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A CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY KOMATU INTEGRAL OPERATOR

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ABSTRACT. In this paper, we introduce a subclasses of analytic functions with negative coefficients defined by Komatu integral operator. We obtain the coefficient bounds, growth distortion properties, extreme points and radii of starlikeness and convexity and close-to-convexity for functions belonging to the class $TS_{a,\delta}^{k}(\vartheta,\hbar,\ell)$. Furthermore, we obtained modified Hardamard product, closure properties for this class.

1. INTRODUCTION

Let A denote the class of all functions $u(z)$ of the form

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $u(0) = u'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions $u(z)$

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which are all univalent in E. A function $u \in A$ is a starlike function of the order $\upsilon, 0 \leq \upsilon < 1$, if it satisfy

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \upsilon, \quad (z \in E). \tag{1.2}$$

We denote this class with $S^*(\upsilon)$.

A function $u \in A$ is a convex function of the order $\upsilon, 0 \leq \upsilon < 1$, if it satisfy

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \upsilon, \quad (z \in E). \tag{1.3}$$

We denote this class with $K(\upsilon)$.

Let T denote the class of functions analytic in E that are of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, \ a_n \geq 0 \quad (z \in E) \tag{1.4}$$

and let $T^*(\upsilon) = T \cap S^*(\upsilon), \ C(\upsilon) = T \cap K(\upsilon)$. The class $T^*(\upsilon)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [17] and others.

Differential operators in a complex domain play a significant role in functions theory and its information. They have used to describe the geometric interpolation of analytic functions in a complex domain. Also, they have utilized to generate new formulas of holomorphic functions. Lately, Lupas [10] presented a amalgamation of two well-known differential operators prearranged by Ruscheweyh [13] and Salagean [14]. Later, these operators are investigated by researchers considering different classes and formulas of analytic functions [5, 6, 11, 12, 15, 16].

For $u \in A$ given by (1.1) and $g(z)$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.5}$$

their convolution (or Hadamard product), denoted by $(u * g)$, is defined as

$$(u * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * u)(z) \quad (z \in E). \tag{1.6}$$

Note that $u * g \in A$.

Salagean [14] introduced the following differential operator for $u(z) \in A$ which is called the Salagean operator:

$$D^0u(z) = u(z),$$
$$D^1u(z) = Du(z) = zu'(z),$$
$$D^ku(z) = D(D^{k-1}u(z)), \quad (k \in \mathbb{N} = 1, 2, 3 \cdots).$$

We note that

$$D^ku(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n \quad (k \in \mathbb{N}_0 = \mathbb{N} \cup 0) \tag{1.7}$$

Recently, Komatu [8] introduced a certain integral operator L_a^δ defined by

$$L_a^\delta u(z) = \frac{a^\delta}{\Gamma(\delta)} \int_0^1 t^{a-2} \left(log\frac{1}{t}\right)^{\delta-1} u(zt)dt \quad 6(a > 0,\ \delta \geq 0,\ u(z) \in A) \tag{1.8}$$

Thus, if $u(z) \in A$ is of the form (1.1), it is easily seen from (1.8) that [8]

$$L_a^\delta u(z) = z + \sum_{n=2}^{\infty} \left(\frac{a}{a+n-1}\right)^\delta a_n z^n \quad (a > 0,\ \delta > 0). \tag{1.9}$$

We note that:

(i). $L_a^0 u(z) = u(z)$;
(ii). $L_1^1 u(z) = A[u](z)$ known as Alexander operator [1];
(iii). $L_2^1 u(z) = A[u](z)$ known as Liberal operator [9];
(iv). $L_{c+1}^1 u(z) = L_c[u](z)$ called Libera operator or Bernardi operator [3];
(v). For $a = 1$ and $\delta = k$ (k is any integer), the multiplier transformation $L_1^{-k} u(z) = I^k u(z)$ was studied by Flett [4] and Salgean [14];
(vi). For $a = 1$ and $\delta = -k$ $(k \in N_0 = N \cup 0,)$ the differential operator $L_1^{-k} u(z) = D^k u(z)$ was studied by Salagean [14];
(vii). For $a = 2$ and $\delta = k$ (k is any integer), $L_2^{-k} u(z) = L^k u(z)$ was studied by Uralegaddi and Somanatha [18];
(viii). $a = 2$, the multiplier transformation $L_2^\delta u(z) = I^\delta u(z)$ was studied by Jung et al [7].

