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A NUMERICAL STUDY OF 2D-LANE-EMDEN PROBLEM USING 2D-BOUBAKER POLYNOMIALS

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ABSTRACT. The paper presents a numerical solution for the two-dimensional Lane-Emden problem using two-dimensional Boubaker polynomials. The method involves utilizing the operational matrix of differentiation and collocation method to convert the problem into a system of algebraic equations. The proposed approach, based on two-dimensional Boubaker polynomials operational matrices, is shown to be straightforward and effective. The validity and applicability of the method are demonstrated through illustrative examples.

1. INTRODUCTION

The Lane-Emden equations have been widely employed in mathematical physics and astronomy to describe various phenomena, particularly in the theory of stellar structure [1, 9–11, 14]. However, the conventional one-dimensional Lane-Emden equations encounter a significant challenge due to the singular behavior occurring at the origin $x = 0$. To address this limitation, this paper focuses on introducing a novel approach by considering the linear and non-linear two-dimensional Lane-Emden type equations. These equations were first introduced by Wazwaz, Rach

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and Duan in [9], as follows:

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{k_x}{x}\frac{\partial u(x,y)}{\partial x} + \frac{\partial^2 u(x,y)}{\partial y^2} + \frac{k_y}{y}\frac{\partial u(x,y)}{\partial y} + f(u(x,y)) = 0, \tag{1.1}$$

subject to the initial conditions

$$u(0,y) = h(y), \quad \frac{\partial u(0,y)}{\partial x} = 0, \quad u(x,0) = h(x), \quad \frac{\partial u(x,0)}{\partial y} = 0, \tag{1.2}$$

where $x > 0, y > 0, k_x > 0, k_y > 0$.

By extending the problem to two dimensions, we aim to explore a more comprehensive and effective solution that overcomes the singularities present in the one-dimensional counterpart. In this work, we propose a numerical solution utilizing the two-dimensional Boubaker polynomials, combined with operational matrix of differentiation and collocation method, to transform the considered two-dimensional Lane-Emden problem into a system of algebraic equations. This innovative method based on the Boubaker polynomials operational matrices promises simplicity and attractiveness in tackling the challenges posed by the 2D-Lane-Emden problem. Throughout the paper, we present illustrative examples to showcase the validity and applicability of our proposed approach, opening new possibilities for modeling diverse phenomena in mathematical physics and beyond.

The problem (1.1) involving the two-dimensional Lane-Emden type equations has been previously investigated using various numerical methods, such as the Adomian decomposition method [8], reduced differential transform method, and modified differential transform method [6]. However, in this paper, we propose a different approach to tackle the problem. Our objective is to employ the two-dimensional Boubaker operational matrix of derivatives for solving singular initial value problems associated with the two-dimensional Lane-Emden type equations (1.1). By leveraging the operational matrix of derivatives, we aim to overcome the challenges posed by the singular behavior at specific initial values. This novel method holds promise in providing more accurate and efficient solutions for the two-dimensional Lane-Emden type equations, contributing to the advancement of mathematical physics and its applications in diverse fields. Throughout the paper, we present illustrative examples to demonstrate the effectiveness and applicability of our proposed approach, showcasing its potential to handle singularities and improve the accuracy of numerical solutions.

The Boubaker polynomials were established for the first time by Boubaker (2007), to solve heat equation inside a physical model. The first monomial definition of the Boubaker polynomials with two variables on interval $\Omega = [0,1] \times [0,1]$, was introduced by [2–5]:

$$B_{i,j}(x,y) = B_i(x)B_j(y), \quad B_i(x) = \sum_{p=0}^{\xi(i)}[\frac{(i-4p)}{(i-p)}C_{i-p}^{p}](-1)^p x^{i-2p}, \tag{1.3}$$

where $\xi(i) = \lfloor \frac{i}{2} \rfloor$ is denotes the floor function.

