \frac{3}{10} + \frac{13\,v^2}{12600} + \frac{59\,v^4}{378000} + \frac{1523\,v^6}{129360000} + \frac{985289\,v^8}{1362160800000}\\
\beta_{22} &= \frac{17}{80} + \frac{31\,v^2}{50400} - \frac{143\,v^4}{6048000} - \frac{14941\,v^6}{4656960000} - \frac{2497739\,v^8}{10897286400000}\\
\gamma_2 &= -\frac{7}{120} - \frac{19\,v^2}{8400} - \frac{47\,v^4}{432000} - \frac{12473\,v^6}{2328480000} - \frac{481139\,v^8}{1816214400000}
\end{aligned}\right.$$

(B.6)
$$\left\{\begin{aligned}
\bar\beta_{20} &= -\frac{1}{48} - \frac{v^2}{360} - \frac{13\,v^4}{57600} - \frac{89\,v^6}{6048000} - \frac{143203\,v^8}{167650560000}\\
\bar\beta_{21} &= \frac{5}{12} + \frac{v^2}{720} + \frac{13\,v^4}{50400} + \frac{121\,v^6}{6048000} + \frac{52133\,v^8}{41912640000}\\
\bar\beta_{22} &= \frac{29}{48} + \frac{v^2}{720} - \frac{13\,v^4}{403200} - \frac{v^6}{189000} - \frac{5939\,v^8}{15240960000}\\
\bar\gamma_2 &= -\frac{1}{8} - \frac{v^2}{240} - \frac{13\,v^4}{67200} - \frac{19\,v^6}{2016000} - \frac{12979\,v^8}{27941760000}
\end{aligned}\right.$$

(B.7)
$$\left\{\begin{aligned}
\beta^0_{20} &= \frac{23}{240} + \frac{293\,v^2}{50400} + \frac{1801\,v^4}{6048000} + \frac{22279\,v^6}{1552320000} + \frac{675643\,v^8}{990662400000}\\
\beta^0_{21} &= \frac{8}{15} - \frac{59\,v^2}{6300} - \frac{23\,v^4}{47250} - \frac{13591\,v^6}{582120000} - \frac{374291\,v^8}{340540200000}\\
\beta^0_{22} &= -\frac{31}{240} + \frac{179\,v^2}{50400} + \frac{127\,v^4}{672000} + \frac{41891\,v^6}{4656960000} + \frac{4545239\,v^8}{10897286400000}\\
\gamma^0_2 &= \frac{7}{120} + \frac{19\,v^2}{8400} + \frac{47\,v^4}{432000} + \frac{12473\,v^6}{2328480000} + \frac{481139\,v^8}{1816214400000}
\end{aligned}\right.$$

(B.8)
$$\left\{\begin{aligned}
\bar\beta^0_{20} &= -\frac{17}{48} - \frac{v^2}{72} - \frac{251\,v^4}{403200} - \frac{169\,v^6}{6048000} - \frac{213203\,v^8}{167650560000}\\
\bar\beta^0_{21} &= -\frac{11}{12} + \frac{17\,v^2}{720} + \frac{53\,v^4}{50400} + \frac{281\,v^6}{6048000} + \frac{87133\,v^8}{41912640000}\\
\bar\beta^0_{22} &= \frac{13}{48} - \frac{7\,v^2}{720} - \frac{173\,v^4}{403200} - \frac{v^6}{54000} - \frac{135329\,v^8}{167650560000}\\
\bar\gamma^0{}_2 &= -\frac{1}{8} - \frac{v^2}{240} - \frac{13\,v^4}{67200} - \frac{19\,v^6}{2016000} - \frac{12979\,v^8}{27941760000}
\end{aligned}\right.$$

APPENDIX C. MATRICES $R\,A_1 - A_0$ FOR $k = 2$

$$[RA_1 - A_0]_{k=2} = \begin{bmatrix} 0 & R-1 & 0 & -R \\ R & R & 0 & R \\ 0 & 0 & 0 & -R-1 \\ 0 & 0 & R & R \end{bmatrix},$$

