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APPLICATION OF THE FOURIER TRANSFORMATION FOR SOLVING A FIRST ORDER PARTIAL DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

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ABSTRACT. This work is devoted to the solution of a linear first-order hyperbolic partial differential equation with constant coefficients. The specific objective is to prove the existence and uniqueness of the solution of the proposed PDE. The existence and uniqueness of the solution have been proved. To demonstrate the existence of the solution, the Fourier transformation was used. The variational formulation was used to prove the uniqueness of the solution. The combination of the Fourier transformation and the variational formulation yielded the expected results: the existence and uniqueness of the solution.

1. INTRODUCTION

PDEs appear frequently in applied sciences to translate fundamental principles and continuously model physical phenomena.

A PDE generally translates physical principles (such as the conservation of mass, energy, momentum) and models (such as the force /deformation relationship in a spring, the law of gravitation), in which one can have reasonable trust [1,3,5,6,9].

EDPs are classified according to three types, namely: Hyperbolic EDPs, Parabolic EDPs and Elliptical EDPs [2, 5, 11, 13, 14].

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This work is devoted to the application of the Fourier transformation for the resolution of a linear hyperbolic PDE of first order, with constant coefficients.

Besides the introduction and the general conclusion, this work presents some essential notions, useful for the good use of the Fourier transformation in this specific case. These notions and these results constitute an essential tool to approach the study of the type of problem of this article. The main activity of this work, is to prove the existence as well as the uniqueness of the solution of a linear hyperbolic partial differential equation (PDE) of the first order with constant coefficients, by the use of the Fourier transformation .

2. Resolution of problem

In this section which is the main one of this work, the activity is to study a case of a hyperbolic system intervening in many branches of science: kinetic theory, mechanics of inviscid fluids, magneto hydrodynamics, dynamics of inviscid gases , road traffic, flow of a river or a glacier, sedimentation process, chemical exchange process, etc [3, 7–10, 12, 15].

2.1. **Presentation of the problem.** This work concerns the resolution by the Fourier transformation of a linear partial differential equation of order 1 with constant coefficients which is presented as follows:

$$u_t + \sum_{j=1}^{n} B_j u_{xj} = 0. \tag{2.1}$$

As:

$$u : \mathbb{R}^n \times [0, \infty[\to \mathbb{R}^m \qquad \text{and} \qquad B_j : \mathbb{R}^n \times [0, \infty[\to \mathbb{M}^{m\times m}.$$

Where the B_j have square matrices of order m with the initial condition:

$$u = g. \tag{2.2}$$

Knowing that

$$g : \mathbb{R}^n \to \mathbb{R}^m.$$

2.2. **Hyperbolic system of partial differential equations, Fourier transform.** Consider the following system of s partial differential equations of the first order for s unknown functions.

$$\vec{u} = (u_1, \dots, u_s), \qquad \vec{u} = (\vec{x}, t), \qquad \text{with} \qquad \vec{x} \in \mathbb{R}^d.$$

$$\frac{\partial \vec{u}}{\partial t} + \sum_{j=1}^{d} \frac{\partial}{\partial x_j} \vec{f^j}(\vec{u}) = 0. \tag{2.3}$$

For $j = 1, \dots, d$, the $\vec{f^j} \in C^1(\mathbb{R}^s, \mathbb{R}^s)$, are continuously differentiable functions, nonlinear in general. We then set each $\vec{f^j}$ the Jacobian matrix $s \times s$.

$$A^j = \begin{pmatrix} \frac{\partial f_1^j}{\partial u_1} \cdots & \frac{\partial f_1^j}{\partial u_s} \\ \vdots & \ddots \vdots \\ \frac{\partial f_s^j}{\partial u_1} \cdots & \frac{\partial f_s^j}{\partial u_s} \end{pmatrix} \qquad \text{for} \qquad j = 1, \dots, d.$$

Thus, the system of equations (2.3) can also be written as follows:

$$\frac{\partial \vec{u}}{\partial t} + \sum_{j=1}^{d} A^j \frac{\partial \vec{u}}{\partial x_j} = 0. \tag{2.4}$$