The space $L^2(\Omega)$ denotes an inner product space that the inner product in this space is defined as follows $\langle u(x,y), v(x,y)\rangle = \int_0^1\int_0^1 u(x,y)v(x,y)dxdy$ and the norm in this space is defined as

$$\|u(x,y)\|_2 = \left(\int_0^1\int_0^1 u(x,y)^2 dxdy\right)^{\frac{1}{2}}.$$

Assume that $\{B_{00}(x,y), B_{01}(x,y), \ldots, B_{0m}(x,y), \ldots, B_{n0}(x,y), B_{n1}(x,y), \ldots, B_{nm}(x,y)\}$ be the set of two-dimensional Boubaker polynomials and let

$$\mathbf{S} = span\left\{B_{00}(x,y), B_{01}(x,y), \ldots, B_{0m}(x,y), \ldots, B_{n0}(x,y), B_{n1}(x,y), \ldots, B_{nm}(x,y)\right\}.$$

Suppose that $u(x,y)$ be an arbitrary function in $L^2(\Omega)$. Because **S** is a finite dimensional vector space, the best approximation of $u(x,y)$ out of **S** exists, that is

$$\exists u_{nm}(x,y) \in \mathbf{S}, \forall v(x,y) \in \mathbf{S}, \|u(x,y) - u_{nm}(x,y)\|_2 \leq \|u(x,y) - v(x,y)\|_2. \tag{1.4}$$

Since $u_{nm}(x,y) \in \mathbf{S}$, there are unique coefficients such that

$$u(x,y) \simeq u_{nm}(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m} u_{ij}B_{ij}(x,y) = \mathbf{U}^T\mathbf{B}(x,y), \tag{1.5}$$

where $\mathbf{U}^T = [u_{00}, u_{01}, \ldots, u_{0m}, u_{10}, u_{11}, \ldots, u_{1m}, \ldots, u_{n0}, u_{n1}, \ldots, u_{nm}]$, and

$$\mathbf{B}(x,y) = [B_{00}(x,y), B_{01}(x,y), \ldots, B_{0m}(x,y), \ldots, B_{n0}(x,y), B_{n1}(x,y), \ldots, B_{nm}(x,y)]^T.$$

Supposing that $E_{n,m} = \int_0^1\int_0^1 (u(x,y) - u_{n,m}(x,y))^2 dxdy$ then $\lim_{n,m\to\infty} E_{n,m} = 0$.

We will construct operational matrix of derivatives $\mathbf{D}_x$ and $\mathbf{D}_y$ for the 2D-BPs are given

$$\frac{\partial \mathbf{B}(x,y)}{\partial x} = \mathbf{D}_x\mathbf{B}(x,y), \quad \frac{\partial \mathbf{B}(x,y)}{\partial y} = \mathbf{D}_y\mathbf{B}(x,y) \tag{1.6}$$

the matrice $\mathbf{D}_x$ and $\mathbf{D}_y$ are of order $(n+1)^2(m+1)^2$. In order to show the high performance of 2D-BPs operational matrix of derivative, we apply it to solve equation (1.1).

The paper is organized as follows. In Section (2), we express two-dimensional Boubaker polynomials in terms of Taylor basis, and operational matrix of derivatives is constructed. In Section(3), we use two-dimensional Boubaker polynomials method for solving two-dimensional Lane-Emden type equations. Section(4) illustrates some numerical examples to show the accuracy of this method. Finally, Section (5) concludes the paper.

2. 2D-BPs OPERATIONAL MATRIX OF DIFFERENTIATION

In the following, we calculate the operational matrices of differentiation based on 2D-BPs. Suppose that $\mathbf{B}(x,y)$ be a set of 2D-BPs defined in Equation (1). Also, $\mathbf{I}$ and $\mathbf{O}$ be the identity and zero matrices respectively, and $\otimes$ denote the Kronecker product as follows $\mathbf{B}(x,y) = \mathbf{A}\mathbf{T}_{nm}(x,y) = (\mathbf{M}\otimes\mathbf{M})\mathbf{T}_{nm}(x,y)$, where

$$\mathbf{T}_{nm}(x,y) = [1, y, y^2, \ldots, y^m, x, xy, \ldots, xy^m, \ldots, x^n, x^n y, \ldots, x^n y^m]^T, \tag{2.1}$$

and if n is odd,

$$\mathbf{M} = \begin{bmatrix} m_{0,0} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & m_{1,0} & 0 & 0 & \cdots & 0 & 0 \\ m_{2,1} & 0 & m_{2,0} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{n-1,\frac{n-1}{2}} & 0 & m_{n-1,\frac{n-3}{2}} & 0 & \cdots & m_{n-1,0} & 0 \\ 0 & m_{n,\frac{n-1}{2}} & 0 & m_{n,\frac{n-3}{2}} & \cdots & 0 & m_{n,0} \end{bmatrix}$$

