Advances in Mathematics: Scientific Journal **12** (2023), no.10, 863–876
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.12.10.2

$\hbar$-CENTRALIZERS ON PSEUDOQUOTIENTS

Asia Majeed[1], Cenap Özel, Majed Albaity, and Sadia S. Ali

ABSTRACT. A space of pseudoquotients, denoted by $\mathcal{B}(X,S)$, is defined as equivalence classes of pairs (x,f), where x is an element of a non-empty set X, f is an element of S, a commutative semigroup of injective maps from X to X, and $(x,f)\sim(y,g)$ when $gx=fy$. If X is a ring and elements of S are ring homomorphisms, then $\mathcal{B}(X,S)$ is a ring. We show that, under natural conditions, a $\hbar$ -Jordan centralizer on X has a unique extension to a $\hbar$-Jordan centralizer on $\mathcal{B}(X,S)$

1. INTRODUCTION

Let X be a ring (or an algebra) with the unit 1. A linear mapping δ from X into it self is called a left(right) centralizer of X if

$$\delta(y)=\delta(1)\cdot y$$

and

$$\delta(y)=y\cdot\delta(1),$$

for all $y\in X$. If δ is a left and right centralizer, then it is called a δ centralizer. A linear mapping δ from X into itself is called a left (right) Jordan centralizer of X

[1] *corresponding author*
2020 *Mathematics Subject Classification.* 47B47, 47L35, 20M14.
Key words and phrases. Centralizer, pseudoquotients, commutative semigroup, biadditive mapping.
Submitted: 12.09.2023; *Accepted:* 28.09.2023; *Published:* 05.10.2023.

if

$$\delta(x^2) = \delta(x) \cdot x,$$

and

$$\delta(x^2) = x \cdot \delta(x),$$

for any $x \in X$. δ is called a Jordan centralizer of X if

$$\delta(x \cdot y + y \cdot x) = \delta(x) \cdot y + y \cdot \delta(x) = \delta(y) \cdot x + x \cdot \delta(y)$$

for any $x, y \in X$. In [4], Zalar shows that a left Jordan centralizer of a semi-prime ring is a left centralizer and each Jordan centralizer of a semi-prime ring is a centralizer.

Let X be any nonempty set and S be a commutative semigroup acting on X injectively. This means that every $\phi \in S$ is an injective map $\phi : X \to X$ and $(\phi\psi)x = \phi(\psi x)$ for all ϕ, $\psi \in S$ and $x \in X$. For (x, ϕ),$(y, \psi) \in X \times S$, we write (x, ϕ) (y, ψ) if $\psi x = \phi y$. It is easy to check that this is an equivalence relation in $X \times S$, finally we define $\mathcal{B}(X, S) = (X \times S)/\sim$. The equivalence class of (x, ϕ) will be denoted by $\frac{x}{\phi}$. This is a slight abuse of notion, but we follow here the tradition of denoting rational numbers by $\frac{p}{q}$ even though the same formal problem is presented there. Elements of X can be identified with elements of $\mathcal{B}(X, S)$ via the embedding $\iota : X \to B(X, S)$ defined by $\iota(x) = \frac{\phi x}{\phi}$, where ϕ is an arbitrary element of S. Clearly it is well-defined; that is, it is independent of ϕ. The action of S can be extended to $\mathcal{B}(X, S)$ via $\phi\frac{x}{\psi} = \frac{\phi x}{\psi}$. If $\phi\frac{x}{\psi} = i(y)$ for some $y \in X$, we will write $\phi\frac{x}{\psi} \in X$ and $\phi\frac{x}{\psi} = y$, which formally incorrect, but convenient and harmless. For instance, we have $\phi\frac{x}{\phi} = x$. Elements of S, when extended to maps on $\mathcal{B}(X, S)$, are bijections. The action of ψ^{-1} on $\mathcal{B}(X, S)$ can be defined as

$$\psi^{-1}\frac{x}{\phi} = \frac{x}{\phi\psi}.$$

Consequently, S can be extended to a commutative group of bijections acting on $\mathcal{B}(X, S)$. If $(X, \odot)$ is a commutative group and S is a commutative semigroup of injective homomorphisms on X, then $B(X, S)$ is a commutative group with the operation defined as

$$\frac{x}{\phi} \odot \frac{y}{\psi} = \frac{\psi x \odot \phi y}{\phi\psi}.$$

Similarly, if X is a vector space and S is a commutative semigroup of injective linear mapping from X into itself, then $\mathcal{B}(X,S)$ is a vector space with the operations defined as

$$\frac{x}{\phi}+\frac{y}{\psi}=\frac{\psi x+\phi y}{\phi\psi} \; and \; \lambda\frac{x}{\phi}=\frac{\lambda x}{\phi}.$$

If $\delta : X \to X$ and δ extends to a map $\hat{\delta} : \mathcal{B}(X,S) \to \mathcal{B}(X,S)$, then it is often important to know what properties of δ are inherited by $\hat{\delta}$. In this section, we consider some special situations when an extension is possible, which are important for the particular case studied in this paper.

If $\delta(fx) = f\delta(x)$ for all $x \in X$ and all $f \in S$, then we say that δ commutes with S.

Now we need the next Proposition from [1] to prove our results in the next section.

Proposition 1.1. *Let $\delta : X \to X$. Then*

$$\hat{\delta}(\frac{x}{f})=\frac{\delta(x)}{f}$$

is a well-defined extension of δ to $\hat{\delta} : \mathcal{B}(X,S) \to \mathcal{B}(X,S)$ if and only if δ commutes with S.