Definition 2.1. *This system* (2.4) *is said to be hyperbolic if, for all* $\alpha_1, \dots \alpha_d \in \mathbb{R}$, *the matrix* $A = \alpha_1 A^1 + \dots + \alpha_d A^d$ *has real eigenvalues and is diagonalizable, if the matrix* A *has two by two distinct real eigenvalues, it is then diagonalizable and we then speak of a strictly hyperbolic system ([14, 18]). Either*

$$\partial_t u(t, x_1, \dots, x_d) + \sum_{i=1}^{d} A_i(u(t, x_1, \dots, x_d)) \partial_{x_i} u(t, x_1, \dots, x_d) = 0. \tag{2.5}$$

Definition 2.2. *The system* (2.4) *is said to be (strictly) hyperbolic in* $u \subset \mathbb{R}^N$ *if, and only if, the matrix*

$$A(u, \xi) = \sum_{i=1}^{d} \xi_i A_i(u)$$

is diagonalizable with real (distinct) eigenvalues for all $(u, \xi) \in u \times \mathbb{R}^d$,*, where we noted* $\xi = (\xi_i)_{i=1}^d \in \mathbb{R}^d$.

Definition 2.3. *If* $u \in L^{'}(\mathbb{R}^n)$, *we define its Fourier transform* $\mathcal{F}[U] = \hat{U}$ *by:*

$$\hat{U}(y) \;= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ixy} u(x) dy \qquad (y \in \mathbb{R}^n). \tag{2.6}$$

Its inverse Fourier transform $\mathcal{F}^{-1} = \hat{U}$ *is given by the relation* (2.7) *below*

$$\hat{U}(y) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{ixy} u(x) dy \qquad (y \in \mathbb{R}^n). \tag{2.7}$$

Since $|e^{\pm ixy}| = 1$ *and* $U \in L^1(\mathbb{R}^n)$, *these integrals converge for each* $y \in \mathbb{R}^n$. *We now intend to extend the definition given by the equations* (2.6) *and* (2.7) *to the function* $U \in L^2(\mathbb{R}^n)$

Theorem 2.1 (Planchevel Theorem, [4, 15–17]). *Suppose* $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. *Next,* $\hat{u}, \hat{u} \in L^2(\mathbb{R}^n)$ *and*

$$\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|\tilde{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}.$$

Theorem 2.2. *Let* k *be a non-negative integer.*

(i) *A function* $u \in L^2(\mathbb{R}^n)$ *is long in* $H^k(\mathbb{R}^n)$ *if and only if:* $\left(1 + |y|^k\right) \hat{u} \in L^2(\mathbb{R}^n)$ *;*

(ii) *Moreover, there is a positive constant* c *such that* $\frac{1}{c}\|u\|_{H^k(\mathbb{R}^n)} \leq \| \left(1 + |y|^k\right) \hat{u}\|_{L^2(\mathbb{R}^n)} \leq \|u\|_{H^k(\mathbb{R}^n)}$, *pour chaque* $u \in H^k(\mathbb{R}^n)$.

Definition 2.4. *Suppose* $0 < s < \infty$ *and* $u \in L^2(\mathbb{R}^n)$. *Afterwards* $u \in H^s(\mathbb{R}^n)$. *If* $(1 + |y|^s) \hat{u} \in L^2(\mathbb{R}^n)$ *for non-integer* s, *we set:* $\|u\|_{H^s(\mathbb{R}^n)} = \| (1 + |y|^s) \hat{u}\|_{L^2(\mathbb{R}^n)}$.

2.3. **Existence of the solution.** For our system to be hyperbolic, we will assume that the $\{B_j\}_{j=1}^n$ are constant and diagonalizable matrices $m \times m$.

Hyperbolicity condition:

For our system to be hyperbolic, we will assume that the $\{B_j\}_{j=1}^n$ are constant and diagonalizable matrices $m \times m$.

