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ON SPHERICAL FUNCTIONS ASSOCIATED WITH MULTIPLICITY-FREE INDUCED REPRESENTATIONS OF A HOMOGENEOUS TREE

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ABSTRACT. Let $T_{q,n}$ be a homogeneous rooted tree, let Σ^n be the set of leaves of $T_{q,n}$ and let $K_{q,n}$ be the stabilizer of the leftmost leaf by the action of $Aut(T_{q,n})$, the group of automorphisms of $T_{q,n}$, on Σ^n.

In this paper, we study spherical functions associated with multiplicity-free induced representation of $K_{q,n}$ and obtain their explicit formula.

1. INTRODUCTION

Homogeneous trees and their automorphisms groups particularly automorphisms groups of rooted trees have been studied intensely for the past few years in connection with their application in geometric group theory [6], theory of dynamic systems [3], theory of probability and statistics [7]. Also, the foundation for interest is that automorphisms of rooted trees contain various interesting subgroups with extremal properties. Moreover, there exists a connection between harmonic analysis on trees and harmonic analysis on hyperbolic spaces by emphasizing the strict analogy between the group of automorphisms of the tree and the real rank one semi-simple Lie groups.

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J. P. Serre's lecture notes [15], In 1970 was the starting point of infatuation for this study. P. Cartier [5], initiated thereafter the study of spherical functions on trees. In the same vein, in 1990, Alessandro Figà-Talamanca [11] has considered and has studied the Gelfand pair $(G; K)$ where G is the group of automorphisms of a locally finite homogeneous tree and K is the stabilizer of a vertex.

The harmonic analysis on a tree and the representation theory of the group of automorphisms acting on the homogeneous tree will be pursued by other researchers such as E. Casadio-Tarabusi [6], E. Axelgaard [2]. Unlike other researchers, T. Cecceherini and al. [8] will focus on harmonic analysis on the group of automorphisms of a finite homogeneous rooted tree $Aut(T_{q,n})$. They describe an application of the theory for the Gelfand pair and spherical functions to define and study a diffusion process in ultrametric spaces (Σ^n, d) where Σ^n is the space of the leaves of the tree $T_{q,n}$ and d is an ultrametric distance.

We know that, for finite Gelfand pair (G, K), the quasi-regular representation $Ind_K^G(\mathbf{1_K})$ is multiplicity-free. The notion of Gelfand pair has been extended to commutative triples [1, 4, 9, 14] and in this case $Ind_K^G\tau$ is multiplicity-free that is the one-dimensional trivial representation of K is remplaced by a non trivial unitary irreducible representation τ of K. In addition, the generalisation of spherical functions to the case of commutative triples implies the consideration of functions with values in the endomorphism of a finite-dimensional vector space [9, 13, 14, 16, 18]. It would be interesting for $Aut(T_{q,n})$ to describe an application of the theory of commutative triples and spherical functions to define and study diffusion process in ultrametric spaces (Σ^n, d).

The aims of our paper is to construct spherical functions with values in the endomorphism of a finite-dimensional vector space V_τ for a finite homogeneous rooted tree $T_{q,n}$. To do so, we adopt the following plan: In section 2, we provide the preliminaries and the basic notions useful for the proper understanding of our paper. In section 3, we construct the τ- spherical functions associated with multiplicity-free induced representation on $Aut(T_{q,n})$. To achieve that, we decompose the generalized permutation and obtain the explicit formula of the τ- spherical functions on Σ^n.

2. Preliminaries and notations

We use the notations and setup of this section in the rest of the paper without mentioning it. Let G be a finite group. Let X be a finite set and let G acts transitively on X. Let $x \in X$ and denote by $K = Stab_G(x) = \{g \in G : g.x = x\}$ the stabilizer of x, a subgroup of G. Then, the map

$$\begin{array}{cccc} \Phi & : & G/K & \rightarrow & X \\ & & gK & \mapsto & g.x \end{array}$$

is a G-equivariant bijection, thus making X and $G/Stab_G(x)$ isomorphic as G-spaces (Lemma 3.1.6 [7], page 80).

Let τ be an irreducible unitary representation of K on finite-dimensional vector space V_τ. Let $\mathbf{I}_{V_\tau}$ be the identity map on V_τ. We denote by $\widehat{G}$ the set of equivalence classes of irreducible unitary representations of G and denote by $m(\tau, \sigma)$ the multiplicity of τ in $\sigma|K$, for all $\sigma \in \widehat{G}$. The character of the representation τ is the function denoted χ^τ on K with complex values defined by: for all $k \in K$, $\chi^\tau(k) = Tr(\tau(k))$, where $Tr(\tau(k))$ is the trace of $\tau(k)$. The induced representation of the representation τ on G is the G-representation generally denoted by $Ind_K^G \tau$. Its space of realization is: $L(G, V_\tau, \tau) = \{f : G \longrightarrow V_\tau \;/\; f(gk) = \tau(k^{-1})f(g),$ $\forall g \in G, k \in K\}$.

The action of G on $L(G, V_\tau, \tau)$ is given by:

$$((Ind_K^G \tau)(g_1)f)(g_2) = f(g_1^{-1}g_2), \quad \forall g_1, g_2 \in G \text{ and } f \in L(G, V_\tau, \tau).$$

Let F be a finite group. We denote by F^X the set of all maps f: $X \longrightarrow F$. The action of G on X induces an action of G on F^X. The set $F^X \times G$ equipped with the following multiplication:

$$(f, g)(f', g') = (f.gf', gg'), \quad \forall\, (f, g)\, ;\, (f', g') \in F^X \times G$$
$$\text{where } (f.gf')(x) = f(x)f'(g^{-1}x), \quad \forall\, x \in X,$$

is a group [8]. The identity element is $(1_F, 1_G)$ and the inverse of (f, g) is given by $(g^{-1}f^{-1}, g^{-1})$. This group is called the wreath product of F by G with respect to X and is denoted by $F \wr_X G$ or by $F \wr G$ if there is no confusion on X. Let N be a normal subgroup of G and $\rho \in \widehat{G/N}$. An inflation representation $\overline{\rho}$ of ρ is the unitary representation of G defined by: $\overline{\rho}(g) = \rho(gN)$, for all $g \in G$.

An extension of $\tau \in \widehat{K}$ is the representation of G denoted $\widetilde{\tau}$ such that the restriction of $\widetilde{\tau}$ to K is equal to τ.

