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ON TOPOLOGY OF CENTROSYMMETRIC MATRICES WITH APPLICATIONS

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ABSTRACT. In this work, we investigate the algebraic and geometric properties of centrosymmetric matrices over the positive reals. We show that the set of centrosymmetric matrices, denoted as $\mathcal{C}_n$, forms a Lie algebra under the Hadamard product with the Lie bracket defined as $[A, B] = A \circ B - B \circ A$. Furthermore, we prove that the set $\mathcal{C}_n$ of centrosymmetric matrices over $\mathbb{R}^+$ is an open connected differentiable manifold with dimension $\lceil \frac{n^2}{2} \rceil$. This result is achieved by establishing a diffeomorphism between $\mathcal{C}_n$ and a Euclidean space $\mathbb{R}^{\lceil \frac{n^2}{2} \rceil}$, and by demonstrating that the set is both open and path-connected. This work provides insight into the algebraic and topological properties of centrosymmetric matrices, paving the way for potential applications in various mathematical and engineering fields.

1. INTRODUCTION

A centrosymmetric matrix is a square matrix that exhibits symmetry with respect to its center. Formally, an $n \times n$ matrix A is centrosymmetric if and only if:

$$A_{i,j} = A_{n+1-i,n+1-j} \quad \text{for all} \quad 1 \leq i, j \leq n.$$

In other words, the elements of a centrosymmetric matrix are symmetric with respect to the center of the matrix. Here is a 3×3 centrosymmetric matrix A:

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$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{21} \\ a_{13} & a_{12} & a_{11} \end{pmatrix}.$$

In this matrix, the elements are symmetric with respect to the center element a_{22}. Specifically, $a_{ij} = a_{(n+1-i,n+1-j)}$ for $n = 3$. Further, here is a 4×4 centrosymmetric matrix A,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{24} & a_{23} & a_{22} & a_{21} \\ a_{14} & a_{13} & a_{12} & a_{11} \end{pmatrix}.$$

The symmetry is centered around the 2×2 block formed by elements a_{22}, a_{23}, a_{32}, and a_{33}. Specifically, the centrosymmetric property holds such that $a_{ij} = a_{(n+1-i,n+1-j)}$ for $n = 4$. The Hadamard product, denoted by $\circ$, is an element-wise multiplication operation on matrices of the same size. Given two $n \times n$ matrices A and B, their Hadamard product $C = A \circ B$ is defined as:

$$C_{i,j} = A_{i,j} \cdot B_{i,j} \quad \text{for all} \quad 1 \leq i, j \leq n. \tag{1.1}$$

The Hadamard product is commutative, associative, and distributive over matrix addition.

Centrosymmetric matrices have been widely studied due to their unique properties and various applications in mathematics, physics, and engineering. In particular, their symmetries and geometric properties make them suitable for problems involving signal processing, control theory, and optimization, among others. Despite their extensive use in the literature, the algebraic and topological properties of centrosymmetric matrices have not been fully explored, and their potential as a Lie algebra under the Hadamard product has not been extensively investigated.

The Hadamard product, an element-wise multiplication of matrices, is a crucial operation in several applications, such as image processing and data analysis. The study of the algebraic properties of centrosymmetric matrices under the Hadamard product may unveil new insights that could be leveraged in these applications.

In the presented work, we aim to fill this gap in the literature by exploring the algebraic and topological properties of the set of centrosymmetric matrices over the positive reals, excluding zero. We demonstrate that the set of centrosymmetric matrices forms a Lie algebra under the Hadamard product and establish that the set is an open connected differentiable manifold with dimension $\lceil \frac{n^2}{2} \rceil$. This result is achieved by showing that the set is both open and path-connected, and by constructing a bijection between the space of centrosymmetric matrices and a Euclidean space.