(i) Either $B(y) = \sum_{j=1}^n y_j B_j$ for each $y \in \mathbb{R}^n$, with real eigenvalues.

(ii) As: $\lambda_1(y) \leq \lambda_2(y) \leq \cdots \leq \lambda_m(y)$.

A priori, there is no assumption concerning the eigenvectors, and therefore we assume here only a kind of very weak hyperbolicity. Then we also make no symmetry assumption for the matrices $\{B_j\}_{j=1}^n$.

Let's take $g \in H^S(\mathbb{R}^n, \mathbb{R}^m)$, $S > \frac{n}{2} + m$. So there is only one solution $u \in \mathcal{C}^1(\mathbb{R}^n \times [0, \infty[, \mathbb{R}^m)$ from problem to initial values (2.1), (2.2).

The function $u = (u^1, \ldots, u^m)$ is assumed to be a smooth solution then set $\hat{u} = (\hat{u}^1, \ldots, \hat{u}^m)$ of the problem. the transformation will be done with respect to the time variable t. Thus the equation (2.1) becomes:

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{n} B_j \frac{\partial u}{\partial x_j} = 0. \tag{2.8}$$

Applying the member-to-member Fourier transform to the equation (2.8) gives:

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right](y) + \mathcal{F}\left[\sum_{j=1}^{n} B_j \frac{\partial u}{\partial x_j}\right](y) = 0. \tag{2.9}$$

This last relation is equivalent to:

$$\frac{\partial \hat{u}}{\partial t}(y,t) + \sum_{j=1}^{n} B_j \mathcal{F}\left[\frac{\partial u}{\partial x_j}\right] = 0. \tag{2.10}$$

Either

$$\frac{\partial \hat{u}}{\partial t} + i \sum_{j=1}^{n} B_j \lambda_j \hat{u} = 0. \tag{2.11}$$

Considering $B(y) = \sum_{j=1}^{n} \lambda_j B_j \hat{u}$, it follows:

$$\frac{\partial \hat{u}}{\partial t} + iB(y)\hat{u} = 0. \tag{2.12}$$

The equation (2.12) is nothing other than a first-order linear ordinary differential equation whose unknown is the function $\hat{u}$. Thus, the general solution is deduced in the form:

$$\hat{u}(y,t) = ke^{-itB(y)}, \qquad k \in \mathbb{R}. \tag{2.13}$$

Determine the value of k by applying the initial condition to the equation (2.13). For $t = 0$, the equation (2.13) becomes:

$$\hat{u}(y,0) = k. \tag{2.14}$$

Taking into account the initial condition (2.2), the following expression will result: (2.2), we have:

$$\hat{u}(y,0) = \hat{g}(y). \tag{2.15}$$

Substituting the equation (2.15) into the equation (2.14) gives $k = \hat{g}(y)$. So the general solution (2.13) becomes:

$$\hat{u}(y,t) = e^{-itB(y)}\hat{g}(y). \tag{2.16}$$

In order to obtain the solution u of the initial problem, the inverse Fourier transform must be applied to the function $\hat{u}(y,t)$. Which would then be defined as follows

$$\mathcal{F}^{-1}\left[\hat{u}(\cdot,t)\right](x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\hat{u}(y,t)e^{ixy}dy. \tag{2.17}$$

By replacing the equation (2.16) in the expression (2.17) above, it is deduced:

$$\mathcal{F}^{-1}\left[\hat{u}(\cdot,t)\right](x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{ixy}e^{-itB(y)}\hat{g}(y)dy. \tag{2.18}$$

Either

$$u(x,t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{ixy}e^{-itB(y)}\hat{g}(y)dy, \quad (x\in\mathbb{R}^n, t\geq 0). \tag{2.19}$$

Let us then show that $u(x,t)$ is well defined. Indeed,

$u(x,t)$ exists if and only if $\int_{\mathbb{R}^n}e^{ixy}e^{-itB(y)}\hat{g}(y)dx < \infty$ and $u\in\mathcal{C}^1\left([0,\infty[\,,\mathbb{R}^m\right)$.