For $D^k u(z)$ given by (1.7) and $L_a^\delta u(z)$ is given by (1.9), Arkan et al [2] defined the differential operator $D^k L_a^\delta u(z)$ as follows:

$$D^k L_a^\delta u(z) = z + \sum_{n=2}^{\infty} \phi(n, a, k, \delta) a_n z^n,$$

where

$$\phi(n, a, k, \delta) = n^k \left(\frac{a}{a+n-1} \right)^\delta . \tag{1.10}$$

Note that, by taking $\delta = 0$ and $k = 0$ in (1.10), the differential operator $D^k L_a^\delta u(z)$ reduces to Salagean differential operator and Komatu integral operator, respectively.

Using the operator $D^k L_a^\delta u$, we now introduce a new subclass of analytic functions as follows:

Definition 1.1. *For* $0 \le \vartheta < 1, 0 \le \hbar < 1,$ *and*$0 < \ell < 1,$ *we let* $TS_{a,\delta}^{k}(\vartheta, \hbar, \ell)$ *be the subclass of* u *consisting of functions of the form* (1.4) *and its geometrical condition satisfy*

$$\left| \frac{\vartheta \left((D^k L_a^\delta u(z))' - \frac{D^k L_a^\delta u(z)}{z} \right)}{\hbar \left(D^k L_a^\delta u(z) \right)' + (1-\vartheta) \frac{D^k L_a^\delta u(z)}{z}} \right| < \ell, \ z \in \mathbb{U},$$

where $D^k L_a^\delta u(z)$, *is given by (1.10).*

2. Coefficient Inequality

In the following theorem, we obtain a necessary and sufficient condition for function to be in the class $TS_{a,\delta}^{k}(\vartheta, \hbar, \ell)$.

Theorem 2.1. *Let the function* u *be defined by* (1.4). *Then* $u \in TS_{a,\delta}^{k}(\vartheta, \hbar, \ell)$ *if, and only if,*

$$\sum_{n=2}^{\infty} [\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)] \phi(n, a, k, \delta) a_n \le \ell(\hbar + (1-\vartheta)), \tag{2.1}$$

where $0 < \ell < 1, 0 \le \vartheta < 1, 0 \le \hbar < 1, \delta \ge 0, \sigma \in [0,1]$ *and* $\wp \in \mathbb{N}$. *The result* (2.1) *is sharp for the function*

$$u(z) = z - \frac{\ell(\hbar + (1-\vartheta))}{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)] \phi(n, a, k, \delta)} z^n, \quad n \ge 2.$$

Proof. Suppose that the inequality (2.1) holds true and $|z| = 1$. Then we obtain

$$\begin{aligned}
&\left|\vartheta\left((D^k L_a^\delta u(z))' - \frac{D^k L_a^\delta u(z)}{z}\right)\right| \\
&\quad - \ell\left|\hbar\left(D^k L_a^\delta u(z))' + (1-\vartheta)\frac{D^k L_a^\delta u(z)}{z}\right)\right| \\
=&\left|-\vartheta\sum_{n=2}^{\infty}(n-1)\phi(n,a,k,\delta)a_n z^{n-1}\right| \\
&\quad - \ell\left|\hbar + (1-\vartheta) - \sum_{n=2}^{\infty}(n\hbar + 1 - \vartheta)\phi(n,a,k,\delta)a_n z^{n-1}\right| \\
\leq&\sum_{n=2}^{\infty}[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\phi(n,a,k,\delta)a_n - \ell(\hbar + (1-\vartheta)) \\
\leq&0.
\end{aligned}$$

Hence, by maximum modulus principle, $u \in TS_{a,\delta}^{\,k}(\vartheta,\hbar,\ell)$.