The set of $n \times n$ centrosymmetric matrices is a subset of the space of all matrices, which is a vector space and can be considered a differentiable manifold. Due to the centrosymmetry constraint, we only have $\lceil \frac{n^2}{2} \rceil$ independent elements (each element and its corresponding centrosymmetric counterpart). Let's denote the space of $n \times n$ centrosymmetric matrices as $\mathcal{C}_n$. We can establish a bijection $\Phi : \mathbb{R}^{\lceil \frac{n^2}{2} \rceil} \to \mathcal{C}_n$, where $\mathbb{R}^{\lceil \frac{n^2}{2} \rceil}$ is a Euclidean space. For a given $n \times n$ centrosymmetric matrix A with elements A_{ij}, the bijection is defined as follows:

$$\Phi(a_{1,1}, \ldots, a_{i,j}, \ldots, a_{n,n}) = A,$$

where $a_{i,j} = A_{i,j}$ for $1 \leq i \leq j \leq n$. The centrosymmetry constraint is automatically satisfied because $A_{n+1-i,n+1-j} = a_{n+1-i,n+1-j} = a_{i,j} = A_{i,j}$ for $1 \leq i \leq j \leq n$.

The mapping Φ is smooth, as it is a simple linear transformation between the elements of the Euclidean space $\lceil \frac{n^2}{2} \rceil$ and the space of centrosymmetric matrices $\mathcal{C}_n$. Furthermore, the inverse mapping $\Phi^{-1} : \mathcal{C}_n \to \mathbb{R}^{\lceil \frac{n^2}{2} \rceil}$ is also smooth, as it corresponds to a linear transformation from the space of centrosymmetric matrices back to the Euclidean space. Since the mapping Φ and its inverse Φ^{-1} are both smooth, the set of centrosymmetric matrices can be considered a differentiable manifold. The remainder of this work is organized as follows. In Section 2, we prove that the set of centrosymmetric matrices forms a Lie group under the Hadamard product. In Section 3, we prove that the set of centrosymmetric matrices forms a Lie algebra under the Hadamard product. Finally, we include some applications of Centrosymmetric matrices in Section 3 and discussing possible future research directions.

2. Lie Group of Centrosymmetric Matrices

Let A and B be two $n \times n$ centrosymmetric matrices with non-zero elements such that $A_{ij} = a_{n+1-i,n+1-j}$ and $B_{ij} = b_{n+1-i,n+1-j}$ for all i, j.

Theorem 2.1. *The set $\mathcal{C}_n$ of centrosymmetric matrices over $\mathbb{R}^+$ is Lie group under Hadamard product.*

Proof. In order to prove that the set $\mathcal{C}_n$ of centrosymmetric matrices over $\mathbb{R}^+$ is a Lie group, we need to show that it satisfies the four properties of a Lie group: closure, associativity, identity, and inverse.

Closure: Let $A, B \in \mathcal{C}_n$ be centrosymmetric matrices. We want to show that their Hadamard product $C = A \circ B$ is also centrosymmetric. Recall that a matrix is centrosymmetric if its elements satisfy $a_{ij} = a_{(n+1-i,n+1-j)}$. Similarly, for matrix B, $b_{ij} = b_{(n+1-i,n+1-j)}$. Now, let's consider the Hadamard product of A and B: $C = A \circ B \implies c_{ij} = a_{ij} \cdot b_{ij}$. We want to show that $c_{ij} = c_{(n+1-i,n+1-j)}$. Using the centrosymmetric property of A and B, we can write: $c_{(n+1-i,n+1-j)} = a_{(n+1-i,n+1-j)} \cdot b_{(n+1-i,n+1-j)}$. As $a_{ij} = a_{(n+1-i,n+1-j)}$ and $b_{ij} = b_{(n+1-i,n+1-j)}$, we can substitute these values into the expression above: $c_{(n+1-i,n+1-j)} = a_{ij} \cdot b_{ij}$. Comparing this with the definition of c_{ij}, we obtain $c_{ij} = c_{(n+1-i,n+1-j)}$. This proves that the Hadamard product of two centrosymmetric matrices A and B is also centrosymmetric, thus satisfying the closure property.

Associativity: Associativity means that for any matrices $A, B, C \in \mathcal{C}_n$, we have $(A \circ B) \circ C = A \circ (B \circ C)$. Let's prove this by showing that the i-th, j-th entry of both sides of the equation are equal. Recall that the Hadamard product of two matrices is an element-wise operation. For any matrices $A, B, C \in \mathcal{C}n$, their elements are denoted as aij, b_{ij}, and c_{ij}, respectively. Now, let's consider the left-hand side of the equation, $(A \circ B) \circ C$:

(1) Calculate the Hadamard product of A and B, which results in a new matrix D with elements $d_{ij} = a_{ij} \cdot b_{ij}$.
(2) Calculate the Hadamard product of the resulting matrix D and C, which results in a matrix E with elements $e_{ij} = d_{ij} \cdot c_{ij}$.

