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ON SOME ERGODIC RATIONAL FUNCTIONS ON Z_5

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ABSTRACT. In this paper, we considered ergodicity conditions of certain rational functions in $\mathbb{Z}_5$. It is given the case when numerator is transitive modulo 5, but not modulo 25, and the case when numerator is not transitive even modulo 5.

1. INTRODUCTION

Necessary and sufficient conditions for 1-Lipschitz functions that are uniformly differentiable modulo p on $\mathbb{Z}_p$, to be ergodic were studied in [15]. Ergodic polynomials were studied in [8], [9], [12], [13]. Besides, rectification of perturbed monomials to construct ergodic transformations was considered in [4], [16], and [14]. It is known that rational functions are not ergodic on the infinite measure set of p-adic numbers [6]. Ergodicity of rational functions on 2-adic spheres was studied in [11]. In Corollaries 2.1 and 2.2, a perturbation of some non-ergodic polynomials is obtained by division by a unit polynomial, to produce ergodic rational functions on the ring of 5-adic integers.

We recall some facts about the ring of p-adic integers $\mathbb{Z}_p$. Every $x \in \mathbb{Z}_p$ has the p-adic representation $x = \sum_{i=0}^{\infty} x_i p^i$, where for each nonnegative integer i, $x_i \in \{0 \ldots, p-1\}$. The p-adic valuation $\nu_p(x)$ of any p-adic integer x is defined as the

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least nonnegative integer i such that $x_i > 0$. It is known that the p-adic norm $|x|$ of any p-adic number x is given by $|x| = p^{-\nu_p(x)}$.

Each set $x + p^n\mathbb{Z}_p$, $n \geq 1$, is a clopen ball of radius p^{-n}. Besides, the set $\mathbb{Z}_p$ is the disjoint union of p^n balls of radius p^{-n}.

The natural probability measure μ defined on $\mathbb{Z}_p$ gives measure p^{-n} to any ball $x + p^n\mathbb{Z}_p$.

It is clear that a function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is 1-Lipschitz if and only if $f(x + p^n\mathbb{Z}_p) \subseteq f(x) + p^n\mathbb{Z}_p$, for all p-adic integer x and every positive integer n. (See [8, Lemma 1]).

A bijective function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is said to be measure preserving if and only if $\mu(f^{-1}(S)) = \mu(S)$ for every measurable subset S of $\mathbb{Z}_p$.

A 1-Lipschitz function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is said to be bijective modulo p^n if the induced mapping modulo p^n is a permutation on $\mathbb{Z}_p/p^n\mathbb{Z}_p$.

It was also proved in [8, Proposition 4.] that a 1-Lipschitz function on $\mathbb{Z}_p$ is measure preserving if and only if it is bijective modulo p^n for all positive integers n. It can be easily seen that such functions are isometric. A 1-Lipschitz function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is said to be transitive modulo p^n if it is bijective modulo p^n and the set $x, f(x), \ldots, f^{p^n-1}(x)$ is composed of only one cycle. In other words, $f^{p^n}(x) = x \pmod{p^n}$, but $f^r(x) \neq x \pmod{p^n}$, for all $r < p^n$. A measure preserving function is said to be ergodic if it has no proper invariant subset. We recall that in [2, Theorem 1.1.] and [3, Proposition 4.35.] it is proved that a 1-Lipschitz measure preserving function is ergodic if and only if it is transitive modulo p^n for every positive integer n. Some equivalent definitions of 1-Lipschitz measure preserving and ergodic functions are presented in [2], [1], [3], [5] and [8].

A function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is said to be uniformly differentiable modulo p^k if there exist a positive integer N and a function $\partial_k f : \mathbb{Z}_p \to \mathbb{Q}_p$ such that for all $r \geq N$ and $h \in \mathbb{Z}_p$, we have

$$f(u + p^r h) = f(u) + p^r h \partial_k f(u) \pmod{p^{k+r}}, \forall u \in \mathbb{Z}_p.$$

The smallest integer N satisfying this property is denoted by $N_k(f)$. In [3, Proposition 3.41.] it was proved that if f is 1-Lipschitz , then $\partial_k f$ takes its values in $\mathbb{Z}_p$.