if n is even,

$$\mathbf{M} = \begin{bmatrix} m_{0,0} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & m_{1,0} & 0 & 0 & \cdots & 0 & 0 \\ m_{2,1} & 0 & m_{2,0} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & m_{n-1,\frac{n-2}{2}} & 0 & m_{n-1,\frac{n-4}{2}} & \cdots & m_{n-1,0} & 0 \\ m_{N,\frac{n}{2}} & 0 & m_{n,\frac{n-2}{2}} & 0 & \cdots & 0 & m_{n,0} \end{bmatrix}$$

where

$$\mathbf{B}_i(x) = \sum_{p=0}^{\xi(n)} m_{i,p} x^{i-2p}, \quad i = 0, 1, \dots, n, p = 0, 1, \dots, \lfloor \frac{i}{2} \rfloor,$$

$$m_{i,p} = [\frac{(i-4p)}{(i-p)} C_{i-p}^{p}](-1)^p.$$

It can be observed, that **M** is a lower triangular matrix, and $|\mathbf{M}| = \Pi_{i=0}^{n} m_{i,0} = 1$ thus it is non-singular and the inverse $\mathbf{M}^{-1}$ exists.

Theorem 2.1. *Let* $\mathbf{B}(x, y)$ *be the Boubaker polynomials in two variables into* $[0, 1] \times [0, 1]$*, then we have,*

$$\frac{\partial \mathbf{B}(x,y)}{\partial x} = \mathbf{D}_x \mathbf{B}(x,y), \quad \frac{\partial \mathbf{B}(x,y)}{\partial y} = \mathbf{D}_y \mathbf{B}(x,y), \tag{2.2}$$

where

$$\mathbf{D}_x = (\mathbf{M}\mathbf{F}\mathbf{M}^{-1}) \otimes \mathbf{I}, \quad \mathbf{D}_y = \mathbf{I} \otimes (\mathbf{M}\mathbf{F}\mathbf{M}^{-1}), \tag{2.3}$$

and

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & \ddots & & \vdots \\ 0 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & n & 0 \end{bmatrix}.$$

Proof. Differentiation of vector $\mathbf{B}(x, y)$ defined in Equation (1) respect to x and y, is approximated as

$$\begin{aligned} \frac{\partial \mathbf{B}(x,y)}{\partial x} = \frac{\partial \mathbf{B}(x)}{\partial x} \otimes \mathbf{B}(y) &\simeq (MFM^{-1}\mathbf{B}(x)) \otimes (I\mathbf{B}(y)) \\ &= (\left(MFM^{-1}\right) \otimes I)(\mathbf{B}(x) \otimes \mathbf{B}(y)) \\ &= \mathbf{D}_x \mathbf{B}(x,y) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \mathbf{B}(x,y)}{\partial y} = \mathbf{B}(x) \otimes \frac{\partial \mathbf{B}(y)}{\partial y} &\simeq (I\mathbf{B}(x)) \otimes (MFM^{-1}\mathbf{B}(y)) \\ &= (I \otimes \left(MFM^{-1}\right))(\mathbf{B}(x) \otimes \mathbf{B}(y)) \\ &= \mathbf{D}_y \mathbf{B}(x,y), \end{aligned}$$

where, $\mathbf{D}_x = (MFM^{-1}) \otimes I$ and $\mathbf{D}_y = I \otimes (MFM^{-1})$ are $(n+1)^2 \times (m+1)^2$ matrices. Also, 2D-BPs operational matrix of differentiation. □

3. Solution of 2D-Lane-Emden type Equations

This section presents the derivation of the method for solving a singular initial value problems of 2D-Lane-Emden type equations.

Let us consider the 2D-Lane-Emden equation of the form [6, 8]

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{k_x}{x}\frac{\partial u(x,y)}{\partial x} + \frac{\partial^2 u(x,y)}{\partial y^2} + \frac{k_y}{y}\frac{\partial u(x,y)}{\partial y} + f(u(x,y)) = 0, \tag{3.1}$$

subject to the following initial conditions

$$u(0,y) = h(y), \quad \frac{\partial u(0,y)}{\partial x} = 0, \quad u(x,0) = h(x), \quad \frac{\partial u(x,0)}{\partial y} = 0. \tag{3.2}$$