For $k=2$, when $n=0$, the equation (3.2) becomes

$$A_1Y_1 = A_0Y_0 + h^2B_0F_0 + h^2B_1F_1, \tag{C.1}$$

where

$$\begin{aligned} Y_1 &= (y_1, y_2, hy_1', hy_2')^T, \\ Y_0 &= \left(y_{-1}, y_0, hy_{-1}', hy_0'\right)^T, \\ F_1 &= (f_1, f_2,\ hg_1,\ hg_2)^T, \\ F_0 &= (f_{-1}, f_0, hg_{-1}, hg_0)^T \end{aligned}$$

Substituting for the square matrices A_0, A_1, B_0 and B_1 in equation (C.1) to obtain

$$\begin{cases} y_2 - hy_2' = y_0 + h^2 \sum\limits_{j=0}^{2} \left(\beta_{2j}^0 (u)\, y_j'' + h\gamma_2^0 (u)\, y_2'''\right) \\ y_1 + y_2 + hy_2' = h^2 \sum\limits_{j=0}^{2} \left(\beta_{2j} (u)\, y_j'' + h\gamma_2 (u)\, y_2'''\right) \\ -hy_1' = hy_0' + h^2 \sum\limits_{j=0}^{2} \left(\bar{\beta}_{2j}^0 y_j'' (u) + h\bar{\gamma}_2^0 (u)\, y_2'''\right) \\ -hy_1' + hy_2' = h^2 \sum\limits_{j=0}^{2} \left(\bar{\beta}_{2j} y_j'' (u) + h\bar{\gamma}_2 (u)\, y_2'''\right). \end{cases} \tag{C.2}$$

Solve equation (C.2) simultaneously to obtain the values of $(y_1, y_2, y'_1, y'_2)^T$ on the block sub-interval $[x_0, x_2]$, as y_0 and y'_0 are known from the IVP (2.1), $y'' = f(x, y, y')$ and y''' is the derivative of y''.

When $n=2$, the equation (3.2) becomes

$$A_1Y_3 = A_0Y_2 + h^2B_0F_2 + h^2B_1F_3, \tag{C.3}$$

where

$$\begin{aligned} Y_3 &= (y_3, y_4, hy_3', hy_4')^T, \\ Y_2 &= (y_1, y_2, hy_1', hy_2')^T, \\ F_3 &= (f_3, f_4,\ hg_3,\ hg_4)^T, \\ F_2 &= (f_1, f_2, hg_1, hg_2)^T \end{aligned}$$

Substitute for the square matrices A_0, A_1, B_0 and B_1 in equation (C.3) to obtain

$$
\text{(C.4)} \qquad \begin{cases}
y_4 - hy_4' = y_2 + h^2 \sum\limits_{j=0}^{2} \left(\beta_{2j}^0 (u)\, y_{j+2}^{''} + h\gamma_2^0 (u)\, y_4^{'''} \right) \\
y_3 + y_4 + hy_4' = h^2 \sum\limits_{j=0}^{2} \left(\beta_{2j} (u)\, y_{j+2}^{''} + h\gamma_2 (u)\, y_4^{'''} \right) \\
-hy_3' = hy_1' + h^2 \sum\limits_{j=0}^{2} \left(\bar{\beta}_{2j}^0 y_{j+2}^{''} (u) + h\bar{\gamma}_2^0 (u)\, y_4^{'''} \right) \\
-hy_3' + hy_4' = h^2 \sum\limits_{j=0}^{2} \left(\bar{\beta}_{2j} y_{j+2}^{''} (u) + h\bar{\gamma}_2 (u)\, y_4^{'''} \right).
\end{cases}
$$

Solve equation (C.4) simultaneously to obtain the values of $(y_3, y_4, {y'}_3, {y'}_4)^T$ on the block sub-interval $[x_2, x_4]$, as y_2 and ${y'}_2$ are known from the previous block.

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