Let (σ, V) and (ρ, W) be two representations of G. The inner tensor product of σ and ρ is the representation of G on $V \otimes W$, defined by:

$$(\sigma \otimes \rho)(g)(v \otimes w) = \sigma(g)v \otimes \rho(g)w, \quad \text{for all } g \in G,\, v \in V,\, w \in W.$$

We denote it by $\sigma \otimes \rho$.

Let (σ, V) be a representation of G_1 and (ρ, W) be a representation of G_2. The outer tensor product of σ and ρ is the representation of $(G_1 \times G_2)$ on $V \otimes W$, defined by:

$$(\sigma \boxtimes \rho)(g_1, g_2)(v \otimes w) = \sigma(g_1)v \otimes \rho(g_2)w, \quad \text{for all } g_1 \in G_1,\, g_2 \in G_2,\, v \in V,\, w \in W.$$

We denote it by $\sigma \boxtimes \rho$.

Let us consider the cartesian product $G \times V_\tau$. The group G acts on $G \times V_\tau$ by:

$$h.(g, v) = (hg, v) \quad \forall h, g \in G, \forall v \in V_\tau.$$

Let us denote by $[g, v]$ the orbit of $(g, v) \in G \times V_\tau$ by above action, by $G \times_\tau V_\tau$ the set of orbits. The action of G on $G \times V_\tau$ induces an action of G on $G \times_\tau V_\tau$ defined by:

$$h.[g, v] = [hg, v], \quad \forall h, g \in G, \forall v \in V_\tau.$$

The subgroup K acts on $G \times V_\tau$ in the following way:

$$k.(g, v) = (gk, \tau(k^{-1})v), \quad \forall k \in K, \forall g \in G, \forall v \in V_\tau.$$

Let us consider a projection

$$\begin{array}{cccc} p & : & G \times_\tau V_\tau & \to & G/K \\ & & [g, v] & \mapsto & gK \end{array}$$

This projection is G-equivariant for the action of G on $G \times_\tau V_\tau$.

We designate by $E^\tau = (G \times_\tau V_\tau, p, G/K)$ the homogeneous vector bundle over $G/K = X$. A cross-section of E^τ may be identified with a vector-valued function f: $G \to V_\tau$ which is right-K-covariant of type τ, that is $f(gk) = \tau(k^{-1})f(g)$ [17]. We designate by $\Gamma(E^\tau)$ the space of cross-sections of E^τ.

In [1], the authors have given a generalization of the permutation representation called a generalized permutation. The generalized permutation representation of G is the representation of G on $\Gamma(E^\tau)$ defined by: $\forall g \in G, \forall s \in \Gamma(E^\tau)$ and $\forall x \in X$,

$$\lambda^\tau(g)(s) = g.s(g^{-1}.x).$$

Let $End(V_\tau)$ be the vector space of all endomorphisms of V_τ and let us denoted by $L(G, End(V_\tau))$= $\{F : G \longrightarrow End(V_\tau)\}$, the space of all $End(V_\tau)$)- valued functions

defined on G. A function $F \in L(G, End(V_\tau))$ is said τ-radial if it satisfies the property: $F(k_1 g k_2) = \tau(k_2^{-1})F(g)\tau(k_1^{-1})$, for all $k_1, k_2 \in K$ and $g \in G$. We denote by $L(G, K, \tau, \tau)$ the space of τ-radial functions.
Let (G, K, τ) be a commutative triple. A non-trivial function φ in the space $L(G, K, \tau, \tau)$ is said to be a τ-spherical function if the map
$\chi : F \mapsto \frac{1}{d_\tau}\sum_{g\in G} Tr[F(g)\varphi(g^{-1})]$ is a character of $L(G, K, \tau, \tau)$ on $\mathbb{C}$. When τ is the trivial representation of K of dimension one, we obtain the notion of spherical functions associated with a Gelfand pairs (G, K).

Let $\varphi \in L(G, K, \tau, \tau)$. The following assertions are equivalent:

(1) φ is a $\tau-$spherical function.
(2) $\forall g, h \in G$, $\frac{1}{|K|}\sum_{g\in G}\varphi(gkh)\chi^\tau(k) = \varphi(h)\varphi(g)$.

Let $q \in \mathbb{N}$. A composition of q of length t is the t-uple $\lambda = (q_1, q_2, \ldots, q_t)$ such that $\sum_{i=1}^{t} q_i = q$ with $q_i \in \mathbb{N}$. If $q_1 \geq q_2 \geq \ldots \geq q_t$, then λ is a partition of q. We write $\lambda \vdash q$. It is well known that the irreducible representations of S_q are labelled by integer partitions of q. Therefore, we denote by S^λ the irreducible representation associated with the partition λ. Let $(q_1, q_2, \ldots, q_t)$ be a composition of q. If $r_1 \vdash q_1; \ldots; r_t \vdash q_t$, then $(r_1, \ldots, r_t)$ is a multipartition of q. We denote it by $(r_1, \ldots, r_t) \Vdash q$. Let $\Sigma = \{0; 1; \ldots; q-1\}$, where $q \in \mathbb{N}^*$. We call the set Σ the alphabet. A word over Σ of length k is a sequence $x = x_1x_2\ldots x_k$ where $x_i \in \Sigma$ for $i = 1,\ldots,k$. The set of all the words of length k is denoted by Σ^k. A tree is a connected graph without circuits. A tree is said to be rooted if it has a fixed vertex called the root of the tree. A leaf is a vertex of degree one. A homogeneous tree is a tree where all the vertices which are not leaves have the same degree. On a tree, we define a distance from two vertices x and y as the length of the shortest path joining x and y. We denote by $T_{q,n}$ a finite homogeneous rooted tree whose root is of degree q and the distance from the root to a leaf is n. We designate by $Aut(T_{q,n})$ the group of automorphisms of the homogeneous rooted tree $T_{q,n}$. The set Σ^n of leaves of $T_{q,n}$ can be endowed with a metric d as follows: for $x = x_1x_2\ldots x_n$ and $y = y_1y_2\ldots y_n$ $d(x, y) = n - max\{k : x_i = y_i$ for all $i \leqslant k\}$. Thus (Σ^n, d) is a metric space, in particular, d is called an ultrametric distance.