Approximating $u(x,t)$, $f(u(x,y))$ by the 2D-BPs as

$$u(x,y) = \sum_{i=0}^{n}\sum_{i=0}^{m} c_{ij} B_{ij}(x,y) = \mathbf{U}^T \mathbf{B}(x,y),$$

$$f(u(x,y)) = f\left(\mathbf{U}^T \mathbf{B}(x,y)\right) = \mathbf{H}^T \mathbf{B}(x,y),$$

where the unknowns are

$$\mathbf{U}^T = [u_{00}, u_{01}, \ldots, u_{0n}, u_{10}, u_{11}, \ldots, u_{1m}, \ldots, u_{n0}, u_{n1}, \ldots, u_{nm}].$$

Using operational matrix of derivative, Eq. (2.2)can be written as

$$\begin{aligned} &\mathbf{U}^T \mathbf{D}_x^2 \mathbf{B}(x,y) + \frac{k_x}{x}\mathbf{U}^T \mathbf{D}_x \mathbf{B}(x,y) + \mathbf{U}^T \mathbf{D}_y^2 \mathbf{B}(x,y) \\ &+ \frac{k_y}{y}\mathbf{U}^T \mathbf{D}_y \mathbf{B}(x,y) + \mathbf{H}^T \mathbf{B}(x,y) = 0. \end{aligned} \tag{3.3}$$

Collocating Eq.(3.3) at $(n-1)(m-1)$ collocation points leads to

$$\begin{aligned} &\mathbf{U}^T \mathbf{D}_x^2 \mathbf{B}(x_i,y_j) + \frac{k_x}{x_i}\mathbf{U}^T \mathbf{D}_x \mathbf{B}(x_i,y_j) + \mathbf{U}^T \mathbf{D}_y^2 \mathbf{B}(x_i,y_j) \\ &+ \frac{k_y}{y_j}\mathbf{U}^T \mathbf{D}_y \mathbf{B}(x_i,y_j) + \mathbf{H}^T \mathbf{B}(x_i,y_j) = 0. \end{aligned} \tag{3.4}$$

We select the collocation nodes as Newton-Cotes points

$$x_i = \frac{2i-1}{2(n+1)}, \quad i = 1, \ldots, (n+1), \quad y_j = \frac{2j-1}{2(m+1)}, \quad j = 1, \ldots, (m+1) \tag{3.5}$$

In addition, the initial conditions (3.2) provide two algebraic equations

$$\text{(3.6)} \qquad \mathbf{U}^T\mathbf{B}\left(0,y_j\right)=h\left(y_j\right), \quad \mathbf{U}^T\mathbf{D}_x\mathbf{B}\left(0,y_j\right)=0,$$

$$\mathbf{U}^T\mathbf{B}\left(x_i,0\right)=h\left(x_i\right), \quad \mathbf{U}^T\mathbf{D}_y\mathbf{B}\left(x_i,0\right)=0.$$

Finally, we can compute the values for the components of $\mathbf{U}$ by solving the system of Eq. (3.4) and Eq. (3.6). Hence, the approximate solution for $u(x,y)$ can be computed by using Eq. (1.5).

4. Applications

In this section, we illustrate the presented method by giving some examples. The results are compared with the exact solutions by calculating the following absolute error

$$e(s,t)=|u(x,t)-u_{n,m}(x,y)|, x,y\in[0,1],$$

where,$u(x,y)$ denotes the exact solution of the given examples, and $u_{n,m}(x,y)$ is the approximate solution by the presented method.

Example 1. *We consider the following 2D-linear Lane-Emden type equation [6, 8]*

$$\text{(4.1)} \qquad \frac{\partial^2 u\left(x,y\right)}{\partial x^2}+\frac{1}{x}\frac{\partial u\left(x,y\right)}{\partial x}+\frac{\partial^2 u\left(x,y\right)}{\partial y^2}+\frac{1}{y}\frac{\partial u\left(x,y\right)}{\partial y}-4\left(x^2+y^2\right)=0,$$

subject to the following initial conditions

$$\text{(4.2)} \qquad u(0,y)=1,\ \frac{\partial u\left(0,y\right)}{\partial x}=0,\ u(x,0)=1,\ \frac{\partial u\left(x,0\right)}{\partial y}=0.$$

The exact solution of Equation (4.1) *is* $u(x,y)=1+x^2y^2$.