We recall the van der Put representation for functions on $\mathbb{Z}_p$ (see [10]). If the p-adic expansion of the positive integer k is given by

$$k = \sum_{i=0}^{s} k_i p^i, \;\; 0 \leq k_i < p, \;\; k_s \neq 0,$$

then we define $q(k) = k_s p^s$.

For every function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ we define the coefficients

$$B_k = \begin{cases} f(k), & k \in \{0, \ldots, p-1\}; \\ f(k) - f(k - q(k)), & k \geq p. \end{cases}$$

In this way the function f can be represented in the so called van der Put basis as follows

$$f(x) = \sum_{k=0}^{\infty} B_k \chi(k, x),$$

where if $k > 0$,

$$\chi(k, x) = \begin{cases} 1, & |x - k| \leq p^{-\lfloor \log_p k \rfloor - 1}; \\ 0, & \text{otherwise.} \end{cases}$$

For $k = 0$ we have

$$\chi(0, x) = \begin{cases} 1, & |x| \leq p^{-1}; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1.1. *[15, Theorem 2.1] Let f be an isometric and uniformly differentiable function modulo p on $\mathbb{Z}_p$, where $N_1(f) = 1$. Then, f is ergodic on $\mathbb{Z}_p$ if and only if the following conditions are satisfied:*

(1) *f is transitive modulo p.*
(2) *For every positive integer k, $f^{p^k}(0) \neq 0 \pmod{p^{k+1}}$.*
(3) *For every positive integer k,*

$$\frac{\prod_{j=0}^{p^k-1} B_{j+p^k}}{(p^k)^{p^k}} = 1 \pmod{p}.$$

Remark 1.1. *Notice that for any analytic function f we have $\frac{B_{i+p^k}}{p^k} = f'(i) \pmod{p^k}$, for every positive integer k and every $i \in \{0, \ldots, p^k - 1\}$, because $f(i + p^k) = f(i) + p^k f'(i) \pmod{p^{2k}}$.*

2. Main result

Theorem 2.1. *Let* $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + 1$ *be an isometric polynomial on* $\mathbb{Z}_5$. *Let* t_i *be representatives of* $5\mathbb{Z}_5$*-cosets such that* $P(t_i) = t_{i+1} \pmod 5$ *and* $P(t_4) = t_0 = 0 \pmod 5$. *Assume that*

$$P^5(t_0) = t_0 \pmod{25} \tag{2.1}$$

and

$$P'(t_0)P'(t_1)P'(t_2)P'(t_3)P'(t_4) = 1 \pmod 5. \tag{2.2}$$

Then $R = \dfrac{P}{Q}$ *is ergodic if the polynomial* $Q(x)$ *satisfies the following conditions*

(1) $Q(\mathbb{Z}_5) \subseteq 1 + 5\mathbb{Z}_5$,
(2) $Q'(x) = 0 \pmod 5$, *for all* $x \in \mathbb{Z}_5$,
(3) $t_4\Big(1 - \dfrac{1}{Q(t_3)}\Big) + t_3P'(t_3)\Big(1 - \dfrac{1}{Q(t_2)}\Big) + t_2P'(t_3)P'(t_2)\Big(1 - \dfrac{1}{Q(t_1)}\Big) \neq 0 \pmod{25}$.

Proof. Since P is transitive modulo 5, from condition (1), it can be easily seen that R satisfies condition (1) of Theorem 1.1. Considering that P' is constant modulo 5 on every $5\mathbb{Z}_5$-coset, according to Remark 1.1 and (2.2), we can see that P satisfies condition (3) of Theorem 1.1, for every nonnegative integer k. From condition (1) we have

$$R' = P' - PQ' \pmod 5.$$

Therefore, from condition (2) we have

$$R'(x) = P'(x) \pmod 5, (\forall x \in \mathbb{Z}_5), \tag{2.3}$$

which means that function R also satisfy the third condition of Theorem 1.1.