3. τ-SPHERICAL FUNCTIONS

In this part, G is the group of automorphisms of the homogeneous rooted tree $T_{q,n}$. Σ^n is the set of the leaves of $T_{q,n}$ and $x_0 = \underbrace{0\ldots0}_{(n)times} = 0^n \in \Sigma^n$ is the leftmost leaf. $Aut(T_{q,n})$ acts transitively on Σ^n. Let $K_{q,n} = \{g \in Aut(T_{q,n}) : g(x_0) = x_0\}$ be the stabilizer of x_0. For $i = 1, \ldots, n-2$, let C_i be an irreducible unitary representation of $Aut(T_{q-1,n-(i+1)}) \wr S_{q-1}$ and let D be a representation of S_{q-1}.

$\tau_{q,n} = C_1 \boxtimes \ldots \boxtimes C_{n-2} \boxtimes D$ is a unitary representation of $K_{q,n}$. In [12], the authors prove that the representation $Ind_{K_{q,n}}^{Aut(T_{q,n})}\tau_{q,n}$ is multiplicity-free. For $i = 1, \ldots, n-2$, let $\sigma_1^i, \ldots, \sigma_{\ell_i}^i$ be a set of pairwise inequivalent irreducible representations of $Aut(T_{q-1,n-(i+1)})$. We set:

$$\begin{aligned} H &= Aut(T_{q-1,n-2}) \wr (S_{1+\alpha_1^1} \times \ldots \times S_{1+\alpha_{\ell_1}^1}) \times \ldots \times Aut(T_{q-1,1}) \wr (S_{1+\alpha_1^{n-2}} \\ &\quad \times \ldots \times S_{1+\alpha_{\ell_{n-2}}^{n-2}}) \times S_{q-1}, \\ \Lambda_1 &= (\sigma_1^{\widetilde{1\boxtimes 1+\alpha_1^1}} \otimes \overline{S^{\beta_1^1}}) \boxtimes \ldots \boxtimes (\sigma_1^{\widetilde{1\boxtimes 1+\alpha_{\ell_1}^1}} \otimes \overline{S^{\beta_{\ell_1}^1}}), \ldots, \Lambda_{n-2} \\ &= (\sigma_1^{\widetilde{n-2\boxtimes 1+\alpha_1^{n-2}}} \otimes \overline{S^{\beta_1^{n-2}}}) \boxtimes \ldots \boxtimes (\sigma_{\ell_{n-2}}^{\widetilde{n-2\boxtimes 1+\alpha_{\ell_{n-2}}^{n-2}}} \otimes \overline{S^{\beta_{\ell_{n-2}}^{n-2}}}). \end{aligned}$$

The decomposition into multiplicity-free irreducible representations of $Ind_{K_{q,n}}^{Aut(T_{q,n})}\tau_{q,n}$ is:

$$Ind_{K_{q,n}}^{Aut(T_{q,n})}\tau_{q,n} = \bigoplus_{(\beta_1^1,\ldots,\beta_{\ell_1}^1)\Vdash q-1} \cdots \bigoplus_{(\beta_1^{n-2},\ldots,\beta_{\ell_{n-2}}^{n-2})\Vdash q-1} \bigoplus_{\lambda \vdash q-1} \delta_{\beta^1,\ldots,\beta^{n-2}}^{\lambda}$$

, where $\delta_{\beta^1,\ldots,\beta^{n-2}}^{\lambda} = Ind_H^{Aut(T_{q,n})}(\Lambda_1 \boxtimes \ldots \boxtimes \Lambda_{n-2} \boxtimes S^\lambda)$ (For terms and notations not mentioned here see [12]).

We denote by $\widehat{Aut(T_{q,n})}(\tau_{q,n})$ the set of those representations $\delta_{\beta^1,\ldots,\beta^{n-2}}^{\lambda}$ in $\widehat{Aut(T_{q,n})}$ which contain $\tau_{q,n}$ upon restriction to $K_{q,n}$. Let $V_{\delta_{\beta^1,\ldots,\beta^{n-2}}^{\lambda}}$ be the realization space of $\delta_{\beta^1,\ldots,\beta^{n-2}}^{\lambda}$ and let $V_{\delta_{\beta^1,\ldots,\beta^{n-2}}^{\lambda}}(\tau_{q,n})$ be the isotypique component of $\tau_{q,n}$. Let

$$P_{\tau_{q,n}}^{\delta_\beta^\lambda} \quad : \quad V_{\delta_{\beta^1,\ldots,\beta^{n-2}}^{\lambda}} \quad \rightarrow \quad V_{\delta_{\beta^1,\ldots,\beta^{n-2}}^{\lambda}}(\tau_{q,n})$$

be the orthogonal projection given by:

$$P_{\tau_{q,n}}^{\delta_\beta^\lambda} = \frac{d_{\tau_{q,n}}}{|K_{q,n}|} \sum_{k \in K_{q,n}} \chi^{\tau_{q,n}}(k) \delta_{\beta^1,\ldots,\beta^{n-2}}^{\lambda}(k^{-1}).$$

Since $m(\tau_{q,n}, \delta^{\lambda}_{\beta^1,\ldots,\beta^{n-2}}) = 1$, $V_{\delta^{\lambda}_{\beta^1,\ldots,\beta^{n-2}}}(\tau_{q,n})$ can be identified with $V_{\tau_{q,n}}$. We assume that $\tau_{q,n}$ extends to a representation π of $Aut(T_{q,n})$ on $V_{\tau_{q,n}}$.
Let us consider the projection:

$$\begin{array}{rcl} p \ : \ \Sigma^n \times_{\tau_{q,n}} V_{\tau_{q,n}} & \rightarrow & \Sigma^n \\ [x, v] & \mapsto & x. \end{array}$$

The homogeneous vector bundle over Σ^n associated with the representation $\tau_{q,n}$ is on the form $E^{\tau_{q,n}} = (\Sigma^n \times_{\tau_{q,n}} V_{\tau_{q,n}}, P, \Sigma^n)$. The space of the sections $\Gamma(E^{\tau_{q,n}})$ is identified with the space $L(Aut(T_{q,n}), V_{\tau_{q,n}}, \tau_{q,n})$ [1]. Also if $\tau_{q,n}$ extends to a representation π of $Aut(T_{q,n})$ on $V_{\tau_{q,n}}$, $\Gamma(E^{\tau_{q,n}})$ is identified with the space $L(\Sigma^n, V_{\tau_{q,n}})$ $= \{f : \Sigma^n \longrightarrow V_{\tau_{q,n}}\}$. Let $f \in L(\Sigma^n, V_{\tau_{q,n}})$ and let $(v_j)_{1 \leqslant j \leqslant d_{\tau_{q,n}}}$ be an orthonormal basis of $V_{\tau_{q,n}}$ where $d_{\tau_{q,n}}$ is the dimension of the representation $\tau_{q,n}$. Then $f = \sum_{j=1}^{d_{\tau_{q,n}}} f_j v_j$ where $f_j \in L(\Sigma^n)$. The action of $Aut(T_{q,n})$ on $L(\Sigma^n, V_{\tau_{q,n}})$ is given by:

$$g.f(x) = \pi(g) f(g^{-1}.x), \text{ for all } g \in Aut(T_{q,n}),\ f \in L(\Sigma^n, V_{\tau_{q,n}}) \text{ and } x \in \Sigma^n.$$

The following result give us $\tau_{q,n}$-spherical functions defined on $Aut(T_{q,n})$ with values in $End(V_{\tau_{q,n}})$.