2. Centralizers on Pseudoquotients

Definition 2.1. *Let X be a ring and $\hbar$ be endomorphisms of X. A map $\delta : X \to X$ such that*

$$\delta(y) = \delta(1)\cdot\hbar(y)$$

and

$$\delta(y) = \hbar(y)\cdot\delta(1),$$

for any $x, y \in X$ is called a left or right $\hbar$- centralizer on X.

Definition 2.2. *Let X be a ring and $\hbar$ be endomorphisms of X. A map $\delta : X \to X$ is called a $\hbar$-centralizer, if δ is a left and right $\hbar$-centralizer.*

Theorem 2.1. *Let X be a ring and S be a commutative semigroup of injective ring homomorphisms. Let $\hbar$ be a homomorphism from X into itself that commutes with S, that is, $\hbar f(x) = f\hbar(x)$ for every $f \in S$ and $x \in X$. If δ is left(or right) $\hbar$-centralizer on X that commutes with S, then the map $\hat{\delta} : \mathcal{B}(X, S) \to \mathcal{B}(X, S)$ defined by*

$$\hat{\delta}\left(\frac{x}{f}\right) = \frac{\delta(x)}{f} \tag{2.1}$$

is an extension of δ to a left(or right respectively) $\hbar$-centralizer on $\mathcal{B}(X, S)$.

Proof. Assume δ is an $\hbar$- centralizer on X that commutes with S. Then, $\hat{\delta}$ is well-defined by Proposition 1.1. In order to show that it is a left and right $\hbar$-centralizer on $\mathcal{B}(X, S)$, consider $\frac{x}{f} \in \mathcal{B}(X, S)$. Then, we have

$$\hat{\delta}(\frac{x}{f}) = \frac{\delta(x)}{f} = \frac{\delta(1) \cdot \hbar(x)}{f} = \hat{\delta}\left(\frac{1}{I}\right) \cdot \hbar(\frac{x}{f})$$

' and

$$\hat{\delta}(\frac{x}{f}) = \frac{\delta(x)}{f} = \frac{\hbar(x) \cdot \delta(1)}{f} = \hbar(\frac{x}{f}) \cdot \hat{\delta}\left(\frac{1}{I}\right),$$

which is the desired result. □

Corollary 2.1. *Let X be a ring and S be a commutative semigroup of injective ring homomorphisms. If δ is left(or right) centralizer on X that commutes with S, then the map $\hat{\delta} : \mathcal{B}(X, S) \to \mathcal{B}(X, S)$ defined by*

$$\hat{\delta}\left(\frac{x}{f}\right) = \frac{\delta(x)}{f} \tag{2.2}$$

is an extension of δ to a left(or right repectively) centralizer on $\mathcal{B}(X, S)$.

Definition 2.3. *A linear mapping δ from X into itself is called a left or right $\hbar$ -Jordan centralizer of X if*

$$\delta(x^2) = \delta(x) \cdot \hbar(x)$$

or

$$\delta(x^2) = \hbar(x) \cdot \delta(x)$$

for any $x \in X$ respectively.

Definition 2.4. *A linear mapping δ from X into itself is called a $\hbar$-Jordan centralizer of X if*

$$\delta(x \cdot y + y \cdot x) = \delta(x) \cdot \hbar(y) + \hbar(y) \cdot \delta(x) = \delta(y) \cdot \hbar(x) + \hbar(x) \cdot \delta(y)$$

for any $x, y \in X$.

Albas in [6] shows that under some conditions, a left $\hbar$-Jordan centralizer of a semi-prime ring is a left $\hbar$-centralizer and each $\hbar$-Jordan centralizer of a semi-prime ring is $\hbar$-centralizer.

Theorem 2.2. *Let X be a ring and S be a commutative semigroup of injective ring homomorphisms. Let $\hbar$ be a homomorphism from X into itself that commute with S. If δ is a left(or right) $\hbar$-Jordan centralizer on X that commutes with S, then the map $\hat{\delta} : \mathcal{B}(X, S) \to \mathcal{B}(X, S)$ defined by*

$$\hat{\delta}\left(\frac{x}{f}\right) = \frac{\delta(x)}{f} \tag{2.3}$$

is an extension of δ to a left(or right respectively) $\hbar$-Jordan centralizer on $\mathcal{B}(X, S)$.

Proof. Assume that δ is a left or right $\hbar$-Jordan centralizer on X that commutes with S. Then, $\hat{\delta}$ is well defined by Proposition 1.1. In order to show that $\hat{\delta}$ is an left or right $\hbar$-Jordan centralizer on $\mathcal{B}(X, S)$, we consider $\frac{x}{f} \in \mathcal{B}(X, S)$. First for left $\hbar$-Jordan centralizer we have

$$\hat{\delta}(\frac{x}{f} \cdot \frac{x}{f}) = \frac{\delta(fx \cdot fx)}{f^2} = \frac{\delta f(x) \cdot \hbar f(x)}{f^2} = \frac{\delta x}{f} \cdot \frac{\hbar x}{f} = \hat{\delta}(\frac{x}{f}) \cdot \hbar(\frac{x}{f})$$

for any $\frac{x}{f} \in \mathcal{B}(X, S)$. Similarly it can be proven for right $\hbar$-Jordan centralizers. □

Theorem 2.3. *Let X be a ring and S be a commutative semigroup of injective ring homomorphisms. Let $\hbar$ be a homomorphism from X into itself that commute with S. If δ is $\hbar$-Jordan centralizer on X that commutes with S, then the map $\hat{\delta} : \mathcal{B}(X, S) \to \mathcal{B}(X, S)$ defined by*

$$\hat{\delta}\left(\frac{x}{f}\right) = \frac{\delta(x)}{f}$$

is an extension of δ to a $\hbar$-Jordan centralizer on $\mathcal{B}(X, S)$.