Now assume that $u \in TS_{a,\delta}^{\,k}(\vartheta,\hbar,\ell)$ so that

$$\left|\frac{\vartheta\left((D^k L_a^\delta u(z))' - \frac{D^k L_a^\delta u(z)}{z}\right)}{\hbar(D^k L_a^\delta u(z))' + (1-\vartheta)\frac{D^k L_a^\delta u(z)}{z}}\right| < \ell, \quad z \in \mathbb{U}.$$

Hence,

$$\left|\vartheta\left((D^k L_a^\delta u(z))' - \frac{D^k L_a^\delta u(z)}{z}\right)\right| < \ell\left|\hbar\left(D^k L_a^\delta u(z))' + (1-\vartheta)\frac{D^k L_a^\delta u(z)}{z}\right)\right|.$$

Therefore, we get

$$\begin{aligned}
&\left|-\sum_{n=2}^{\infty}\vartheta(n-1)\phi(n,a,k,\delta)a_n z^{n-1}\right| \\
&< \ell\left|\hbar + (1-\vartheta) - \sum_{n=2}^{\infty}(n\hbar + 1 - \vartheta)\phi(n,a,k,\delta)a_n z^{n-1}\right|.
\end{aligned}$$

Thus,

$$\sum_{n=2}^{\infty}[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\phi(n,a,k,\delta)a_n \leq \ell(\hbar + (1-\vartheta)),$$

and this completes the proof. □

Corollary 2.1. *Let the function* $u \in TS^{k}_{a,\delta}(\vartheta, \hbar, \ell)$. *Then*

$$a_n \leq \frac{\ell(\hbar + (1-\vartheta))}{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\phi(n,a,k,\delta)} z^n, \quad n \geq 2.$$

3. Distortion and Covering Theorem

We introduce the growth and distortion theorems for the functions in the class $TS^{k}_{a,\delta}(\vartheta, \hbar, \ell)$.

Theorem 3.1. *Let the function* $u \in TS^{k}_{a,\delta}(\vartheta, \hbar, \ell)$. *Then*

$$\begin{aligned} |z| - \frac{\ell(\hbar + (1-\vartheta))}{[\vartheta + \ell(2\hbar + 1 - \vartheta)]\phi(2,a,k,\delta)}|z|^2 &\leq |u(z)| \\ \leq |z| + \frac{\ell(\hbar + (1-\vartheta))}{[\vartheta + \ell(2\hbar + 1 - \vartheta)]\phi(2,a,k,\delta)}|z|^2. \end{aligned}$$

The result is sharp and attained for

$$u(z) = z - \frac{\ell(\hbar + (1-\vartheta))}{[\vartheta + \ell(2\hbar + 1 - \vartheta)]\phi(2,a,k,\delta)} z^2.$$

Proof. First,

$$|u(z)| = \left| z - \sum_{n=2}^{\infty} a_n z^n \right| \leq |z| + \sum_{n=2}^{\infty} a_n |z|^n \leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n.$$

By Theorem 2.1, we get

$$\sum_{n=2}^{\infty} a_n \leq \frac{\ell(\hbar + (1-\vartheta))}{[\vartheta + \ell(2\hbar + 1 - \vartheta)]\phi(n,a,k,\delta)}. \tag{3.1}$$

Thus,

$$|u(z)| \leq |z| + \frac{\ell(\hbar + (1-\vartheta))}{[\vartheta + \ell(2\hbar + 1 - \vartheta)]\phi(2,a,k,\delta)}|z|^2.$$