According to [8, Proposition 9], it remains to prove that function R is transitive modulo 5^2 to prove that it is ergodic. Therefore, it remains to prove that condition (2) of Theorem 1.1 satisfied for $k = 1$.

Using the notations introduced in the proof of [15, Theorem 2.1] with the additional notation that $t_5 = t_0 = 0$, applying formula (2.4) ([15, Theorem 2.1]) on the function P for $k=1$, we have

$$\begin{aligned} P^5(0) &= P(t_4) + \sum_{i=1}^{4} \frac{P(t_{4-i}) - t_{5-i}}{p^i} \prod_{j=1}^{i} B_{t_{5-j}+5} \pmod{5^2} \\ &= P(t_4) + \frac{P(t_3) - t_4}{5} B_{t_4+5} + \frac{P(t_2) - t_3}{5^2} B_{t_4+5} B_{t_3+5} \\ &+ \frac{P(t_1) - t_2}{5^3} B_{t_4+5} B_{t_3+5} B_{t_2+5} \pmod{5^2}. \end{aligned}$$

According Remark (1.1) now we have

$$\begin{aligned} P^5(0) =& P(t_4) + (P(t_3) - t_4)P'(t_4) + (P(t_2) - t_3)P'(t_4)P'(t_3) \\ &+ (P(t_1) - t_2)P'(t_4)P'(t_3)P'(t_2) \pmod{5^2} \\ =& R(t_4)Q(t_4) + (R(t_3)Q(t_3) - t_4)R'(t_4) + (R(t_2)Q(t_2) - t_3)R'(t_4)R'(t_3) \\ &+ (R(t_1)Q(t_1) - t_2)R'(t_4)R'(t_3)R'(t_2) \pmod{5^2} \\ =& R(t_4)(Q(t_4) - 1) + R(t_4) + (R(t_3) - t_4)R'(t_4) + (R(t_2) - t_3)R'(t_4)R'(t_3) \\ &+ (R(t_1) - t_2)R'(t_4)R'(t_3)R'(t_2) + R'(t_4)R(t_3)(Q(t_3) - 1) \\ &+ R'(t_4)R'(t_3)R(t_2)(Q(t_2) - 1) \\ &+ R'(t_4)R'(t_3)R'(t_2)R(t_1)(Q(t_1) - 1) \pmod{5^2} \\ =& R^5(t_0) + R'(t_4)R(t_3)(Q(t_3) - 1) + R'(t_4)R'(t_3)R(t_2)(Q(t_2) - 1) \\ &+ R'(t_4)R'(t_3)R'(t_2)R(t_1)(Q(t_1) - 1) \pmod{5^2}, \end{aligned}$$

because

$$R(t_4) = Q(t_4) - 1 \pmod{5} = 0 \pmod{5}.$$

Hence

$$\begin{aligned} P^5(0) &= R^5(t_0) + R'(t_4)\Big(\frac{P(t_3)}{Q(t_3)}(Q(t_3) - 1) + P'(t_3)\frac{P(t_2)}{Q(t_2)}(Q(t_2) - 1) \\ &+ P'(t_3)P'(t_2)\frac{P(t_1)}{Q(t_1)}(Q(t_1) - 1)\Big) \pmod{5^2}. \end{aligned} \tag{2.4}$$

From (2.2) and (2.3) it follows $R'(t_4) \neq 0 \pmod{5}$. Now, according to this, condition (2.1) and (2.4) we conclude that R satisfies condition (2) of Theorem 1.1, so R is ergodic. □