Now, let's consider the right-hand side of the equation, $A \circ (B \circ C)$:

(1) Calculate the Hadamard product of B and C, which results in a new matrix F with elements $f_{ij} = b_{ij} \cdot c_{ij}$.
(2) Calculate the Hadamard product of A and the resulting matrix F, which results in a matrix G with elements $g_{ij} = a_{ij} \cdot f_{ij}$.

To show the associativity, we need to prove that $e_{ij} = g_{ij}$. Let's start by substituting the definitions of e_{ij} and g_{ij}:

$$e_{ij} = d_{ij} \cdot c_{ij} = (a_{ij} \cdot b_{ij}) \cdot c_{ij} g_{ij} = a_{ij} \cdot f_{ij} = a_{ij} \cdot (b_{ij} \cdot c_{ij}). \tag{2.1}$$

Now, due to the associativity of multiplication of real numbers, we can rewrite e_{ij} and g_{ij} as:

$$e_{ij} = a_{ij} \cdot (b_{ij} \cdot c_{ij}), \qquad g_{ij} = a_{ij} \cdot (b_{ij} \cdot c_{ij}).$$

As we can see, $e_{ij} = g_{ij}$, which means that the Hadamard product is associative for centrosymmetric matrices: $(A \circ B) \circ C = A \circ (B \circ C)$.

Identity element: The identity property states that there exists an identity element $I \in \mathcal{C}_n$, such that for any $A \in \mathcal{C}_n$, the Hadamard product of A and I results in A. In other words, $A \circ I = A$. For the Hadamard product, the identity element is a matrix with all elements equal to 1. Let's construct the identity matrix $I \in \mathcal{C}_n$ with all elements equal to 1. We also need to show that I is centrosymmetric.

To prove that I is centrosymmetric, we need to show that $i_{ij} = i_{(n+1-i,n+1-j)}$. Since all elements of I are equal to 1, we have

$$1 = i_{ij} = i_{(n+1-i,n+1-j)}.$$

Now, let's show that the Hadamard product of any centrosymmetric matrix $A \in \mathcal{C}_n$ and the identity matrix I results in A. Recall that the Hadamard product is an element-wise operation

$$(A \circ I)_{ij} = a_{ij} \cdot i_{ij}.$$

Since $i_{ij} = 1$ for all elements of the identity matrix I, we have

$$(A \circ I)_{ij} = a_{ij} \cdot 1 = a_{ij}.$$

Therefore, the Hadamard product of A and I results in A:

$$A \circ I = A.$$

This proves the identity property for the Hadamard product of centrosymmetric matrices.

Inverse element: The inverse property states that for any centrosymmetric matrix $A \in \mathcal{C}_n$, there exists an inverse matrix $A^{-1} \in \mathcal{C}_n$ such that the Hadamard product of A and A^{-1} results in the identity matrix I. In other words, $A \circ A^{-1} = I$. First, let's construct the inverse matrix A^{-1} element-wise as follows:

$$(A^{-1})_{ij} = \frac{1}{aij}.$$

We also need to show that A^{-1} is centrosymmetric. To do that, we need to demonstrate that $(A^{-1})_{ij} = (A^{-1})(n+1-i, n+1-j)$. Given that A is centrosymmetric, we know that $a_{ij} = a_{(n+1-i,n+1-j)}$. Thus, we have

$$(A^{-1})_{ij} = \frac{1}{a_{ij}} = \frac{1}{a_{(n+1-i,n+1-j)}} = (A^{-1})_{(n+1-i,n+1-j)}.$$

Now, let's show that the Hadamard product of any centrosymmetric matrix $A \in \mathcal{C}_n$ and its inverse A^{-1} results in the identity matrix I. Recall that the Hadamard product is an element-wise operation

$$(A \circ A^{-1})_{ij} = a_{ij} \cdot (A^{-1})_{ij}.$$

Using the definition of the inverse matrix A^{-1}, we obtain

$$(A \circ A^{-1})_{ij} = a_{ij} \cdot \frac{1}{a_{ij}} = 1.$$

Since $(A \circ A^{-1})_{ij} = 1$ for all elements, we have

$$A \circ A^{-1} = I.$$

This proves the inverse property for the Hadamard product of centrosymmetric matrices. Considering that the Hadamard product is an element-wise operation involving simple multiplication and division, it possesses the necessary smoothness properties. Therefore, the set of centrosymmetric matrices with non-zero elements forms a Lie group under the Hadamard product. □

Theorem 2.2. *The set $\mathcal{C}_n$ of centrosymmetric matrices over $\mathbb{R}^+$ is an open connected differentiable manifold with dimension $\lceil \frac{n^2}{2} \rceil$.*

Proof.