Also,

$$\begin{aligned} |u(z)| &\geq |z| - \sum_{n=2}^{\infty} a_n |z|^n \geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq |z| - \frac{\ell(\hbar + (1-\vartheta))}{[\vartheta + \ell(2\hbar + 1 - \vartheta)]\phi(2,a,k,\delta)}|z|^2. \end{aligned}$$

Then proof of the theorem follows. ∎

Theorem 3.2. *Let $u \in TS_{a,\delta}^{k}(\vartheta, \hbar, \ell)$. Then*

$$1-\frac{2\ell(\hbar+(1-\vartheta))}{[\vartheta+\ell(2\hbar+1-\vartheta)\phi(2,a,k,\delta)]}|z| \leq |u'(z)| \leq 1+\frac{2\ell(\hbar+(1-\vartheta))}{[\vartheta+\ell(2\hbar+1-\vartheta)]\phi(2,a,k,\delta)}|z|$$

with equality for

$$u(z) = z - \frac{2\ell(\hbar+(1-\vartheta))}{[\vartheta+\ell(2\hbar+1-\vartheta)]\phi(2,a,k,\delta)}z^2.$$

Proof. Notice that

$$\begin{aligned}(3.2)\qquad & [\vartheta+\ell(2\hbar+1-\vartheta)]\phi(2,a,k,\delta)\sum_{n=2}^{\infty} na_n \\ & \leq \sum_{n=2}^{\infty} n[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,a,k,\delta)a_n \leq \ell(\hbar+(1-\vartheta)),\end{aligned}$$

from Theorem 2.1. Thus

$$\begin{aligned}|u'(z)| &= \left|1-\sum_{n=2}^{\infty} na_n z^{n-1}\right| \leq 1+\sum_{n=2}^{\infty} na_n|z|^{n-1} \leq 1+|z|\sum_{n=2}^{\infty} na_n \\ &\leq 1+|z|\frac{2\ell(\hbar+(1-\vartheta))}{[\vartheta+\ell(2\hbar+1-\vartheta)]\phi(2,a,k,\delta)}. \qquad (3.3)\end{aligned}$$

On the other hand

$$\begin{aligned}|u'(z)| &= \left|1-\sum_{n=2}^{\infty} na_n z^{n-1}\right| \geq 1-\sum_{n=2}^{\infty} na_n|z|^{n-1} \geq 1-|z|\sum_{n=2}^{\infty} na_n \\ &\geq 1-|z|\frac{2\ell(\hbar+(1-\vartheta))}{[\vartheta+\ell(2\hbar+1-\vartheta)]\phi(2,a,k,\delta)}. \qquad (3.4)\end{aligned}$$

Combining (3.3) and (3.4), we get the result. □

4. Radii of Starlikeness, Convexity and Close-to-Convexity

In the following theorems, we obtain the radii of starlikeness, convexity and close-to-convexity for the class $TS_{a,\delta}^{k}(\vartheta, \hbar, \ell)$.

Theorem 4.1. *Let $u \in TS_{a,\delta}^{k}(\vartheta, \hbar, \ell)$. Then u is starlike in $|z| < R_1$ of order $\vartheta, 0 \leq \vartheta < 1$, where*

$$(4.1)\qquad R_1 = \inf_n \left\{\frac{(1-\vartheta)(\vartheta(n-1)+\ell(n\hbar+1-\vartheta))\phi(n,a,k,\delta)}{(n-\vartheta)\ell(\hbar+(1-\vartheta))}\right\}^{\frac{1}{n-1}}, \quad n \geq 2.$$

Proof. The function u is starlike of order $\vartheta, 0 \leq \vartheta < 1$, if

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \vartheta.$$

Thus, it is enough to show that

$$\left|\frac{zu'(z)}{u(z)} - 1\right| = \left|\frac{-\sum\limits_{n=2}^{\infty}(n-1)a_n z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_n z^{n-1}}\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_n|z|^{n-1}}.$$