Remark 2.1. *Let* $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + 1$. *If we introduce the notations*

$$\sum_{i\in 1+4\mathbb{N}} a_i = A_1, \quad \sum_{i\in 2+4\mathbb{N}} a_i = A_2, \quad \sum_{i\in 3+4\mathbb{N}} a_i = A_3, \quad \sum_{i\in 4\mathbb{N}} a_i = A_4,$$

then, according [7, Proposition 4.2], we have six classes of transitive polynomials modulo 5. Hence, the third condition from previous Theorem for all six classes , can be written in the following way:

$$\begin{cases} 4\Big(1-\dfrac{1}{Q(3)}\Big) + 3P'(3)\Big(1-\dfrac{1}{Q(2)}\Big) + 2P'(2)P'(3)\Big(1-\dfrac{1}{Q(1)}\Big) \neq 0 \pmod{25}, \\ \quad \textit{if} \quad A_1 \equiv 1, \quad A_2 \equiv 0, \quad A_3 \equiv 0, \qquad A_4 \equiv 0 \pmod 5; \\ 3\Big(1-\dfrac{1}{Q(4)}\Big) + 4P'(4)\Big(1-\dfrac{1}{Q(2)}\Big) + 2P'(2)P'(4)\Big(1-\dfrac{1}{Q(1)}\Big) \neq 0 \pmod{25}, \\ \quad \textit{if} \quad A_1 \equiv 4, \quad A_2 \equiv 4, \quad A_3 \equiv 3, \qquad A_4 \equiv 0 \pmod 5, \\ 4\Big(1-\dfrac{1}{Q(2)}\Big) + 2P'(2)\Big(1-\dfrac{1}{Q(3)}\Big) + 3P'(2)P'(3)\Big(1-\dfrac{1}{Q(1)}\Big) \neq 0 \pmod{25}, \\ \quad \textit{if} \quad A_1 \equiv 1, \quad A_2 \equiv 3, \quad A_3 \equiv 3, \qquad A_4 \equiv 0 \pmod 5, \\ 2\Big(1-\dfrac{1}{Q(4)}\Big) + 4P'(4)\Big(1-\dfrac{1}{Q(3)}\Big) + 3P'(3)P'(4)\Big(1-\dfrac{1}{Q(1)}\Big) \neq 0 \pmod{25}, \\ \quad \textit{if} \quad A_1 \equiv 1, \quad A_2 \equiv 4, \quad A_3 \equiv 2, \qquad A_4 \equiv 0 \pmod 5, \\ 3\Big(1-\dfrac{1}{Q(2)}\Big) + 2P'(2)\Big(1-\dfrac{1}{Q(4)}\Big) + 4P'(2)P'(4)\Big(1-\dfrac{1}{Q(1)}\Big) \neq 0 \pmod{25}, \\ \quad \textit{if} \quad A_1 \equiv 4, \quad A_2 \equiv 2, \quad A_3 \equiv 2, \qquad A_4 \equiv 0 \pmod 5, \\ 2\Big(1-\dfrac{1}{Q(3)}\Big) + 3P'(2)\Big(1-\dfrac{1}{Q(4)}\Big) + 4P'(2)P'(3)\Big(1-\dfrac{1}{Q(1)}\Big) \neq 0 \pmod{25}, \\ \quad \textit{if} \quad A_1 \equiv 0, \quad A_2 \equiv 0, \quad A_3 \equiv 3, \qquad A_4 \equiv 0 \pmod 5. \end{cases}$$

Example 1. *Let* $P(x) = 2x^7 + 3x^6 + 5x^5 + 5x^4 + 3x^3 + 2x^2 + x + 1$. *This is an isometric polynomial, transitive modulo 5 and it satisfies conditions*

i) $P(i) = i + 1$ *za* $i \in \{0, 1, 2, 3, 4\}$,
ii) $P^5(0) = 0 \pmod{25}$,
iii) $P'(0)P'(1)P'(2)P'(3)P'(4) = 1 \pmod 5$.

Function $R(x) = \dfrac{2x^7 + 3x^6 + 5x^5 + 5x^4 + 3x^3 + 2x^2 + x + 1}{Q(x)}$ *would be ergodic if the polynomial* $Q(x)$ *satisfies conditions of Theorem 2.1. One of the polynomials which* *satisfies these conditions is* $Q(x) = 10x^4 + 5x^2 + 1$. *Hence, the function* $R(x) = \dfrac{2x^7 + 3x^6 + 5x^5 + 5x^4 + 3x^3 + 2x^2 + x + 1}{10x^4 + 5x^2 + 1}$ *is ergodic.*

The next result is in the case when the numerator is not transitive modulo 5.