Proposition 3.1. *Let $\delta^{\lambda}_{\beta^1,\ldots,\beta^{n-2}} \in \widehat{Aut(T_{q,n})}$. The function $\phi \in L(Aut(T_{q,n}), End(V_{\tau_{q,n}}))$ defined by*

$$\phi(g) = P^{\delta^{\lambda}_{\beta}}_{\tau_{q,n}} \delta^{\lambda}_{\beta^1,\ldots,\beta^{n-2}}(g^{-1}) P^{\delta^{\lambda}_{\beta}}_{\tau_{q,n}}$$

is a $\tau_{q,n}$-spherical function associated with $Ind^{Aut(T_{q,n})}_{K_{q,n}} \tau_{q,n}$.

In the following result, we show that the $\tau_{q,n}$-spherical functions are positive definite.

Theorem 3.1. *Let ϕ be a $\tau_{q,n}$-spherical function on $Aut(T_{q,n})$. Then, ϕ is a positive definite function.*

Proof. Let $c_1, c_2, \dots, c_n \in \mathbb{C}$, $g_1, \dots, g_n \in Aut(T_{q,n})$ and $v \in V_{\tau_{q,n}}$. Then,

$$\begin{aligned}\sum_{i,k} c_i\overline{c_k} < \phi(g_k^{-1}g_i)v, v > &= \sum_{i,k} c_i\overline{c_k} < P_{\tau_{q,n}}^{\delta_\beta^\lambda}\delta^\lambda_{\beta^1,\dots,\beta^{n-2}}(g_i^{-1}g_k)P_{\tau_{q,n}}^{\delta_\beta^\lambda}v, v > \\ &= \sum_{i,k} c_i\overline{c_k} < P_{\tau_{q,n}}^{\delta_\beta^\lambda}\delta^\lambda_{\beta^1,\dots,\beta^{n-2}}(g_i^{-1})\delta^\lambda_{\beta^1,\dots,\beta^{n-2}}(g_k)v, v > \\ &= \sum_{i,k} c_i\overline{c_k} < \delta^\lambda_{\beta^1,\dots,\beta^{n-2}}(g_i^{-1})\delta^\lambda_{\beta^1,\dots,\beta^{n-2}}(g_k)v, P_{\tau_{q,n}}^{\delta_\beta^\lambda}v > \\ &= \sum_{i,k} c_i\overline{c_k} < \delta^\lambda_{\beta^1,\dots,\beta^{n-2}}(g_k)v, \delta^\lambda_{\beta^1,\dots,\beta^{n-2}}(g_i)v > \\ &= < \sum_{i=1} \overline{c_k}\delta^\lambda_{\beta^1,\dots,\beta^{n-2}}(g_k)v, \sum_{k=1} \overline{c_i}\delta^\lambda_{\beta^1,\dots,\beta^{n-2}}(g_i)v > \\ &\geqslant 0,\end{aligned}$$

thus ϕ is a positive definite $\tau_{q,n}$-spherical function. □

A function $f \in L(\Sigma^n, V_{\tau_{q,n}})$ can be considered as a function $f(x_1, x_2, \dots, x_n)$ of the Σ-valued variables $x_1, x_2, \dots, x_n$. We set $W_0 = L(\emptyset, V_{\tau_{q,n}}) \simeq V_{\tau_{q,n}}$ and for $j = 1, \dots, n$, $W_j = \{f \in L(\Sigma^n, V_{\tau_{q,n}}) : f = f(x_1, x_2, \dots, x_j)$ and $\sum_{x=0}^{q-2} f(x_1, x_2, \dots, x_{j-1}, x) \equiv 0\}$.
In the following result, we give a decomposition of the space $L(\Sigma^n, V_{\tau_{q,n}})$.

Theorem 3.2. *We have that* $L(\Sigma^n, V_{\tau_{q,n}}) = \bigoplus_{j=0}^{n} W_j$ *is the decomposition of* $L(\Sigma^n, V_{\tau_{q,n}})$ *into* $Aut(T_{q,n})$*-irreducible subrepresentations.*

Proof. We know that $Aut(T_{q,n})$ acts transitively on Σ^n. Also, the condition that $f(x_1, x_2, \dots, x_j)$ depends on the first j variables is invariant because $Aut(T_{q,n})$ preserves the levels of the tree. The action of $Aut(T_{q,n})$ on Σ^n is given by: $g(x_1 \dots x_n)$ $= g_\emptyset(x_1)g_{x_1}(x_2)\dots g_{x_1\dots x_{n-1}}(x_n)$, and the action of $Aut(T_{q,n})$ on $L(\Sigma^n, V_{\tau_{q,n}})$ is given by :

$$g^{-1}.f(x_1, \dots, x_n) = \pi(g^{-1})f(g(x_1, \dots, x_n)),$$

for all $g \in Aut(T_{q,n})$, $f \in L(\Sigma^n, V_{\tau_{q,n}})$ and $x_1 \dots x_n \in \Sigma^n$. So,

$$g^{-1}.f(x_1,\dots,x_n) = \pi(g^{-1})f(g(x_1,\dots,x_n))$$
$$= \pi(g^{-1})f(g_\emptyset(x_1), g_{x_1}(x_2),\dots,g_{x_1\dots x_{n-1}}(x_n)).$$

Therefore,

$$\sum_{x=0}^{q-2} g^{-1}.f(x_1,\dots,x_{j-1},x)$$
$$= \sum_{x=0}^{q-2} \pi(g^{-1})f(g_\emptyset(x_1), g_{x_1}(x_2),\dots,g_{x_1\dots x_{j-2}}(x_{j-1}), g_{x_1\dots x_{j-1}}(x)).$$
$$= \pi(g^{-1})\sum_{x=0}^{q-2} f(g_\emptyset(x_1), g_{x_1}(x_2),\dots,g_{x_1\dots x_{j-2}}(x_{j-1}), g_{x_1\dots x_{j-1}}(x))$$
$$= \pi(g^{-1})\sum_{x'=0}^{q-2} f(g_\emptyset(x_1), g_{x_1}(x_2),\dots,g_{x_1\dots x_{j-2}}(x_{j-1}), x') \equiv 0.$$

Thus W_j are $Aut(T_{q,n})$-invariant.