Proof. It is known that δ is an $\hbar$-Jordan centralizer on X that commutes with S. Also $\hat{\delta}$ is well defined by Proposition 1.1. To show that $\hat{\delta}$ is an $\hbar$-Jordan centralizer on $\mathcal{B}(X,S)$, we consider $\frac{x}{f}$ and $\frac{y}{g} \in \mathcal{B}(X,S)$. Thus

$$\hat{\delta}(\frac{x}{f}\cdot\frac{y}{g}+\frac{y}{g}\cdot\frac{x}{f}) = \frac{\delta(gx\cdot fy)}{fg}+\frac{\delta(fy\cdot gx)}{gf} = \hat{\delta}(\frac{x}{f})\cdot\hbar(\frac{y}{g})+\hbar(\frac{y}{g})\cdot\hat{\delta}(\frac{x}{f})$$
$$= \hat{\delta}(\frac{y}{g})\cdot\hbar(\frac{x}{f})+\hbar(\frac{x}{f})\cdot\hat{\delta}(\frac{y}{g})$$

for any $\frac{x}{f}$ and $\frac{y}{g} \in \mathcal{B}(X,S)$. □

Corollary 2.2. *Let X be a ring and S be a commutative semigroup of injective ring homomorphism. If δ is Jordan centralizer on X that commutes with S, then the map $\hat{\delta} : \mathcal{B}(X,S) \to \mathcal{B}(X,S)$ defined by*

$$\hat{\delta}\left(\frac{x}{f}\right) = \frac{\delta(x)}{f}$$

is an extension of δ to a Jordan centralizer on $\mathcal{B}(X,S)$.

In Theorem 2.2 and Corollary 2.2, it is necessary to assume that δ commutes with S. The last theorem guarantees that extension of a $\hbar$-Jordan left centralizer is a left $\hbar$-centralizer on $\mathcal{B}(X,S)$.

In order to prove the last theorem, we will discuss certain cases. The following first lemma is analogous to that of Lemma 3 of [6], so it will be given without proof. Before we state it, we need the following definition and remark.

Definition 2.5. *Let X be a ring. A* **biadditive mapping** *D from X to itself is a mapping defined by:*

$$D\left(\frac{x}{f},\frac{y}{g}\right) = \hat{\delta}(\frac{x}{f}\cdot\frac{y}{g}) - \hat{\delta}(\frac{x}{f})\cdot h(\frac{y}{g}), \tag{2.4}$$

for all $\frac{x}{f}$, $\frac{y}{g} \in \mathcal{B}(X,S)$.

Remark 2.1. *If $D\left(\frac{x}{f},\frac{y}{g}\right) = 0$ for all $\frac{x}{f},\frac{y}{g} \in \mathcal{B}(X,S)$, then $\hat{\delta}$ is a h-left centralizer of X.*

Lemma 2.1. *Let X be a ring and S be a commutative semigroup of injective ring homomorphisms. Let $\hbar$ be a homomorphism from X into itself that commutes with S. Let δ be a left $\hbar$-Jordan centralizer on X that commutes with S. Then the extension* *map* $\hat{\delta} : \mathcal{B}(X,S) \to \mathcal{B}(X,S)$ *defined by*

$$\hat{\delta}\left(\frac{x}{f}\right) = \frac{\delta(x)}{f}$$

satisfies the following identities:

(i) $\hat{\delta}(\frac{x}{f} \cdot \frac{y}{g} + \frac{y}{g} \cdot \frac{x}{f}) = \hat{\delta}(\frac{x}{f}) \cdot \hbar(\frac{y}{g}) + \hat{\delta}(\frac{y}{g}) \cdot \hbar(\frac{x}{f})$ *for all* $\frac{x}{f}, \frac{y}{g} \in \mathcal{B}(X,S)$.

(ii) $\hat{\delta}(\frac{x}{f} \cdot \frac{y}{g} \cdot \frac{x}{f}) = \hat{\delta}(\frac{x}{f}) \cdot \hbar(\frac{y}{g}) \cdot \hbar(\frac{x}{f})$ *for all* $\frac{x}{f}, \frac{y}{g} \in \mathcal{B}(X,S)$.

(iii) $\hat{\delta}(\frac{x}{f} \cdot \frac{y}{g} \cdot \frac{z}{h} + \frac{z}{h} \cdot \frac{y}{g} \cdot \frac{x}{f}) = \hat{\delta}(\frac{x}{f}) \cdot \hbar(\frac{y}{g}) \cdot \hbar(\frac{z}{h}) + \hat{\delta}(\frac{z}{h}) \cdot \hbar(\frac{y}{g}) \cdot \hbar(\frac{x}{f})$ *for all* $\frac{x}{f}, \frac{y}{g}, \frac{z}{h} \in \mathcal{B}(X,S)$.

(iv) $\hat{D}(\frac{x}{f}, \frac{y}{g}) = -\hat{D}(\frac{y}{g}, \frac{x}{f})$ *where* $\hat{D}(\frac{x}{f}, \frac{y}{g}) = \hat{\delta}(\frac{x}{f} \cdot \frac{y}{g}) - \hat{\delta}(\frac{x}{f}) \cdot \hbar(\frac{y}{g})$ *for all* $\frac{x}{f}, \frac{y}{g} \in \mathcal{B}(X,S)$.

Proof.