Let $n=m=2$. *By applying the technique described in Section* (3) *for Equation* (4.1)*, we get:*

$$\text{(4.3)} \qquad u_{2,2}(x,y)=\sum_{i=0}^{2}\sum_{j=0}^{2}u_{ij}B_{ij}\left(x,y\right)=\mathbf{U}^T\mathbf{B}\left(x,y\right),$$

where

$$\mathbf{U^T}=\left[u_{00},u_{10},u_{20},u_{01},u_{11},u_{20},u_{02},u_{12},u_{22}\right],$$

$$\mathbf{B}(x,y)=[1,y,y^2+2,x,xy,xy^2+2x,x^2+2,yx^2+2y,x^2y^2+2x^2+2y^2+4]^T,$$

and

$$\mathbf{D}_x = \begin{bmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&2&0&0&0&0&0 \\ 0&0&0&0&2&0&0&0&0 \\ 0&0&0&0&0&2&0&0&0 \end{bmatrix}, \quad \mathbf{D}_y = \begin{bmatrix} 0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&2&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&2&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&2&0 \end{bmatrix},$$

$$\mathbf{H}^T = [16, 0, -4, 0, 0, 0, -4, 0, 0].$$

By solving the linear system (3.4)*, we get*

$$\mathbf{U}^T = [5, 0, -2, 0, 0, 0, -2, 0, 1],$$

then

$$u_{2,2}(x, y) = \mathbf{U^T B}(x, y) = 1 + x^2 y^2,$$

which is the exact solution of Equation (4.1)*. The absolute errors* $e(x, y)$ *for* $n = m = 3$ *and* $n = m = 4$ *reported at some selected points in Table* (1)

TABLE 1. Result for Example (1)

(x,y)	$e(x,y)$ for $n=m=3$	$e(x,y)$ for $n=m=4$
$(0.1, 0.1)$	$6.6276e{-5}$	$1.4177e{-7}$
$(0.2, 0.2)$	$2.6512e{-5}$	$5.6708e{-7}$
$(0.3, 0.3)$	$5.9648e{-5}$	$1.2759e{-6}$
$(0.4, 0.4)$	$1.0604e{-4}$	$2.2683e{-6}$
$(0.5, 0.5)$	$1.6569e{-4}$	$3.5443e{-6}$
$(0.6, 0.6)$	$2.3859e{-5}$	$5.1037e{-7}$
$(0.7, 0.7)$	$3.2475e{-4}$	$6.9467e{-6}$
$(0.8, 0.8)$	$4.2417e{-4}$	$9.0733e{-6}$
$(0.9, 0.9)$	$5.3684e{-5}$	$1.1483e{-6}$
$(1.0, 1.0)$	$6.6276e{-4}$	$1.4177e{-5}$

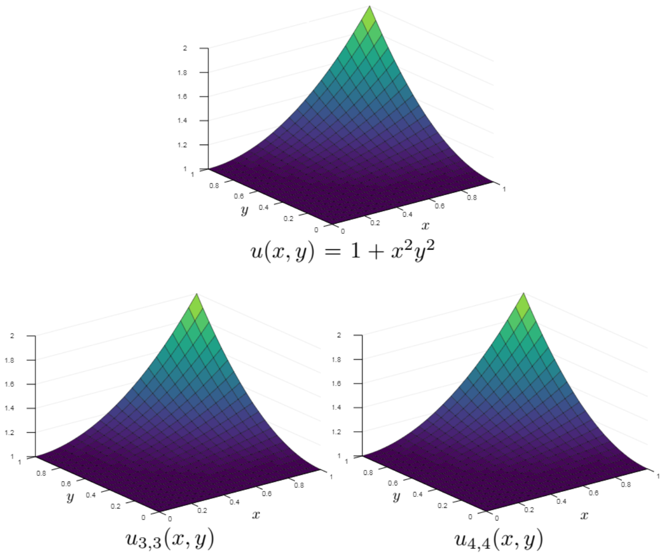

FIGURE 1. Graph of exact and approximate solution for $n = m = 3$ and $n = m = 4$.