Let $f \in W_j$, $f' \in W_{j'}$, $j \neq j'$ such that $f(x) = \sum_{k=1}^{d_{\tau_{q,n}}} f_k(x)v_k$, $f_k \in L(\Sigma^n)$ and $f'(x) = \sum_{k'=1}^{d_{\tau_{q,n}}} f'_{k'}(x)v_{k'}$, $f'_{k'} \in L(\Sigma^n)$. We have that $f' \in W_{j'}$ thus $\sum_{t=0}^{q-2} f'(x_1,x_2,\dots, x_{j'-1},t) = 0$. That is $\sum_{k'=1}^{d_{\tau_{q,n}}}(\sum_{t=0}^{q-2} f'_{k'}(x_1,\dots,x_{j'-1},t))v_{k'} = 0$. Since $(v_{k'})_{1\leqslant k'\leqslant d_{\tau_{q,n}}}$ is an orthonormal basis of $V_{\tau_{q,n}}$ then, they are linearly independent so $\sum_{t=0}^{q-2} f'_{k'}(x_1,x_2,\dots, x_{j-1},t) = 0$.

Suppose that $j < j'$. We have for $x = x_1 \dots x_n$,

$$< f, f' > = \sum_{x\in\Sigma^n} < f(x), f'(x) >$$
$$= \sum_{x_1\dots x_n\in\Sigma^n} < \sum_{k=1}^{d_{\tau_{q,n}}} f_k(x_1,\dots,x_n)v_k, \sum_{k'=1}^{d_\tau} f'_{k'}(x_1,\dots,x_n)v_{k'} >$$

$$= \sum_{x_1=0}^{q-1}\sum_{x_2=0}^{q-1}\cdots\sum_{x_{n-1}=0}^{q-1}\sum_{x_n=0}^{q-2} < \sum_{k=1}^{d_{\tau_{q,n}}} f_k(x_1,\ldots,x_n)v_k, \sum_{k'=1}^{d_{\tau_{q,n}}} f'_{k'}(x_1,\ldots,x_n)v_{k'} >$$

$$= \sum_{x_1=0}^{q-1}\sum_{x_2=0}^{q-1}\cdots\sum_{x_{n-1}=0}^{q-1}\sum_{x_n=0}^{q-2}\sum_{k,k'=1}^{d_{\tau_{q,n}}} < f_k(x_1,\ldots,x_n)v_k, f'_{k'}(x_1,\ldots,x_n)v_{k'} >$$

$$= \sum_{x_1=0}^{q-1}\sum_{x_2=0}^{q-1}\cdots\sum_{x_{n-1}=0}^{q-1}\sum_{x_n=0}^{q-2}\sum_{k,k'=1}^{d_{\tau_{q,n}}} f_k(x_1,\ldots,x_n)\overline{f'_{k'}(x_1,\ldots,x_n)} < v_k, v_{k'} >$$

$$= q^{n-j'}\sum_{x_1=0}^{q-1}\sum_{x_2=0}^{q-1}\cdots\sum_{x_{j'-1}=0}^{q-1}\sum_{k,k'=1}^{d_{\tau_{q,n}}} f_k(x_1,\ldots,x_j)\sum_{t=0}^{q-2}\overline{f'_{k'}(x_1,\ldots,x_{j'-1},t)} < v_k, v_{k'} >$$

$$= q^{n-j'}\sum_{x_1=0}^{q-1}\sum_{x_2=0}^{q-1}\cdots\sum_{x_{j'-1}=0}^{q-1} f_{k'}(x_1,\ldots,x_j)\sum_{k'=1}^{d_{\tau_{q,n}}}\sum_{t=0}^{q-2}\overline{f'_{k'}(x_1,\ldots,x_{j'-1},t)}.$$

As $\sum_{t=0}^{q-2} f'_{k'}(x_1,\ldots,x_{j'-1},t) = 0$, we conclude that $< f, f' > = 0$ and the spaces W_j are pairwise orthogonal.

The final step is to prove that the $W_j's$ fill all the space $L(\Sigma^n, V_{\tau_{q,n}})$. We prove it by induction on n.

For $n = 1$, f depends only on one variable and we have $f(x) = \sum_{k=1}^{d_{\tau_{q,n}}} f_k(x)v_k$, where v_k is an orthonormal basis of $V_{\tau_{q,n}}$ and $f_k \in L(\Sigma^1)$. Any function f_k can be expressed as: $f_k(x) = c_k + g_k(x)$, where $\sum_{x=0}^{q-2} g_k(x) = 0$ and $c_k = \frac{1}{q-1}\sum_{x=0}^{q-2} f_k(x)$. We have:

$f(x) = \sum_{k=1}^{d_{\tau_{q,n}}}(c_k + g_k(x))v_k = \sum_{k=1}^{d_{\tau_{q,n}}} c_k v_k + \sum_{k=1}^{d_{\tau_{q,n}}} g_k(x)v_k$. $\sum_{k=1}^{d_{\tau_{q,n}}} c_k v_k \in W_0$, also $\sum_{x=0}^{q-2}\sum_{k=1}^{d_{\tau_{q,n}}} g_k(x)v_k = \sum_{k=1}^{d_{\tau_{q,n}}}\sum_{x=0}^{q-2} g_k(x)v_k = 0$. Thus $\sum_{k=1}^{d_{\tau_{q,n}}} g_k(x)v_k \in W_1$. Therefore the assertion is true on rank one.