At this level, it is first a question of proving that the integral defined in the relation (2.19) converges.

Since $g \in H^S(\mathbb{R}^n,\mathbb{R}^m)$, then according to the theory on fractional spaces of Sobolev, there exists $f\in L^2\left(\mathbb{R}^n,\mathbb{R}^m\right)$ such as

$$|\hat{g}(y)| \leq c(1+|y|^s)^{-1}|f(y)| \qquad (y\in\mathbb{R}^n). \tag{2.20}$$

Indeed, in order to study the convergence of the integral (2.19), we have to estimate $\|e^{-itB(y)}\|$, for a fixed y.

Let Γ be the path $\partial B(0,r)$ in the complex plane, traversed counter-clockwise; the radius r chosen as large as the eigenvalues $\lambda_1(y),\ldots,\lambda_m$ they are located in Γ. It results:

$$e^{-itB(y)} = \frac{1}{2\pi i}\int_\Gamma e^{-itz}(zI-B(y))^{-1}dz. \tag{2.21}$$

From all the above, it follows $A(t,y)$ the right side of equality (2.21) and fix $x\in\mathbb{R}^m$. Then, by multiplying on the left by $B\left(y\right)$ and on the right by x the equality

(2.21) the following result is obtained. To verify this, denote $A(t, y)$ the right side of equality (2.21) and fix $x \in \mathbb{R}^m$. Next, we multiply on the left by $B\,(y)$ and on the right by x the equality (2.21)

$$B(y)A(t,y)x = \frac{1}{2\pi i}\int_\Gamma e^{-itz}B(y)(zI - B(y))^{-1}xdz \tag{2.22}$$

$$= \frac{1}{2\pi i}\int_\Gamma e^{-itz}(z(zI - B(y))^{-1}x - x)dz \tag{2.23}$$

$$= -\frac{1}{i}\frac{d}{dt}A(t,y)x. \tag{2.24}$$

Since

$$\int_\Gamma e^{-itz}dz = 0,$$

therefore:

$$\left(\frac{d}{dt} + iB(y)\right)A(t,y) = 0. \tag{2.25}$$

In addition:

$$A(o,y)x = \frac{1}{2\pi i}\int_\Gamma (zI - B(y))^{-1}\,xdz \tag{2.26a}$$

$$A(o,y)x = \frac{1}{2i\pi}\int_\Gamma \frac{x + B(y)\,(zI - B(y))^{-1}}{z}dz \tag{2.26b}$$

$$A(o,y)x = x + \frac{1}{2\pi i}\int_\Gamma \frac{B(y)\,(zI - B(y))^{-1}\,x}{z}dz \tag{2.26c}$$

Let's say:

$$w = (zI - B(y))^{-1}\,x,$$

so that $zw - B(y)w = x$. By taking the product with $\bar{w}$, it follows:

$$|w| \leq \frac{c}{|z|}, \quad \text{for a constant } c. \tag{2.27}$$

Using this estimate and letting r tend to infinity, we conclude from the equation (2.26c) that

$$A(0,y)x = x. \tag{2.28}$$

This equality (2.25) verifies the presentation formula (2.21). Furthermore, let's define a new path Δ in the complex plane as follows: For fixed y, draw the circles $B_k = B(\lambda_k(y), 1)$ of radius equal to 1, centered at $\lambda_k(y), \quad (k = 1, \dots, m)$. Next, Δ is chosen as the boundary of $U_{k=1}^{m} B_k$, whose travel counterclockwise. Thus, deforming the path Γ into Δ, from the expression (2.21) is deduced

$$e^{-tB(y)} = \frac{1}{2i\pi} \int_{\Delta} e^{-itz} (zI - B(y))^{-1} dz. \tag{2.29}$$