We consider the following 2D-non linear Lane-Emden problem [6, 8]

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{3}{x}\frac{\partial u(x,y)}{\partial x} + \frac{\partial^2 u(x,y)}{\partial y^2} + \frac{3}{y}\frac{\partial u(x,y)}{\partial y} - 7u^{-1}(x,y) = 0, \tag{4.4}$$

subject to the following initial conditions

$$u(0,y) = y,\ \frac{\partial u(0,y)}{\partial x} = 0,\ u(x,0) = x,\ \frac{\partial u(x,0)}{\partial y} = 0. \tag{4.5}$$

The exact solution is $u(x,y) = \sqrt{x^2+y^2}$. *Let* $n = m = 3$ *we get*

$$\mathbf{D}_x = \begin{bmatrix} \mathbf{I}_{4\times4} & \mathbf{O}_{4\times4} & \mathbf{O}_{4\times4} & \mathbf{O}_{4\times4} \\ \mathbf{I}_{4\times4} & \mathbf{O}_{4\times4} & \mathbf{O}_{4\times4} & \mathbf{O}_{4\times4} \\ \mathbf{O}_{4\times4} & 2\mathbf{I}_{4\times4} & \mathbf{O}_{4\times4} & \mathbf{O}_{4\times4} \\ -5\mathbf{I}_{4\times4} & \mathbf{O}_{4\times4} & 3\mathbf{I}_{4\times4} & \mathbf{O}_{4\times4} \end{bmatrix}_{16\times16},$$

$$\mathbf{D}_y = \begin{bmatrix} \mathbf{T}_{4\times4} & \mathbf{O}_{4\times4} & \mathbf{O}_{4\times4} & \mathbf{O}_{4\times4} \\ \mathbf{O}_{4\times4} & \mathbf{T}_{4\times4} & \mathbf{O}_{4\times4} & \mathbf{O}_{4\times4} \\ \mathbf{O}_{4\times4} & \mathbf{O}_{4\times4} & \mathbf{T}_{4\times4} & \mathbf{O}_{4\times4} \\ \mathbf{O}_{4\times4} & \mathbf{O}_{4\times4} & \mathbf{O}_{4\times4} & \mathbf{T}_{4\times4} \end{bmatrix}_{16\times16},$$

where

$$\mathbf{T}_{4\times 4}=\begin{bmatrix} 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ -5 & 0 & 3 & 0 \end{bmatrix},\quad \mathbf{O}_{4\times 4}=\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The absolute errors $e(s,t)$ *for* $n=m=3, n=m=5$ *reported in Table* (2).

TABLE 2. Result for Example(2)

(x,y)	$e(x,y)$ for $n=m=3$	$e(x,y)$ for $n=m=5$
$(0.1,0.1)$	$2.3286e-3$	$1.5474e-5$
$(0.2,0.2)$	$4.6572e-3$	$3.0948e-5$
$(0.3,0.3)$	$6.9858e-3$	$4.6422e-5$
$(0.4,0.4)$	$9.3144e-3$	$6.1896e-5$
$(0.5,0.5)$	$1.1643e-2$	$7.7371e-5$
$(0.6,0.6)$	$1.3972e-2$	$9.2844e-5$
$(0.7,0.7)$	$1.6300e-2$	$1.0832e-4$
$(0.8,0.8)$	$1.8629e-2$	$1.2379e-4$
$(0.9,0.9)$	$2.0957e-2$	$1.3927e-4$
$(1.0,1.0)$	$2.3286e-2$	$1.5474e-4$

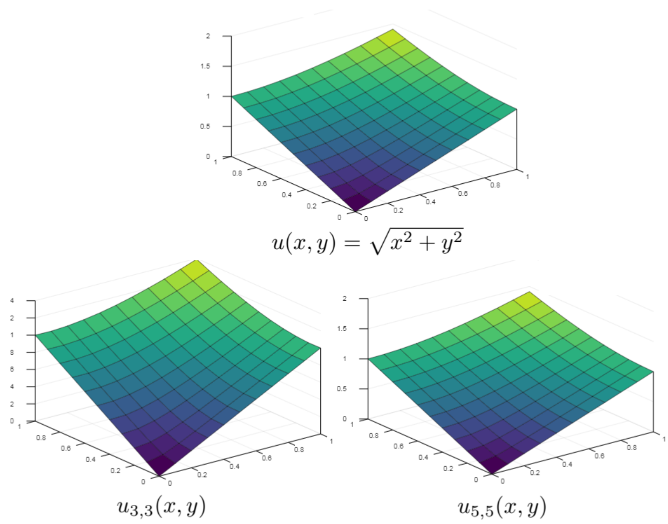

FIGURE 2. Graph of exact and approximate solution for $n=m=3$ and $n=m=5$.