Suppose the assertion is true for $(n-1)$. A function $f \in L(\Sigma^n, V_{\tau_{q,n}})$ can be expressed as: $f(x_1,\ldots,x_{n-1},x_n) = \sum_{k=1}^{d_{\tau_{q,n}}} f_k(x_1,\ldots,x_{n-1},x_n)v_k$ with $f_k(x_1,\ldots,x_{n-1},x_n)$

$= c_k(x_1, \ldots, x_{n-2}, x_{n-1}) + g_k(x_1, \ldots, x_{n-1}, x_n)$, where $\sum_{x_n=0}^{q-2} g_k(x_1, x_2, \ldots, x_{n-1}, x_n) = 0$, and $c_k(x_1, \ldots, x_{n-1}) = \frac{1}{q-1} \sum_{x_n=0}^{q-2} f_k(x_1, \ldots, x_{n-1}, x_n)$. Therefore, $f(x_1, \ldots, x_{n-1}, x_n) = \sum_{k=1}^{d_{\tau_{q,n}}} (c_k(x_1, \ldots, x_{n-2}, x_{n-1}) + g_k(x_1, \ldots, x_{n-1}, x_n))v_k = \sum_{k=1}^{d_{\tau_{q,n}}} c_k(x_1, \ldots, x_{n-2}, x_{n-1})v_k + \sum_{k=1}^{d_{\tau_{q,n}}} g_k(x_1, \ldots, x_{n-1}, x_n)v_k$.

Set $H(x_1, \ldots, x_{n-2}, x_{n-1}) = \sum_{k=1}^{d_{\tau_{q,n}}} c_k(x_1, \ldots, x_{n-2}, x_{n-1})v_k$ and $R(x_1, \ldots, x_{n-1}, x_n) = \sum_{k=1}^{d_{\tau_{q,n}}} g_k(x_1, \ldots, x_{n-1}, x_n)v_k$. H depends only on $(n-1)$ variables, then according to induction, $H \in \bigoplus_{j=0}^{n-1} W_j$. Also, $\sum_{x=0}^{q-2} R(x_1, \ldots, x_{n-1}, x) = \sum_{x=0}^{q-2} (\sum_{k=1}^{d_{\tau_{q,n}}} g_k(x_1, \ldots, x_{n-1}, x)v_k)$ and $\sum_{k=1}^{d_{\tau_{q,n}}} (\sum_{x=0}^{q-2} g_k(x_1, \ldots, x_{n-1}, x))v_k = 0$. Thus $\sum_{k=1}^{d_{\tau_{q,n}}} g_k(x_1, \ldots, x_{n-1}, x_n)v_k \in W_n$, so the assertion is true for n.

We observe that $Aut(T_{q,n})$ acts on Σ^n transitively, and $d(\Sigma^n, \Sigma^n) = \{0, 1, \ldots, n\}$. Set $S(x_0, j) = \{x \in \Sigma^n : d(x_0, x) = j\}$ the sphere of radius j centred at x_0, these are the $K_{q,n}$-orbits. Since $d(\Sigma^n, \Sigma^n) = \{0, 1, \ldots, n\}$, so the number of $K_{q,n}$-orbits is exactly $n + 1$. By virtue of Theorem 3.6 [1], we have that the $W_j's$ are irreducible subspaces. □

We consider the action of $Aut(T_{q,n})$ on $L(\Sigma^n, End(V_{\tau_{q,n}}))$ defined by: $g.F(x) = g^{-1}.F(g^{-1}.x)$, for all $g \in Aut(T_{q,n})$, $F \in L(\Sigma^n, End(V_{\tau_{q,n}}))$ and $x \in \Sigma^n$. We designate by $L_{\tau_{q,n}}(\Sigma^n, End(V_{\tau_{q,n}})) = \{F : \Sigma^n \to End(V_{\tau_{q,n}}) : k.F(x) = \tau(k^{-1})F(x)$, for all $k \in K_{q,n}\}$.

The following result establishes an isomorphism between $L_{\tau_{q,n}}(\Sigma^n, End(V_{\tau_{q,n}}))$ and $L(Aut(T_{q,n}), End(V_{\tau_{q,n}}), \tau_{q,n}, \tau_{q,n})$.

Theorem 3.3. *Let us consider the map*

$$\begin{array}{rccc}\Theta & : & L_{\tau_{q,n}}(\Sigma^n, End(V_{\tau_{q,n}})) & \to & L(Aut(T_{q,n}), End(V_{\tau_{q,n}}), \tau_{q,n}, \tau_{q,n}) \\ & & F & \mapsto & \widetilde{F}\end{array}$$

defined by $\widetilde{F}(g) = g^{-1}.F(g^{-1}.x_0)$ *that is for all* $v \in V_{\tau_{q,n}}$*,* $\widetilde{F}(g)[v] = g^{-1}.F(g^{-1}.x_0)[v]$ $= F(g^{-1}.x_0)[g^{-1}.v]$ *is an isomorphism.*

Proof. Let us prove that $\widetilde{F}$ is a $\tau_{q,n}-$ radial function. For all $k_1, k_2 \in K_{q,n}$, $g \in Aut(T_{q,n})$ and $v \in V_{\tau_{q,n}}$ we have

$$\begin{aligned}
&\widetilde{F}(k_1 g k_2)[v] \\
&= (k_1 g k_2)^{-1}.F((k_1 g k_2)^{-1}.x_0)[v] = (k_2^{-1} g^{-1} k_1^{-1}).F(k_2^{-1} g^{-1} k_1^{-1}.x_0)[v] \\
&= (k_2^{-1} g^{-1} k_1^{-1}).F(k_2^{-1} g^{-1}.x_0)[v] = (k_2^{-1} g^{-1}).F(k_2^{-1} g^{-1}.x_0)(k_1^{-1}.v) \\
&= k_2^{-1}.g^{-1}.F(k_2^{-1} g^{-1}.x_0)(\tau_{q,n}(k_1^{-1})v) = k_2^{-1}.F(k_2^{-1} g^{-1}.x_0)(g^{-1}.\tau_{q,n}(k_1^{-1})v) \\
&= \tau_{q,n}(k_2^{-1})F(g^{-1}.x_0)(g^{-1}.\tau_{q,n}(k_1^{-1})v) = \tau_{q,n}(k_2^{-1})g^{-1}.F(g^{-1}.x_0)(\tau_{q,n}(k_1^{-1})v) \\
&= \tau_{q,n}(k_2^{-1})\widetilde{F}(g)\tau_{q,n}(k_1^{-1})[v]
\end{aligned}$$