Thus

$$\left|\frac{zu'(z)}{u(z)} - 1\right| \leq 1 - \vartheta \; if \; \sum_{n=2}^{\infty}\frac{(n-\vartheta)}{(1-\vartheta)}a_n|z|^{n-1} \leq 1. \tag{4.2}$$

Hence, by Theorem 2.1, (4.2) will be true, if

$$\frac{n-\vartheta}{1-\vartheta}|z|^{n-1} \leq \frac{(\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta))\phi(n,a,k,\delta)}{\ell(\hbar + (1-\vartheta)},$$

or, if

$$|z| \leq \left[\frac{(1-\vartheta)(\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta))\phi(n,a,k,\delta)}{(n-\vartheta)\ell(\hbar + (1-\vartheta))}\right]^{\frac{1}{n-1}}, \quad n \geq 2. \tag{4.3}$$

The theorem follows easily from (4.3). □

Theorem 4.2. *Let $u \in TS_{a,\delta}^{k}(\vartheta,\hbar,\ell)$. Then u is convex in $|z| < R_2$ of order $\vartheta, 0 \leq \vartheta < 1$,, where*

$$R_2 = \inf_n \left\{\frac{(1-\vartheta)(\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta))\phi(n,a,k,\delta)}{n(n-\vartheta)\ell(\hbar + (1-\vartheta))}\right\}^{\frac{1}{n-1}}, \quad n \geq 2. \tag{4.4}$$

Proof. The function u is convex of order $\vartheta, 0 \leq \vartheta < 1$, if

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \vartheta.$$

Thus, it is enough to show that

$$\left|\frac{zu''(z)}{u'(z)}\right| = \left|\frac{-\sum\limits_{n=2}^{\infty}n(n-1)a_n z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} na_n z^{n-1}}\right| \leq \frac{\sum\limits_{n=2}^{\infty}n(n-1)a_n|z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} na_n|z|^{n-1}}.$$

Thus,

$$\left|\frac{zu''(z)}{u'(z)}\right| \le 1-\vartheta \; if \; \sum_{n=2}^{\infty}\frac{n(n-\vartheta)}{(1-\vartheta)}a_n|z|^{n-1} \le 1. \tag{4.5}$$

Hence, by Theorem 2.1, (4.5) will be true, if

$$\frac{n(n-\vartheta)}{1-\vartheta}|z|^{n-1} \le \frac{(\vartheta(n-1)+\ell(n\hbar+1-\vartheta))\phi(n,a,k,\delta)}{\ell(\hbar+(1-\vartheta)},$$

or, if

$$|z| \le \left[\frac{(1-\vartheta)(\vartheta(n-1)+\ell(n\hbar+1-\vartheta))\phi(n,a,k,\delta)}{n(n-\vartheta)\ell(\hbar+(1-\vartheta))}\right]^{\frac{1}{n-1}}, \quad n \ge 2. \tag{4.6}$$

The theorem follows easily from (4.6). □

Theorem 4.3. *Let $u \in TS_{a,\delta}^{k}(\vartheta,\hbar,\ell)$. Then u is close-to-convex in $|z| < R_3$ of order $\vartheta, 0 \le \vartheta < 1,$ where*

$$R_3 = \inf_n \left\{\frac{(1-\vartheta)(\vartheta(n-1)+\ell(n\hbar+1-\vartheta))\phi(n,a,k,\delta)}{n\ell(\hbar+(1-\vartheta))}\right\}^{\frac{1}{n-1}}, \quad n \ge 2. \tag{4.7}$$

Proof. The function u is close-to-convex of order $\vartheta, 0 \le \vartheta < 1$, if $\Re\{u'(z)\} > \vartheta$. Thus, it is enough to show that

$$|u'(z)-1| = \left|-\sum_{n=2}^{\infty} na_n z^{n-1}\right| \le \sum_{n=2}^{\infty} na_n|z|^{n-1}.$$