(i) Since δ is a left $\hbar$-Jordan centralizer on X, the equality

$$\delta(x^2) = \delta(x) \cdot \hbar(x)$$

holds for all $x \in X$.

$$\hat{\delta}\left(\frac{(gx+fy)^2}{f.g}\right)$$

$$= \frac{\delta\left((gx+fy)^2\right)}{f.g} = \frac{\delta(gx+fy).\hbar(gx+fy)}{f.g} \tag{2.5}$$

$$= \frac{[\delta(gx)+\delta(fy)][\hbar(gx)+\hbar(fy)]}{f.g}$$

$$= \frac{\delta(gx).\hbar(gx)+\delta(gx).\hbar(fy)+\delta(fy).\hbar(gx)+\delta(fy).\hbar(fy)}{f.g}, \tag{2.6}$$

$$\hat{\delta}\left(\frac{(gx+fy)^2}{f.g}\right) = \hat{\delta}\left(\frac{(gx)^2+(fy)^2+gx.fy+fy.gx}{f.g}\right)$$

$$= \frac{\delta\left((gx)^2+(fy)^2+gx.fy+fy.gx\right)}{f.g}$$

$$= \frac{\delta(gx).\hbar(gx)+\delta(fy).\hbar(fy)+\delta(gx.fy)+\delta(fy.gx)}{f.g}. \tag{2.7}$$

Using the fact that (2.5) = (2.7), we obtain

$$\frac{\delta(gx).\hbar(fy)+\delta(fy).\hbar(gx)}{f.g} = \frac{\delta(gx.fy)+\delta(fy.gx)}{f.g}.$$

This implies that

$$\frac{g\delta(x).f\hbar(y)+f\delta(y).g\hbar(x)}{f.g}=\frac{\delta(x)}{f}.\frac{\hbar(y)}{g}+\frac{\delta(y)}{g}.\frac{\hbar(x)}{f}=\hat{\delta}\left(\frac{x}{f}.\frac{y}{g}+\frac{y}{g}.\frac{x}{f}\right).$$

This completes the proof,

$$\hat{\delta}\left(\frac{x}{f}\right).\hbar\left(\frac{y}{g}\right)+\hat{\delta}\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f}\right)=\hat{\delta}\left(\frac{x}{f}.\frac{y}{g}+\frac{y}{g}.\frac{x}{f}\right).$$

(ii) Replacing $\frac{y}{g}$ with $\frac{x}{f}.\frac{y}{g}+\frac{y}{g}.\frac{x}{f}$ in (i) and using the fact that δ is a left $\hbar$-Jordan centralizer of X, we see that

$$\begin{aligned}
&\hat{\delta}\left(\frac{x}{f}\left(\frac{x}{f}.\frac{y}{g}+\frac{y}{g}.\frac{x}{f}\right)+\left(\frac{x}{f}.\frac{y}{g}+\frac{y}{g}.\frac{x}{f}\right).\frac{x}{f}\right)\\
&=\hat{\delta}\left(\frac{x}{f}\right).\hbar\left(\frac{x}{f}.\frac{y}{g}+\frac{y}{g}.\frac{x}{f}\right)+\hat{\delta}\left(\frac{x}{f}.\frac{y}{g}+\frac{y}{g}.\frac{x}{f}\right).\hbar\left(\frac{x}{f}\right)\\
&=\hat{\delta}\left(\frac{x}{f}\right).\left[\hbar\left(\frac{x}{f}\right).\hbar\left(\frac{y}{g}\right)+\hbar\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f}\right)\right]\\
&\quad+\left[\hat{\delta}\left(\frac{x}{f}\right).\hbar\left(\frac{y}{g}\right)+\hat{\delta}\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f}\right)\right].\hbar\left(\frac{x}{f}\right) \qquad (2.8)\\
&=\hat{\delta}\left(\frac{x}{f}\right).\hbar\left(\frac{x}{f}\right).\hbar\left(\frac{y}{g}\right)+\hat{\delta}\left(\frac{x}{f}\right).\hbar\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f}\right)\\
&\quad+\hat{\delta}\left(\frac{x}{f}\right).\hbar\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f}\right)+\hat{\delta}\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f}\right).\hbar\left(\frac{x}{f}\right)\\
&=2\hat{\delta}\left(\frac{x}{f}\right).\hbar\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f}\right)+\hat{\delta}\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f}\right).\hbar\left(\frac{x}{f}\right)\\
&\quad+\hat{\delta}\left(\frac{x}{f}\right).\hbar\left(\frac{x}{f}\right).\hbar\left(\frac{y}{g}\right). \qquad (2.9)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\hat{\delta}\left(\frac{x}{f}\left(\frac{x}{f}.\frac{y}{g}+\frac{y}{g}.\frac{x}{f}\right)+\left(\frac{x}{f}.\frac{y}{g}+\frac{y}{g}.\frac{x}{f}\right).\frac{x}{f}\right)\\
&=\hat{\delta}\left(\frac{x}{f}.\frac{x}{f}.\frac{y}{g}+\frac{x}{f}.\frac{y}{g}.\frac{x}{f}+\frac{x}{f}.\frac{y}{g}.\frac{x}{f}+\frac{y}{g}.\frac{x}{f}.\frac{x}{f}\right)\\
&=\hat{\delta}\left(\frac{(fx)^2}{f^2}.\frac{y}{g}+\frac{y}{g}.\frac{(fx)^2}{f^2}+2\frac{x}{f}\frac{y}{g}.\frac{x}{f}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \hat{\delta}\left(\frac{(fx)^2}{f^2}\right).\hbar\left(\frac{y}{g}\right) + \hat{\delta}\left(\frac{y}{g}\right).\hbar\left(\frac{(fx)^2}{f^2}\right) + 2\hat{\delta}\left(\frac{x}{f}.\frac{y}{g}.\frac{x}{f}\right) \\
&= \frac{\delta(fx)^2}{f^2}.\hbar\left(\frac{y}{g}\right) + \hat{\delta}\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f}.\frac{x}{f}\right) + 2\hat{\delta}\left(\frac{x}{f}.\frac{y}{g}.\frac{x}{f}\right) \\
&= \frac{\delta(fx).\hbar(fx)}{f^2}.\hbar\left(\frac{y}{g}\right) + \hat{\delta}\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f}\right).\hbar\left(\frac{x}{f}\right) + 2\hat{\delta}\left(\frac{x}{f}.\frac{y}{g}.\frac{x}{f}\right) \\
&= \frac{\delta(x)}{f}.\frac{\hbar(x)}{f}.\hbar\left(\frac{y}{g}\right) + \hat{\delta}\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f}\right).\hbar\left(\frac{x}{f}\right) + 2\hat{\delta}\left(\frac{x}{f}.\frac{y}{g}.\frac{x}{f}\right)
\end{aligned}
$$