Theorem 2.2. *Let* P *be an isometric polynomial on* $\mathbb{Z}_5$. *Assume that* P *is not transitive modulo* P *and let* $2 \le i \le 4$ *be a fixed number such that*

(i) $P(k) = (k+1)i \pmod 5$, $0 \le k \le 4$ *and*
(ii) $P'(0)P'(1)P'(2)P'(3)P'(4) = i \pmod 5$.

Then $R = \dfrac{P}{Q}$ *is ergodic if the polynomial* $Q(x)$ *satisfies conditions*

(1) $Q(\mathbb{Z}_5) \subseteq i + 5\mathbb{Z}_5$,
(2) $Q'(x) = 0 \pmod 5$, *for all* $x \in \mathbb{Z}_5$,
(3) $1 + \sum_{s=0}^{4} \dfrac{l_{4-s}}{i^s} \prod_{j=1}^{s} P'(t_{5-j}) \neq 0 \pmod 5$, *where* $l_k \in \{0, \ldots, 4\}$ *satisfy* $P(k) = (k+1+5l_k)Q(k) \pmod{25}$.

Proof. From (i) and (1) it follows that

$$R(x) = x + 1 \pmod 5 \quad (\forall x \in \mathbb{Z}_5),$$

so R satisfies condition (1) of Theorem 1.1. According to condition (2) we have $R' = \dfrac{P'(x)}{Q'(x)} \pmod 5$, wherence using condition (1) we get that for all $x \in \mathbb{Z}_5$

$$R'(x) = \frac{P'(x)}{i} \pmod 5. \tag{2.5}$$

Now, according to (ii) we have

$$R'(0)R'(1)R'(2)R'(3)R'(4) = \frac{P'(0)P'(1)P'(2)P'(3)P'(4)}{i^5} = 1 \pmod 5,$$

so R satisfies third condition of Theorem 1.1. We should now show that second condition of Theorem 1.1 is also satisfied i.e. $R^5(0) \neq 0 \pmod{25}$.
Now, since $P(k) = (k+1)i \pmod 5 = (k+1)Q(k) \pmod 5$, we have, that for all $k \in \{0, \ldots 4\}$ there is $l_k \in \{0, \ldots 4\}$ such that

$$P(k) = (k+1+5l_k)Q(k) \pmod{25},$$

so $R(k) = k+1+5l_k \pmod{25}$, for all $k \in \{0, \ldots 4\}$. Applying [15, Formula (2.4)] on the function R for $k = 1$ and $t_s = s$, $s \in \{0, \ldots 4\}$ we have

$$\begin{aligned} R^5(0) &= R(t_4) + \sum_{s=1}^{4}(R(t_{4-s}) - t_{5-s}) \prod_{j=1}^{s} R'(t_{5-j}) \pmod{25} \\ &= R(t_4) + \sum_{s=1}^{4}(t_{4-s} + 1 + 5l_{t_{4-s}} - t_{5-s}) \prod_{j=1}^{s} R'(t_{5-j}) \pmod{25}. \end{aligned}$$

For $s \in \{1, \ldots, 4\}$, $t_{5-s} \in \{1, \ldots 4\}$, so $t_{5-s} = t_{4-s}+1$. It follows that $R(t_4) = 5+5l_4$, so we have

$$\begin{aligned} R^5(0) &= 5 + 5l_4 + 5\sum_{s=1}^{4} l_{4-s} \prod_{j=1}^{s} R'(t_{5-j}) \pmod{25} \\ &= 5 + 5\sum_{s=0}^{4} l_{4-s} \prod_{j=1}^{s} R'(t_{5-j}) \pmod{25}, \end{aligned}$$

where first product is taken to be 1 when $s = 0$. The result follows now from (3) and (2.5). □

Example 2. *Let* $P(x) = 2 + 3x + x^3 + 4x^5 + 4x^7$. *This polynomial satisfies conditions from previous Theorem. One of the polynomials* $Q(x)$ *such that the function* $R = \dfrac{P}{Q}$ *is ergodic would be* $Q(x) = 2 + 5x^4$.

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