Thus,

$$|u'(z)-1| \le 1-\vartheta \; if \; \sum_{n=2}^{\infty}\frac{n}{(1-\vartheta)}a_n|z|^{n-1} \le 1. \tag{4.8}$$

Hence, by Theorem 2.1, (4.8) will be true, if

$$\frac{n}{1-\vartheta}|z|^{n-1} \le \frac{(\vartheta(n-1)+\ell(n\hbar+1-\vartheta))\phi(n,a,k,\delta)}{\ell(\hbar+(1-\vartheta)}$$

or, if

$$|z| \le \left[\frac{(1-\vartheta)(\vartheta(n-1)+\ell(n\hbar+1-\vartheta))\phi(n,a,k,\delta)}{n\ell(\hbar+(1-\vartheta))}\right]^{\frac{1}{n-1}}, \quad n \ge 2. \tag{4.9}$$

The theorem follows easily from (4.9). □

5. Extreme points

In the following theorem, we obtain extreme points for the class $TS_{a,\delta}^{k}(\vartheta, \hbar, \ell)$.

Theorem 5.1. *Let* $u_1(z) = z$ *and*

$$u_n(z) = z - \frac{\ell(\hbar + (1-\vartheta))}{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\phi(n,a,k,\delta)} z^n, \text{ for } n = 2, 3, \cdots.$$

Then $u \in TS_{a,\delta}^{k}(\vartheta, \hbar, \ell)$ *if and only if it can be expressed in the form*

$$u(z) = \sum_{n=1}^{\infty} \theta_n u_n(z), \text{ where } \theta_n \geq 0 \text{ and } \sum_{n=1}^{\infty} \theta_n = 1.$$

Proof. Assume that $u(z) = \sum\limits_{n=1}^{\infty} \theta_n u_n(z)$, hence we get

$$u(z) = z - \sum_{n=2}^{\infty} \frac{\ell(\hbar + (1-\vartheta))\theta_n}{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\phi(n,a,k,\delta)} z^n.$$

Now, $u \in TS_{a,\delta}^{k}(\vartheta, \hbar, \ell)$, since

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\phi(n,a,k,\delta)}{\ell(\hbar + (1-\vartheta))} \\ &\times \frac{\ell(\hbar + (1-\vartheta))\theta_n}{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\phi(n,a,k,\delta)} \\ &= \sum_{n=2}^{\infty} \theta_n = 1 - \theta_1 \leq 1. \end{aligned}$$

Conversely, suppose $u \in TS_{a,\delta}^{k}(\vartheta, \hbar, \ell)$. Then we show that u can be written in the form $\sum\limits_{n=1}^{\infty} \theta_n u_n(z)$.

Now $u \in TS_{a,\delta}^{k}(\vartheta, \hbar, \ell)$ implies from Theorem 2.1

$$a_n \leq \frac{\ell(\hbar + (1-\vartheta))}{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\phi(n,a,k,\delta)}.$$

Setting $\theta_n = \frac{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,a,k,\delta)}{\ell(\hbar+(1-\vartheta))} a_n, n = 2, 3, \cdots$ and $\theta_1 = 1 - \sum\limits_{n=2}^{\infty} \theta_n$, we obtain $u(z) = \sum\limits_{n=1}^{\infty} \theta_n u_n(z)$. $\square$

6. Hadamard product

In the following theorem, we obtain the convolution result for functions belongs to the class $TS_{a,\delta}^{k}(\vartheta,\hbar,\ell)$.