Thus, $\widetilde{F}$ is a $\tau_{q,n}$-radial function. It is straightforward to prove that Θ is linear. Let $F_1, F_2 \in L(\Sigma^n, End(V_{\tau_{q,n}}))$ such that $\Theta(F_1) = \Theta(F_2)$. We have

$$\begin{aligned}
\Theta(F_1) = \Theta(F_2) &\Rightarrow \widetilde{F_1} = \widetilde{F_2} \Rightarrow \widetilde{F_1}(g) = \widetilde{F_2}(g) \\
&\Rightarrow g^{-1}.F_1(g^{-1}.x_0) = g^{-1}.F_2(g^{-1}.x_0), \quad \forall g \in Aut(T_{q,n}) \\
&\Rightarrow F_1(g^{-1}.x_0) = F_1(g^{-1}.x_0) \Rightarrow F_1 = F_2.
\end{aligned}$$

Thus Θ is injective. Let $H \in L(Aut(T_{q,n}), End(V_{\tau_{q,n}}), \tau_{q,n}, \tau_{q,n})$ and set $F(g.x_0) = g^{-1}.H(g^{-1})$. For all $k \in K_{q,n}$ and $v \in V_{\tau_{q,n}}$,

$$\begin{aligned}
(k.F(g.x_0))(v) &= k^{-1}.F(k^{-1}g.x_0)(v) = F(k^{-1}g.x_0)(k^{-1}v) = (g^{-1}k).H(g^{-1}k)(k^{-1}.v) \\
&= H(g^{-1}k)(g^{-1}kk^{-1}.v) = H(g^{-1}k)(g^{-1}.v) = \tau(k^{-1})H(g^{-1})(g^{-1}.v) \\
&= \tau(k^{-1})(g^{-1}.H(g^{-1}))(v) = \tau(k^{-1})F(g.x_0)(v).
\end{aligned}$$

Therefore, $F \in L_{\tau_{q,n}}(\Sigma^n, End(V_{\tau_{q,n}}))$.
Also, $\Theta(F)(g) = g^{-1}.F(g^{-1}.x_0) = g^{-1}.g.H(g) = H(g)$. Thus Θ is surjective and is an isomorphism. □

In the sequel, we set $S(x_0, j) = \{z \in \Sigma^n : d(x_0, z) = j\}$. We know that if $k \in K_{q,n}$, k is uniquely determined by labelling. We denote by $(k_\emptyset, k_{x_1}, k_{x_2}, \ldots, k_{x_1 x_2 \ldots x_{n-1}})$

the labelling of $k \in K_{q,n}$. Each label $k_{x_1x_2\dots x_j}$ is an automorphism of the subtree rooted at $x_1x_2\dots x_j$ [12].

Remark 3.1. *If $F \in L_{\tau_{q,n}}(\Sigma^n, End(V_{\tau_{q,n}}))$ and $x = x_1x_2\dots x_{n-1}x_n$, $y = y_1y_2\dots y_{n-1}y_n \in S(x_0, j)$ then there exists $k \in Aut(T_{q,n})$ such that $F(y) = F(x)\tau_{q,n}(k^{-1})$. In particular, if x and y are brothers we can identify k with its label $k_{x_1x_2\dots x_{n-1}}$ and write $F(y) = F(x)\tau_{q,n}(k^{-1}_{x_1x_2\dots x_{n-1}})$.*

In fact, the action of $K_{q,n}$ on Σ^n is transitive so there exists $k \in Aut(T_{q,n})$ such that $y = kx$. Thus $F(y) = F(kx) = F(x)\tau_{q,n}(k^{-1})$. Now if x and y are brothers we have

$$\begin{aligned} k.x &= k_\emptyset(x_1)k_{x_1}(x_2)\dots k_{x_1x_2\dots x_{n-2}}(x_{n-1})k_{x_1x_2\dots x_{n-1}}(x_n) \\ &= x_1x_2\dots x_{n-1}k_{x_1x_2\dots x_{n-1}}(x_n) \\ &= x_1x_2\dots x_{n-1}y_n, \end{aligned}$$

that is all the labels are identity map except the last one $k_{x_1x_2\dots x_{n-1}}$. So we can identify k with $k_{x_1x_2\dots x_{n-1}}$

In the next result, we obtain the explicit formula of $\tau_{q,n}$-spherical functions defined on Σ^n.

Theorem 3.4. *The $\tau_{q,n}$-spherical function is given by:*

$$\Psi_j(x_1, x_2, \dots, x_n)$$
$$= \begin{cases} I_{V_{\tau_{q,n}}} & \text{if} \quad x_1 = x_2 = \dots = x_j = 0; \\ -\big(I_{V_{\tau_{q,n}}} + \sum_{i=2}^{q-2} \tau_{q,n}((k^i_{x_1x_2\dots x_{j-1}})^{-1})\big)^{-1} & \\ \qquad \tau_{q,n}((k^{x_j}_{x_1\dots x_{j-1}})^{-1}), & \text{if} x_1 = x_2 = \dots = x_{j-1} = 0 \text{ and } x_j \neq 0; \\ 0, & otherwise; \end{cases},$$

where for $t \in \{1, 2, \dots, q-2\}$, $k^t_{x_1x_2\dots x_{j-1}}$ is the j^{ieme} label of $k^t \in Aut(T_{q,n})$ such that $k^t(x_1, x_2, \dots, x_{j-1}, 1) = x_1x_2\dots x_{j-1}t$.

Proof. The functions $\Psi_j \in L(\Sigma^n, End(V_{\tau_{q,n}}))$ are left $K_{q,n}-$covariant of type $\tau_{q,n}$. Moreover, we observe that if $x_1x_2\dots x_{j-1} \neq 00\dots0$, all the points of the form $x_1x_2\dots x_n$ have the same distance from the base point $x_0 = 00\dots0$. Since $\Psi_j(\bullet)v \in L(\Sigma^n, V_{\tau_{q,n}})$ is left $K_{q,n}$-covariant of type $\tau_{q,n}$, then

$\Psi_j(x_1,\dots,x_{j-1},\ell)v = \Psi_j(x_1,\dots,x_{j-1},0)\tau_{q,n}((k^{\ell}_{x_1x_2\dots x_{j-1}})^{-1})v$, for all $\ell = 1,\dots,q-2$. As $\Psi_j(\bullet)v \in W_j$ we have

$$\sum_{\ell=0}^{q-2}\Psi_j(x_1x_2\dots x_{j-1},\ell)v = \big(\sum_{\ell=0}^{q-2}\Psi_j(x_1,\dots,x_{j-1},\ell)\big)v$$
$$= \big(\Psi_j(x_1,\dots,x_{j-1},0) + \Psi_j(x_1,\dots,x_{j-1},0)\tau_{q,n}((k^{1}_{x_1x_2\dots x_{j-1}})^{-1})$$
$$+ \dots + \Psi_j(x_1,\dots,x_{j-1},0)\tau_{q,n}((k^{q-2}_{x_1x_2\dots x_{j-1}}))^{-1}\big)v$$
$$= (\Psi_j(x_1,\dots,x_{j-1},0)\big(I_{V_{\tau_{q,n}}} + \sum_{\ell=1}^{q-2}\tau_{q,n}((k^{\ell}_{x_1x_2\dots x_{j-1}})^{-1})\big)v = 0,$$