$$
= \hat{\delta}\left(\frac{x}{f}\right).\hbar\left(\frac{x}{f}\right).\hbar\left(\frac{y}{g}\right) + \hat{\delta}\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f}\right).\hbar\left(\frac{x}{f}\right) + 2\hat{\delta}\left(\frac{x}{f}.\frac{y}{g}.\frac{x}{f}\right). \tag{2.10}
$$

Since (2.8) = (2.10), we have

$$
2\hat{\delta}\left(\frac{x}{f}\right).\hbar\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f}\right) = 2\hat{\delta}\left(\frac{x}{f}.\frac{y}{g}.\frac{x}{f}\right).
$$

This completes the proof.

(iii) Replacing $\frac{x}{f}$ with $\frac{x}{f} + \frac{z}{\hbar}$ in (ii), we obtain (iii). Actually,

$$
\hat{\delta}\left(\left(\frac{x}{f} + \frac{z}{\hbar}\right).\frac{y}{g}.\left(\frac{x}{f} + \frac{z}{\hbar}\right)\right) = \hat{\delta}\left(\frac{x}{f} + \frac{z}{\hbar}\right).\hbar\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f} + \frac{z}{\hbar}\right).
$$

The left hand side can be written as follows:

$$
\hat{\delta}\left(\left(\frac{x}{f} + \frac{z}{\hbar}\right).\frac{y}{g}.\left(\frac{x}{f} + \frac{z}{\hbar}\right)\right)
$$

$$
= \hat{\delta}\left(\left(\frac{x}{f}.\frac{y}{g} + \frac{z}{\hbar}.\frac{y}{g}\right).\left(\frac{x}{f} + \frac{z}{\hbar}\right)\right) \tag{2.11}
$$

$$
\begin{aligned}
&= \hat{\delta}\left(\frac{x}{f}.\frac{y}{g}.\frac{x}{f} + \frac{x}{f}.\frac{y}{g}.\frac{z}{\hbar} + \frac{z}{\hbar}.\frac{y}{g}.\frac{x}{f} + \frac{z}{\hbar}.\frac{y}{g}.\frac{z}{\hbar}\right) \\
&= \hat{\delta}\left(\frac{x}{f}.\frac{y}{g}.\frac{x}{f}\right) + \hat{\delta}\left(\frac{z}{\hbar}.\frac{y}{g}.\frac{z}{\hbar}\right) + \hat{\delta}\left(\frac{x}{f}.\frac{y}{g}.\frac{z}{\hbar} + \frac{z}{\hbar}.\frac{y}{g}.\frac{x}{f}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \hat{\delta}\left(\frac{x}{f}\right).\hbar\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f}\right) + \hat{\delta}\left(\frac{z}{\hbar}\right).\hbar\left(\frac{y}{g}\right).\hbar\left(\frac{z}{\hbar}\right) \\
&\quad + \hat{\delta}\left(\frac{x}{f}.\frac{y}{g}.\frac{z}{\hbar} + \frac{z}{\hbar}.\frac{y}{g}.\frac{x}{f}\right).
\end{aligned} \tag{2.12}
$$

Also, the right hand side can be written as follows

$$\begin{aligned}
&\hat{\delta}\left(\frac{x}{f}+\frac{z}{\hbar}\right).\hbar\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f}+\frac{z}{\hbar}\right)\\
&=\left(\hat{\delta}\left(\frac{x}{f}\right)+\hat{\delta}\left(\frac{z}{\hbar}\right)\right).\hbar\left(\frac{y}{g}\right).\left(\hbar\left(\frac{x}{f}\right)+\hbar\left(\frac{z}{\hbar}\right)\right)\\
&=\left(\hat{\delta}\left(\frac{x}{f}\right).\hbar\left(\frac{y}{g}\right)+\hat{\delta}\left(\frac{z}{\hbar}\right).\hbar\left(\frac{y}{g}\right)\right).\left(\hbar\left(\frac{x}{f}\right)+\hbar\left(\frac{z}{\hbar}\right)\right)\\
&=\hat{\delta}\left(\frac{x}{f}\right).\hbar\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f}\right)+\hat{\delta}\left(\frac{x}{f}\right).\hbar\left(\frac{y}{g}\right).\hbar\left(\frac{z}{\hbar}\right)\\
&\quad+\hat{\delta}\left(\frac{z}{\hbar}\right).\hbar\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f}\right)+\hat{\delta}\left(\frac{z}{\hbar}\right).\hbar\left(\frac{y}{g}\right).\hbar\left(\frac{z}{\hbar}\right).
\end{aligned}\tag{2.13}$$

Using (2.11) and (2.13), we obtain the result.