Now,

$$|e^{-itz}| \leq e^{t} \quad (z \in \Delta). \tag{2.30}$$

Furthermore,

$$det(zI - B(y)) = \prod_{k=1}^{m} (z - \lambda_k(y)). \tag{2.31}$$

From where:

$$|det(zI - B(y)| \geq 1 \quad \text{if } z \ \in \Delta. \tag{2.32}$$

Right now,

$$(zI - B(y))^{-1} = \frac{Com(zI - B(y))^{T}}{det(zI - B(y))}, \tag{2.33}$$

where "Com" denotes the matrix of cofactors. Thus it is deduced:

$$\|(zI - B(y))^{-1}\| \leq \|Com(zI - B(y)\| \tag{2.34a}$$

$$\text{Either:} \quad \|(zI - B(y))^{-1}\| \leq c(1 + |z|^{m-1} + \|B(y)\|^{m-1}), \tag{2.34b}$$

$$\text{Or again:} \quad \|(zI - B(y))^{-1}\| \leq c(1 + |y|^{m-1}) \quad \text{if} \quad z \in \Delta. \tag{2.34c}$$

In this calculation, a the following elementary inequality was used:

$$|\lambda_k(y)| \leq C|y|, \quad (k = 1, \dots, m).$$

By combining the inequalities (2.30) and (2.34c), the derivation of the estimate is favored. Either:

$$\|e^{-itB(y)}\| \leq ce^{t}(1 + |y|^{m-1}), \quad (y \in \mathbb{R}^{n}). \tag{2.35}$$

Let's now return to our solution (2.19). Referring to the existence condition of $u(x,t)$ and the inequality (2.34c), we deduce that:

$$\int_R^n |e^{ixy}e^{-itB(y)}\hat{g}(y)|dy \leq c\int_{\mathbb{R}^n} \|e^{-itB(y)}\|(1+|y|^S)^{-1}|f(y)|dy, \tag{2.36a}$$

$$\int_R^n |e^{ixy}e^{-itB(y)}\hat{g}(y)|dy \leq ce^t\int_{\mathbb{R}^n} |f(y)|(1+|y|^{m-1})(1+|y|^S)^{-1}dy, \tag{2.36b}$$

$$\int_R^n |e^{ixy}e^{-itB(y)}\hat{g}(y)|dy \leq c(\int_{\mathbb{R}^n} |f(y)|^2dy)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n} \frac{dy}{1+|y|^{2(S-m+1)}}\right)^{\frac{1}{2}}, \tag{2.36c}$$

$$\int_R^n |e^{ixy}e^{-itB(y)}\hat{g}(y)|dy < \infty. \tag{2.36d}$$

As $S > \frac{n}{2}+m-1$, this result proves that the integral defined in the relation (2.19) converges; and it easily follows that the function

$$u(x,t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{ixy}e^{-itB(y)}\hat{g}(y)dy \tag{2.37}$$

is continuous on $\mathbb{R}^n \times [0,\infty[$.

Now let's show that u is of class C^1. Observe for $0 < |h| \leq 1$ that

$$\frac{u(x,t)-u(x,t)}{h} = \frac{1}{(2\pi)^{\frac{n}{2h}}}\int_{\mathbb{R}^n} e^{ixy}(e^{-i(t+h)B(y)} - e^{-itB(y)})\hat{g}(y)dy. \tag{2.38}$$

Since:

$$e^{-i(t+h)B(y)} - e^{-itB(y)} = -i\int_t^{t+h} B(y)e^{-iSB(y)}ds, \tag{2.39}$$

allowing ourselves to estimate as below that:

$$|\frac{1}{h}(e^{-i(t+h)B(y)} - e^{-itB(y)})| \leq Ce^{t+1}(1+|y|^m), \tag{2.40}$$

it follows that

$$|\frac{u(x,t+h)-u(x,t)}{h}| \leq ce^{t+1}\int_{\mathbb{R}^n} |f(y)|(1+|y|^S)^{-1}dy. \tag{2.41}$$

The function u is therefore differentiable. This result allows us to conclude that u is well defined.

So u_t exists and is continuous on $\mathbb{R}$. A similar argument shows that $u_{x_i}(i = 1, \dots, n)$ exists and is continuous by the dominated convergence theorem, one can further differentiate under the integral sign in (2.19), to confirm that u solves the system

$$u_t + \sum_{j=1}^{n} B_j u_{xj} = 0.$$

u is therefore a solution of the system above. Hence the existence of the solution.