Theorem 6.1. *Let* $u,g\in TS_{a,\delta}^{k}(\vartheta,\hbar,\ell,\lambda)$. *Then* $u*g\in TS_{a,\delta}^{k}(\vartheta,\hbar,\zeta,\lambda)$ *for*

$$u(z)=z-\sum_{n=2}^{\infty}a_nz^n,\ g(z)=z-\sum_{n=2}^{\infty}b_nz^n \text{ and } (u*g)(z)=z-\sum_{n=2}^{\infty}a_nb_nz^n,$$

where

$$\zeta\geq\frac{\ell^2(\hbar+(1-\vartheta))\vartheta(n-1)}{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]^2\phi(n,a,k,\delta)-\ell^2(\hbar+(1-\vartheta))(n\hbar+1-\vartheta)}.$$

Proof. $u\in TS_{a,\delta}^{k}(\vartheta,\hbar,\ell)$ and so

$$\sum_{n=2}^{\infty}\frac{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,a,k,\delta)}{\ell(\hbar+(1-\vartheta))}a_n\leq 1, \tag{6.1}$$

and

$$\sum_{n=2}^{\infty}\frac{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,a,k,\delta)}{\ell(\hbar+(1-\vartheta))}b_n\leq 1. \tag{6.2}$$

We have to find the smallest number ζ such that

$$\sum_{n=2}^{\infty}\frac{[\vartheta(n-1)+\zeta(n\hbar+1-\vartheta)]\phi(n,a,k,\delta)}{\zeta(\hbar+(1-\vartheta))}a_nb_n\leq 1. \tag{6.3}$$

By Cauchy-Schwarz inequality

$$\sum_{n=2}^{\infty}\frac{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,a,k,\delta)}{\ell(\hbar+(1-\vartheta))}\sqrt{a_nb_n}\leq 1. \tag{6.4}$$

Therefore it is enough to show that

$$\begin{aligned}&\frac{[\vartheta(n-1)+\zeta(n\hbar+1-\vartheta)]\phi(n,a,k,\delta)}{\zeta(\hbar+(1-\vartheta))}a_nb_n\\ \leq&\frac{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,a,k,\delta)}{\ell(\hbar+(1-\vartheta))}\sqrt{a_nb_n}.\end{aligned}$$

That is

$$\sqrt{a_nb_n}\leq\frac{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\zeta}{[\vartheta(n-1)+\zeta(n\hbar+1-\vartheta)]\ell}. \tag{6.5}$$

From (6.4)

$$\sqrt{a_n b_n} \leq \frac{\ell(\hbar + (1-\vartheta))}{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\phi(n,a,k,\delta)}.$$

Thus it is enough to show that

$$\begin{aligned}&\frac{\ell(\hbar + (1-\vartheta))}{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\phi(n,a,k,\delta)}\\ \leq &\frac{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\zeta}{[\vartheta(n-1) + \zeta(n\hbar + 1 - \vartheta)]\ell},\end{aligned}$$

which simplifies to

$$\zeta \geq \frac{\ell^2(\hbar + (1-\vartheta))\vartheta(n-1)}{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]^2\phi(n,a,k,\delta) - \ell^2(\hbar + (1-\vartheta))(n\hbar + 1 - \vartheta)}.$$

$\square$

7. Closure theorems

We shall prove the following closure theorems for the class $TS_{a,\delta}^{k}(\vartheta, \hbar, \ell)$.

Theorem 7.1. *Let* $u_j \in TS_{a,\delta}^{k}(\vartheta, \hbar, \ell), j = 1, 2, \cdots, s$. *Then*

$$g(z) = \sum_{j=1}^{s} c_j u_j(z) \in TS_{a,\delta}^{k}(\vartheta, \hbar, \ell).$$

For $u_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n$, *where* $\sum_{j=1}^{s} c_j = 1$.