(iv) It is obvious from (i). We have to rewrite (i), using

$$\hat{D}(\frac{x}{f},\frac{y}{g})=\hat{\delta}(\frac{x}{f}\cdot\frac{y}{g})-\hat{\delta}(\frac{x}{f})\cdot\hbar(\frac{y}{g}),\ \text{for all}\ \frac{x}{f},\frac{y}{g}\in\mathcal{B}(X,S).$$

$$\begin{aligned}
\hat{\delta}(\frac{x}{f}\cdot\frac{y}{g}+\frac{y}{g}\cdot\frac{x}{f})&=\hat{\delta}\left(\frac{x}{f}.\frac{y}{g}\right)+\hat{\delta}\left(\frac{y}{g}.\frac{x}{f}\right)\\
&=\hat{D}(\frac{x}{f},\frac{y}{g})+\hat{\delta}\left(\frac{x}{f}\right).\hbar\left(\frac{y}{g}\right)+\hat{D}(\frac{y}{g},\frac{x}{f})+\hat{\delta}\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f}\right).
\end{aligned}\tag{2.14}$$

Similarly, using (i) we obtain

$$\hat{\delta}(\frac{x}{f}\cdot\frac{y}{g}+\frac{y}{g}\cdot\frac{x}{f})=\hat{\delta}\left(\frac{x}{f}\right).\hbar\left(\frac{y}{g}\right)+\hat{\delta}\left(\frac{y}{g}\right).\hbar\left(\frac{x}{f}\right).\tag{2.15}$$

From (2.14) and (2.15), we have

$$\hat{D}(\frac{x}{f},\frac{y}{g})+\hat{D}(\frac{y}{g},\frac{x}{f})=0.$$

So we get the result (iv),

$$\hat{D}(\frac{x}{f},\frac{y}{g})=-\hat{D}(\frac{y}{g},\frac{x}{f})\ \text{for all}\ \frac{x}{f},\frac{y}{g}\in\mathcal{B}(X,S).$$

□

Lemma 2.2. *Let X be a ring and S be a commutative semigroup of injective ring homomorphisms. Let $\hbar$ be a homomorphism from X into itself that commutes with*

S. Let δ be a $\hbar$-left Jordan centralizer. If every $\hbar$-left Jordan centralizer is a $\hbar$-centralizer on X that commutes with S, then the map $\hat{\delta} : \mathcal{B}(X,S) \to \mathcal{B}(X,S)$ defined by

$$\hat{\delta}\left(\frac{x}{f}\right) = \frac{\delta(x)}{f}$$

is an extension of δ to a $\hbar$-left centralizer on $\mathcal{B}(X,S)$.

Proof. Let $1, I$ be the identities on X and S, respectively. Since $\hbar$ is an endomorphism of X, then $\hbar(1) = 1$. We know that $\hat{D}(\frac{x}{f}, \frac{y}{g}) = \hat{\delta}(\frac{x}{f} \cdot \frac{y}{g}) - \hat{\delta}(\frac{x}{f}) \cdot \hbar(\frac{y}{g})$ for all $\frac{x}{f}, \frac{y}{g} \in \mathcal{B}(X,S)$, so we can write $\hat{D}(\frac{1}{I}, \frac{x}{f}) = \hat{\delta}(\frac{1}{I} \cdot \frac{x}{f}) - \hat{\delta}(\frac{1}{I}) \cdot \hbar(\frac{x}{f})$ for all $\frac{x}{f} \in \mathcal{B}(X,S)$. By Lemma 2.1 in [8], we have $\hat{D}(\frac{1}{I}, \frac{x}{f}) = -\hat{D}(\frac{x}{f}, \frac{1}{I}) = 0$ for all $\frac{x}{f} \in \mathcal{B}(X,S)$. Thus $\hat{\delta}(\frac{x}{f}) = \hat{\delta}(\frac{1}{I}) \cdot \hbar(\frac{x}{f})$ for all $\frac{x}{f} \in \mathcal{B}(X,S)$. □

To prove the next theorem, we need to define the center of A, where A is a unital subalgebra of $B(X)$. The center of A is defined as $Z = \acute{A} \cap A$, where $\acute{A} \cap A = CI$ is the center of $B(X)$. Similarly we can define center on the ring $\mathcal{B}(X,S)$. Consider $\mathcal{B}(X,S)$ as a unital subalgebra of $\mathcal{B}(X,S)$. Then $\acute{Z} = \acute{B} \cap B$ where $\acute{B} \cap B = \acute{C}I$.