2.4. **Uniqueness of the solution.** Let's first find the variational formulation of the equation (2.1)

$$u_t + \sum_{j=1}^{n} B_j u_{xj} = 0. \qquad \text{Either} \qquad \sum_{j}^{n} B_j u_{xj} = -u_t.$$

The initial condition says that: $u = g$. Therefore:

$$\sum_{J=1}^{n} B_j u_{xj} = -g, \tag{2.42}$$

$$g \in H^1(\mathbb{R}^n, \mathbb{R}^m) \quad \text{implies that} \quad \sum_{j=1}^{n} B_j u_{xj} \in H^1(\mathbb{R}^n, \mathbb{R}^m).$$

By multiplying equation (2.15) by $v \in H^1(\Omega)$ and integrating over an open Ω of $\mathbb{R}^n$ we easily obtain

$$\int_\Omega \sum_{j=1}^{n} B_j u_{xj} v d\Omega = \int_\Omega -gv d\Omega,$$

$$\sum_{j=1}^{n} B_j \int_\Omega \frac{\partial u}{\partial x_i} v d\Omega = -\int_\Omega gv d\Omega. \tag{2.43}$$

Now,

$$\int_\Omega \frac{\partial u}{\partial x_i} v d\Omega = -\int_\Omega u \frac{\partial v}{\partial x_i} d\Omega + \int_{\partial\Omega} uv\eta_i ds$$

and by integration by parts, follows:

$$\sum_{j=1}^{n} B_j \left(-\int_{\Omega} gu \frac{\partial u}{\partial x_i} d\Omega + \int_{\partial\Omega} uv\eta_i ds \right)$$

$$= -\int_{\Omega} gvd\Omega - \sum_{j=1}^{n} B_j \int_{\Omega} u \frac{\partial v}{\partial x_i} d\Omega + \sum_{j=1}^{n} B_j \int_{\partial\Omega} uv\eta_i ds = -\int_{\Omega} gvd\Omega$$

Hence the variational formulation

$$\sum_{j=1}^{n} B_j \int_{\Omega} u \frac{\partial v}{\partial x} d\Omega - \sum_{j=1}^{n} B_j \int_{\partial\Omega} uv\eta ds = \int_{\Omega} gvd\Omega.$$

Let's prove uniqueness. Let u_1, u_2 be two solutions of (2.1), then

$$\sum_{j=1}^{n} B_j \int_{\Omega} (u_1 - u_2) \frac{\partial v}{\partial x_i} d\Omega - \sum_{j=1}^{n} B_j \int_{\partial\Omega} (u_1 - u_2) v\eta ds = \int_{\Omega} gvd\Omega.$$

Choose the test function $v = u_1 - u_2$. Then,

$$\sum_{j=1}^{n} B_j \int v \frac{\partial v}{\partial x_i} d\Omega - \sum_{j=1}^{n} B_j \int_{\partial\Omega} vv\eta ds = \int_{\Omega} gvd\Omega,$$

and η is the unit normal component outside Ω and, $\displaystyle\int_{\partial\Omega} \eta ds = 0$ because η is zero on $\partial\Omega$. The following results are easily obtained

$$\sum_{j=1}^{n} B_j \|u_1 - u_2\|^2_{H^1_\Omega} = 0, \qquad \forall\ v \in H^1(\Omega),$$

$\|u_1 - u_2\|_{H^1(\Omega)} = 0$ and So $\quad u_1 = u_2$. Hence the uniqueness of the solution.

3. Conclusion

The general aim of our study was to solve a first-order linear hyperbolic partial differential equation with constant coefficients. The specific objective was to prove the existence and uniqueness of the solution. The existence and uniqueness of the solution were proved. To achieve this specific objective, two methods were exploited: the Fourier transform method to show the existence of the solution; and the variational formulation method, with the use of a test function to justify the uniqueness of the solution. In conclusion, the combination of these two methods delivered the expected results

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