Proof. First

$$g(z) = \sum_{j=1}^{s} c_j u_j(z) = z - \sum_{n=2}^{\infty} \sum_{j=1}^{s} c_j a_{n,j} z^n = z - \sum_{n=2}^{\infty} e_n z^n,$$

where $e_n = \sum_{j=1}^{s} c_j a_{n,j}$. Thus $g(z) \in TS_{a,\delta}^{k}(\vartheta, \hbar, \ell)$ if

$$\sum_{n=2}^{\infty} \frac{[\vartheta(n-1) + \ell(n\hbar + 1 - \vartheta)]\phi(n,a,k,\delta)}{\ell(\hbar + (1-\vartheta))} e_n \leq 1,$$

that is, if

$$\begin{aligned}&\sum_{n=2}^{\infty}\sum_{j=1}^{s}\frac{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,a,k,\delta)}{\ell(\hbar+(1-\vartheta))}c_j a_{n,j}\\ &=\sum_{j=1}^{s}c_j\sum_{n=2}^{\infty}\frac{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]\phi(n,a,k,\delta)}{\ell(\hbar+(1-\vartheta))}a_{n,j}\\ &\leq\sum_{j=1}^{s}c_j=1.\end{aligned}$$

□

Theorem 7.2. *Let* $u,g\in TS^{\,k}_{a,\delta}(\vartheta,\hbar,\ell)$. *Then*

$$h(z)=z-\sum_{n=2}^{\infty}(a_n^2+b_n^2)z^n\in TS^{\,k}_{a,\delta}(\vartheta,\hbar,\ell,\zeta),$$

where

$$\zeta\geq\frac{2\vartheta(n-1)\ell^2(\hbar+(1-\vartheta))}{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]^2\phi(n,a,k,\delta)-2\ell^2(\hbar+(1-\vartheta))(n\hbar+1-\vartheta)}.$$

Proof. Since $u,g\in TS^{\,k}_{a,\delta}(\vartheta,\hbar,\ell)$, so Theorem 2.1 yields

$$\sum_{n=2}^{\infty}\left[\frac{(\vartheta(n-1)+\ell(n\hbar+1-\vartheta))\phi(n,a,k,\delta)}{\ell(\hbar+(1-\vartheta))}a_n\right]^2\leq 1,$$

and

$$\sum_{n=2}^{\infty}\left[\frac{(\vartheta(n-1)+\ell(n\hbar+1-\vartheta))\phi(n,a,k,\delta)}{\ell(\hbar+(1-\vartheta))}b_n\right]^2\leq 1.$$

We obtain from the last two inequalities

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{(\vartheta(n-1)+\ell(n\hbar+1-\vartheta))\phi(n,a,k,\delta)}{\ell(\hbar+(1-\vartheta))}\right]^2(a_n^2+b_n^2)\leq 1. \tag{7.1}$$

But $h(z)\in TS^{\,k}_{a,\delta}(\vartheta,\hbar,\ell,\zeta)$, if and only, if

$$\sum_{n=2}^{\infty}\frac{[\vartheta(n-1)+\zeta(n\hbar+1-\vartheta)]\phi(n,a,k,\delta)}{\zeta(\hbar+(1-\vartheta))}(a_n^2+b_n^2)\leq 1, \tag{7.2}$$

where $0 < \zeta < 1$, however (7.1) implies (7.2) if

$$\begin{aligned}&\frac{[\vartheta(n-1)+\zeta(n\hbar+1-\vartheta)]\phi(n,a,k,\delta)}{\zeta(\hbar+(1-\vartheta))}\\ \leq&\frac{1}{2}\left[\frac{(\vartheta(n-1)+\ell(n\hbar+1-\vartheta))\phi(n,a,k,\delta)}{\ell(\hbar+(1-\vartheta))}\right]^2.\end{aligned}$$

Simplifying, we get

$$\zeta \geq \frac{2\vartheta(n-1)\ell^2(\hbar+(1-\vartheta))}{[\vartheta(n-1)+\ell(n\hbar+1-\vartheta)]^2\phi(n,a,k,\delta)-2\ell^2(\hbar+(1-\vartheta))(n\hbar+1-\vartheta)}.$$

□

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