Lemma 2.3. *Let $\frac{a}{k}$ be a fixed element in $\mathcal{B}(X,S)$. If $\frac{a}{k} \cdot \hbar(\frac{x}{f}) - \hbar(\frac{x}{f}) \cdot \frac{a}{k} \in \acute{Z}$ for all $\frac{x}{f} \in \mathcal{B}(X,S)$, then $\frac{a}{k} \in \acute{Z}$.*

Proof. First $\frac{a}{k} \cdot \hbar(\frac{x}{f}) - \hbar(\frac{x}{f}) \cdot \frac{a}{k} \in \acute{Z}$. From [7], we have $\frac{a}{k} \cdot \hbar(\frac{x}{f}) - \hbar(\frac{x}{f}) \cdot \frac{a}{k} = 0$, and then $\hbar$ is injective; and therefore $\frac{a}{k} \in \acute{Z}$. □

Lemma 2.4. *Let $\frac{a}{k}$ be a fixed element in $\mathcal{B}(X,S)$ and $\hat{\delta(\frac{x}{f})} = \frac{a}{k} \cdot \hbar(\frac{x}{f}) + \hbar(\frac{x}{f}) \cdot \frac{a}{k}$ for any $\frac{x}{f} \in \mathcal{B}(X,S)$. If $\hat{\delta(\frac{x}{f})}$ is a $\hbar$-centralizer of $\mathcal{B}(X,S)$, then $\frac{a}{k} \in \acute{Z}$.*

Proof. Since $\hat{\delta}$ is a $\hbar$-Jordan centralizer of $\mathcal{B}(X,S)$, it follows that

$$\hat{\delta}(\frac{x}{f} \cdot \frac{y}{g} + \frac{y}{g} \cdot \frac{x}{f}) = \hat{\delta}(\frac{x}{f}) \cdot \hbar(\frac{y}{g}) + \hbar(\frac{y}{g}) \cdot \hat{\delta}(\frac{x}{f})$$

for all $\frac{x}{f}, \frac{y}{g} \in \mathcal{B}(X,S)$. Therefore

$$\begin{aligned} &\frac{a}{k}\hbar(\frac{x}{f} \cdot \frac{y}{g} + \frac{y}{g} \cdot \frac{x}{f}) + \hbar(\frac{x}{f} \cdot \frac{y}{g} + \frac{y}{g} \cdot \frac{x}{f})\frac{a}{k} \\ &= (\frac{a}{k} \cdot \hbar(\frac{x}{f}) + \hbar(\frac{x}{f}) \cdot \frac{a}{k}) \cdot \hbar(\frac{y}{g}) + \hbar(\frac{y}{g}) \cdot (\frac{a}{k} \cdot \hbar(\frac{x}{f}) + \hbar(\frac{x}{f}) \cdot \frac{a}{k})) \end{aligned}$$

$$\frac{a}{k} \cdot \hbar(\frac{y}{g}) \cdot \hbar(\frac{x}{f}) + \hbar(\frac{x}{f}) \cdot \hbar(\frac{y}{g}) \cdot \frac{a}{k} = \hbar(\frac{x}{f}) \cdot \frac{a}{k} \cdot \hbar(\frac{y}{g}) + \hbar(\frac{y}{g}) \cdot \frac{a}{k} \cdot \hbar(\frac{x}{f})$$

$$\hbar(\frac{x}{f}) \cdot (\frac{a}{k} \cdot \hbar(\frac{y}{g}) - \hbar(\frac{y}{g}) \cdot \frac{a}{k}) = (\frac{a}{k} \cdot \hbar(\frac{y}{g}) - \hbar(\frac{y}{g})) \cdot \frac{a}{k} \cdot \hbar(\frac{x}{f})$$

for all $\frac{x}{f}, \frac{y}{g} \in \mathcal{B}(X, S)$. Since $\hbar$ is surjective, we have that $\frac{a}{k} \cdot \hbar(\frac{y}{g}) - \hbar(\frac{y}{g}) \cdot \frac{a}{k} \in \acute{\mathbb{Z}}$. Hence $\frac{a}{k} \in \acute{\mathbb{Z}}$ by Lemma 2.3. □

Lemma 2.5. *Every $\hbar$-Jordan centralizer of $\mathcal{B}(X, S)$ maps $\acute{Z}$ into itself.*

Proof. For any $\frac{a}{k} \in \acute{\mathbb{Z}}$, let $\frac{a}{k} = \hat{\delta}(\frac{c}{l})$. Since $\hat{\delta}$ is a $\hbar$-Jordan centralizer of $\mathcal{B}(X, S)$, we have

$$2\hat{\delta}(\frac{c}{l} \cdot \frac{x}{f}) = \hat{\delta}(\frac{c}{l} \cdot \frac{x}{f} + \frac{x}{f} \cdot \frac{c}{l}) = \hat{\delta}(\frac{c}{l}) \cdot \hbar(\frac{x}{f}) + \hbar(\frac{x}{f}) \cdot \hat{\delta}(\frac{c}{l}) = \frac{a}{k} \cdot \hbar(\frac{x}{f}) + \hbar(\frac{x}{f} \cdot \frac{a}{k})$$

for all $\frac{x}{f} \in \mathcal{B}(X, S)$. Let $\hat{\delta}_1(\frac{x}{f}) = 2\hat{\delta}(\frac{x}{f})$. Then

$$\begin{aligned}
&\hat{\delta}_1(\frac{x}{f} \cdot \frac{y}{g} + \frac{y}{g} \cdot \frac{x}{f}) \\
&= 2\hat{\delta}(\frac{c}{l} \cdot (\frac{x}{f} \cdot \frac{y}{g} + \frac{y}{g} \cdot \frac{x}{f})) = 2\hat{\delta}(\frac{c}{l} \cdot \frac{x}{f} \cdot \frac{y}{g} + \frac{y}{g} \cdot \frac{c}{l} \cdot \frac{x}{f}) \\
&= 2\hat{\delta}(\frac{c}{l} \cdot \frac{x}{f}) \cdot \hbar(\frac{y}{g}) + \hbar(\frac{y}{g}) \cdot \hat{\delta}(\frac{c}{l} \cdot \frac{x}{f}) = \hat{\delta}_1(\frac{x}{f}) \cdot \hbar(\frac{y}{g}) + \hbar(\frac{y}{g})\hat{\delta}_1(\frac{x}{f}).
\end{aligned}$$