As $\tau_{q,n}$ is an irreducible unitary representation of $K_{q,n}$ then $(\tau_{q,n}(k)v,v) \geqslant 0$, for all $v \in V_{\tau_{q,n}}$, $k \in K_{q,n}$. Also $(I_{V_{\tau_{q,n}}}v,v) \geqslant 0$, for all $v \in V_{\tau_{q,n}}$. Since $I_{V_{\tau_{q,n}}}$ is invertible and $I_{V_{\tau_{q,n}}} \leqslant I_{V_{\tau_{q,n}}} + \sum_{\ell=1}^{q-2}\tau_{q,n}((k^{\ell}_{x_1x_2\dots x_{j-1}})^{-1})$ then by Theorem 2.3 [10], $I_{V_{\tau_{q,n}}} + \sum_{\ell=1}^{q-2}\tau_{q,n}((k^{\ell}_{x_1x_2\dots x_{j-1}})^{-1})$ is inversible. Therefore $\Psi_j(x_1,\dots,x_{j-1},0) = 0$.

Furthermore, $\Psi_j(x_1,\dots,x_{j-1},\ell)v = \Psi_j(x_1,\dots,x_{j-1},0)\tau_{q,n}(k^{\ell}_{x_1x_2\dots x_{j-1}})v$, for all $\ell = 1,\dots,q-2$ thus $\Psi_j(x_1,\dots,x_{j-1},\ell) = \Psi_j(x_1,\dots,x_{j-1},0) = 0$.

Similarly, all leaves of form $\underbrace{00\dots0}_{j-1}ty_{j+1}\dots y_n$ with $t = 1,2,\dots,q-2$ constitute the ball of radius $n-j+1$. By definition of $\tau_{q,n}$-spherical function, $\Psi_j(0,0,\dots,0) = I_{V_{\tau_{q,n}}}$. As $\Psi_j(\bullet)v \in W_j$, for all $v \in V_{\tau_{q,n}}$ then, $(\Psi_j(0,0,\dots,0) + \sum_{t=1}^{q-2}\Psi_j(0,0,\dots0,t))v$ $= 0$. Also,

$$\big(\Psi_j(0,0,\dots,0) + \sum_{t=1}^{q-2}\Psi_j(0,0,\dots0,t)\big)v = \big(I_{V_{\tau_{q,n}}} + \Psi_j(0,0,\dots0,1)(I_{V_{\tau_{q,n}}} + \sum_{i=2}^{q-2}\tau_{q,n}((k^{i}_{x_1\dots x_{j-1}}))^{-1})\big)v = 0,$$

and $\big(I_{V_{\tau_{q,n}}} + \Psi_j(0,0,\dots0,1)(I_{V_{\tau_{q,n}}} + \sum_{i=2}^{q-2}\tau_{q,n}((k^{i}_{x_1x_2\dots x_{j-1}})^{-1}))\big)v = 0$ implies $I_{V_{\tau_{q,n}}} + \Psi_j(0,0,\dots0,1)\big(I_{V_{\tau_{q,n}}} + \sum_{i=2}^{q-2}\tau_{q,n}((k^{i}_{x_1x_2\dots x_{j-1}})^{-1})\big) = 0$ which implies $-\Psi_j(0,0,\dots0,1)$

$\big(I_{V_{\tau_{q,n}}} + \sum_{i=2}^{q-2} \tau_{q,n}((k^i_{x_1x_2\ldots x_{j-1}})^{-1})\big) = I_{V_{\tau_{q,n}}}$. Thus $-\Psi_j(0,0,\ldots 0,1)$ admits an inverse on the right. As $V_{\tau_{q,n}}$ is a finite-dimensional vector space then $\Psi_j(0,0,\ldots 0,1)$ is invertible and $\Psi_j(0,0,\ldots 0,1) = -\big(I_{V_{\tau_{q,n}}} + \sum_{i=2}^{q-2} \tau_{q,n}((k^i_{x_1x_2\ldots x_{j-1}})^{-1})\big)^{-1}$.

For $t = 2,\ldots,q-2$, $\Psi_j(0,0,\ldots 0,t) = \Psi_j(0,0,\ldots 0,1)\tau_{q,n}((k^t_{x_1\ldots x_{j-1}})^{-1})$, it follows that $\Psi_j(0,0,\ldots 0,t) = -\big(I_{V_{\tau_{q,n}}} + \sum_{i=2}^{q-2} \tau_{q,n}((k^i_{x_1x_2\ldots x_{j-1}})^{-1})\big)^{-1}\tau_{q,n}((k^t_{x_1\ldots x_{j-1}})^{-1})$.

Finally, if $d(x,x_0) < n-j+1$, then $x = \underbrace{00\ldots0}_{h} y_{h+1}\ldots y_n$ with $h > j+1$. Thus $\Psi_j(x_1,x_2,\ldots,x_n) = \Psi_j(0,0,\ldots 0,0) = I_{V_{\tau_{q,n}}}$. $\square$

Remark 3.2. *If $\tau_{q,n}$ is the one-dimensional trivial representation of $K_{q,n}$, then we can identifty $\tau_{q,n}$ with $\mathbb{C}$. So, $\sum_{i=2}^{q-2} \tau_{q,n}((k^i_{x_1x_2\ldots x_{j-1}})^{-1})v = \sum_{i=2}^{q-2} v = (q-3)v$, for all $v \in V_{\tau_{q,n}}$. Therefore, $\sum_{i=2}^{q-2} 1 = (q-3)$ and $-(I_{V_{\tau_{q,n}}} + \sum_{i=2}^{q-2} \tau_{q,n}((k^i_{x_1x_2\ldots x_{j-1}})^{-1}))^{-1} = -(1+q-3)^{-1} = -\dfrac{1}{q-2}$. We obtain the spherical functions for the Gelfand pair $(Aut(T_{q,n}), K_{q,n})$ [7].*

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