We consider the following 2D-linear Lane-Emden problem

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{2}{x}\frac{\partial u(x,y)}{\partial x}+\frac{\partial^2 u(x,y)}{\partial y^2}+\frac{2}{y}\frac{\partial u(x,y)}{\partial y}-6(2+x^2+y^2)=0, \tag{4.6}$$

subject to the following initial conditions

$$u(0,y)=1+y^2,\ \frac{\partial u(0,y)}{\partial x}=0,\ u(x,0)=1+x^2,\ \frac{\partial u(x,0)}{\partial y}=0. \tag{4.7}$$

The exact solution is $u(x,y)=1+x^2+y^2+x^2y^2$. *Table* (3) *shows the absolute error between the approximate solution obtained for* $n=m=5$ *and* $n=m=10$.

TABLE 3. Result for Example (3)

(x,y)	$e(x,y)$ for $n=m=5$	$e(x,y)$ for $n=m=10$
$(0.1,0.1)$	$2.0443e-17$	$2.7406e-23$
$(0.2,0.2)$	$8.1773e-17$	$2.7406e-23$
$(0.3,0.3)$	$1.8399e-16$	$6.1663e-23$
$(0.4,0.4)$	$3.2709e-16$	$1.0962e-22$
$(0.5,0.5)$	$5.1108e-16$	$1.7129e-22$
$(0.6,0.6)$	$7.3596e-17$	$2.4665e-23$
$(0.7,0.7)$	$1.0017e-15$	$3.3572e-22$
$(0.8,0.8)$	$1.3084e-15$	$4.3849e-22$
$(0.9,0.9)$	$1.6559e-15$	$5.5496e-22$
$(1.0,1.0)$	$2.0443e-15$	$6.8514e-22$

Example 4. *We consider the following 2D-non linear Lane-Emden problem [6, 8]*

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{4}{x}\frac{\partial u(x,y)}{\partial x}+\frac{\partial^2 u(x,y)}{\partial y^2}+\frac{4}{y}\frac{\partial u(x,y)}{\partial y}-\left(5+4x^2y^2\right)\left(x^2+y^2\right)u^{-3}(x,y)=0,$$

subject to the following initial conditions

$$u(0,y)=1,\ \frac{\partial u(0,y)}{\partial x}=0,\ u(x,0)=1,\ \frac{\partial u(x,0)}{\partial y}=0.$$

The exact solution is $u(x,y)=\sqrt{1+x^2y^2}$. *Approximate , exact solutions and the absolute error* $e(s,t)$ *for* $n=m=4, n=m=6$ *reported in Table* (4).

TABLE 4. Result for Example (4)

(x,y)	Present method with $n=4$,	$n=6$,	Exact solution	$e(x,y)$ for $n=4$	$e(x,y)$ for $n=6$
$(0.1,0.1)$	1.00010	1.00012	1.000049	$5.11e-5$	$7.32e-5$
$(0.2,0.2)$	1.00027	1.00081	1.000799	$5.29e-4$	$1.87e-5$
$(0.3,0.3)$	1.00503	1.00403	1.004041	$9.89e-4$	$6.30e-6$
$(0.4,0.4)$	1.01802	1.01262	1.012719	$5.31e-3$	$9.30e-5$
$(0.5,0.5)$	1.03169	1.03084	1.030776	$9.14e-4$	$6.83e-5$
$(0.6,0.6)$	1.06604	1.06253	1.062826	$3.21e-3$	$2.92e-4$
$(0.7,0.7)$	1.11100	1.11357	1.113597	$2.59e-3$	$1.71e-5$
$(0.8,0.8)$	1.18686	1.18702	1.187265	$4.05e-4$	$2.45e-4$
$(0.9,0.9)$	1.28341	1.28639	1.286895	$3.48e-3$	$5.01e-4$
$(1.0,1.0)$	1.41562	1.41415	1.414213	$1.41e-3$	$6.11e-5$

5. CONCLUSIONS

In this study, we successfully derived operational matrices of derivatives based on Two-dimensional Boubaker polynomials. Leveraging these matrices, we efficiently transformed the singular Two-dimensional Lane-Emden type equations into a system of algebraic equations. This transformation allowed us to devise a simple and effective numerical solution for the problem at hand. Through a numerical example, we demonstrated the validity and applicability of our proposed method, showcasing its capability to handle singularities and deliver accurate results. In conclusion, the introduced method based on Two-dimensional Boubaker polynomials operational matrices provides a promising and effective approach to address singularities in the Two-dimensional Lane-Emden type equations. As we continue to explore and refine this method, it has the potential to make valuable contributions to the field of numerical analysis and mathematical modeling.

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