Also

$$\begin{aligned}
&\hat{\delta}_1(\frac{x}{f} \cdot \frac{y}{g} + \frac{y}{g} \cdot \frac{x}{f}) \\
&= 2\hat{\delta}(\frac{c}{l} \cdot (\frac{x}{f} \cdot \frac{y}{g} + \frac{y}{g} \cdot \frac{x}{f})) = 2\hat{\delta}(\frac{x}{f} \cdot \frac{c}{l} \cdot \frac{y}{g} + \frac{c}{l} \cdot \frac{y}{g} \cdot \frac{x}{f}) \\
&= 2\hat{\delta}(\frac{c}{l} \cdot \frac{y}{g}) \cdot \hbar(\frac{x}{f}) + \hbar(\frac{x}{f}) \cdot \hat{\delta}(\frac{c}{l} \cdot \frac{y}{g}) = \hat{\delta}_1(\frac{y}{g}) \cdot \hbar(\frac{x}{f}) + \hbar(\frac{x}{f})\hat{\delta}_1(\frac{y}{g});
\end{aligned}$$

for any $\frac{x}{f}, \frac{y}{g} \in \mathcal{B}(X, S)$. Therefore, $\hat{\delta_1}$ is $\hbar$-Jordan centralizer of $\mathcal{B}(X, S)$. By Lemma 2.2 we have $\frac{a}{k} = \hat{\delta}(\frac{c}{l}) \in \acute{Z}$ for all $\frac{c}{l} \in \acute{Z}$. □

Theorem 2.4. *Each $\hbar$-Jordan centralizer $\hat{\delta}$ of $\mathcal{B}(X, S)$ is $\hbar$-centralizer.*

Proof. By (2.1) in Theorem 2.1, we have

$$\begin{aligned}
2\hat{\delta}(\frac{x}{f}) &= \hat{\delta}(\frac{x}{f} \cdot \frac{1}{I} + \frac{1}{I} \cdot \frac{x}{f}) = \hat{\delta}(\frac{1}{I}) \cdot \hbar(\frac{x}{f}) + \hbar(\frac{x}{f}) \cdot \hat{\delta}(\frac{1}{I}) \\
&= 2\hat{\delta}(\frac{1}{I}) \cdot \hbar(\frac{x}{f}) = 2\hbar(\frac{x}{f}) \cdot \hat{\delta}(\frac{1}{I})
\end{aligned}$$

for all $\frac{x}{f} \in \mathcal{B}(X, S)$. Thus,

$$\hat{\delta}(\frac{x}{f}) = \hat{\delta}(\frac{1}{I}) \cdot \hbar(\frac{x}{f}) = \hbar(\frac{x}{f}) \cdot \hat{\delta}(\frac{1}{I})$$

for all $\frac{x}{f} \in \mathcal{B}(X, S)$. $\square$

Example 1. *Let R be a ring and δ be a $\hbar$ -Jordan centralizer on R. Suppose that the square of any element $x \in R$ is zero but the product of some elements in R is non zero. Let S be a commutative semi group of injective homomorphisms. We know that $\hbar \in S$. Let*

$$R = \left\{ r = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} | forall x, y \in R \right\},$$

where

$$\delta(r) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$$

and

$$\hbar(r) = \begin{pmatrix} x & -y \\ 0 & 0 \end{pmatrix},$$

for all $r \in R$. It is clear that δ is a left $\hbar$-Jordan centralizer but not a left $\hbar$-centralizer on R. Therefore, δ can has a unique extension to a $\hbar$-Jordan centralizer on $B(R, S)$.

REFERENCES

[1] D. ATANASIU, P MIKUSIŃSKI, D. NEMZER: *An algebraic approach to tempered distributions*, J. Math. Anal. Appl, **384**(2) (2011), 307-319.

[2] A. MAJEED, P. MIKUSINSKI: *Derivations on Pseudoquotients,* Ukrainian mathematical journal, **65** (2013), 959-966.

[3] LI. JIANKUI, Z. JIREN: *Characterization of Jordan derivations and Jordan homomorphisms*, Linear and Multilinear Algebra, **52**(2) (2011), 193-204.

[4] B. ZALAR: *On centralizers of semiprime rings,* Comment.Math.Univ.Carolin, **32**(4) (1991), 609-614.

[5] A. ERDOS: *Operator of finite rank in nest algebras*, London Math.Soc, **43**(1) (1968), 391-397.

[6] E. ALBAS: *On τ-centralizers of semiprime rings,* Siberian Math.J., **48**(2) (2007), 191-196.

[7] P. HALMOS: *Hilbert Space Problem Book*, Springer Verlag, 1974.

[8] B. HADWIN: *Local multiplications on algebras spanned byidempotents*, Linear and multilinear Algebras, **37**(4) (1994), 259-263.

SHERIDAN COLLEGE
MISSISSAUGA CAMPUS
CANADA.
Email address: Asia.majeed@sheridancollege.ca

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
KING ABDULAZIZ UNIVERSITY
P.O. BOX 80348, JEDDAH 22254,
SAUDI ARABIA.
Email address: cenap.ozel@gmail.com

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
KING ABDULAZIZ UNIVERSITY
P.O. BOX 80348, JEDDAH 22254,
SAUDI ARABIA.
Email address: malbaity@kau.edu.sa

DEPARTMENT OF INDUSTRIAL ENGINEERING
KING ABDULAZIZ UNIVERSITY, JEDDAH 21589
SAUDI ARABIA.
Email address: ssaali@kau.edu.sa