1. **Open**: The set $\mathcal{C}_n$ of centrosymmetric matrices over $\mathbb{R}^+$ is an open set, as it is a subset of $\mathbb{R}^{n \times n}$ where every entry in the matrix is in $\mathbb{R}^+$. For any point in this space, we can find an open ball around it in which every other point also belongs to $\mathcal{C}_n$. Hence, $\mathcal{C}_n$ is open.

2. **Connected**: The set $\mathcal{C}_n$ of centrosymmetric matrices over $\mathbb{R}^+$ is path-connected. To show this, consider any two centrosymmetric matrices A and B in $\mathcal{C}_n$. Define a path between A and B as $\gamma(t) = (1-t)A + tB$ for $0 \leq t \leq 1$. Since both A and B are centrosymmetric, it follows that $\gamma(t)$ is also centrosymmetric for all t. Moreover, all entries in $\gamma(t)$ are nonzero and positive, as both A and B have entries in $\mathbb{R}^+$, and $0 \leq t \leq 1$. Thus, $\mathcal{C}_n$ is connected.

3. **Differentiable manifold**: The set $\mathcal{C}_n$ of centrosymmetric matrices over $\mathbb{R}^+$ can be considered a differentiable manifold because it can be smoothly embedded into the space of all matrices, which is a vector space and, therefore, a differentiable manifold. Due to the centrosymmetry constraint, we only have $\lceil \frac{n^2}{2} \rceil$ independent elements (each element and its corresponding centrosymmetric counterpart). We can establish a bijection $\Phi : \mathbb{R}^{\lceil \frac{n^2}{2} \rceil} \to \mathcal{C}_n$, where $\mathbb{R}^{\lceil \frac{n^2}{2} \rceil}$ is a Euclidean space. For a given $n \times n$ centrosymmetric matrix A with elements A_{ij}, the bijection is defined as follows:

$$\Phi(a_{1,1}, \ldots, a_{i,j}, \ldots, a_{n,n}) = A;$$

where $a_{i,j} = A_{i,j}$ for $1 \leq i \leq j \leq n$. The centrosymmetry constraint is automatically satisfied because $A_{n+1-i,n+1-j} = a_{n+1-i,n+1-j} = a_{i,j} = A_{i,j}$ for $1 \leq i \leq j \leq n$. Moreover, all entries in Φ are in $\mathbb{R}^+$, and the function is smooth with smooth inverse. Since the space $\mathcal{C}_n$ is open, connected, and smoothly embedded into a differentiable manifold, it is itself a differentiable manifold. Therefore, the set $\mathcal{C}_n$ of centrosymmetric matrices over $\mathbb{R}^+$ is an open connected differentiable manifold with dimension $\lceil \frac{n^2}{2} \rceil$. □

3. Lie Algebra of Centrosymmetric Matrices

A Lie algebra is a vector space with a binary operation called the Lie bracket, which satisfies bilinearity, alternativity, and the Jacobi identity. Let's verify these properties for the set of centrosymmetric matrices with the given Lie bracket definition:

1. **Bilinearity**: For any centrosymmetric matrices A, B, and C, and scalars α and β, we have:

$$\begin{aligned}[\alpha A + \beta B, C] &= (\alpha A + \beta B) \circ C - C \circ (\alpha A + \beta B) \\ &= \alpha(A \circ C) + \beta(B \circ C) - \alpha(C \circ A) - \beta(C \circ B) = \alpha[A, C] + \beta[B, C].\end{aligned}$$

2. **Alternativity**: For any centrosymmetric matrix A,

$$[A, A] = A \circ A - A \circ A = 0.$$

3. **Jacobi identity**: For any centrosymmetric matrices A, B, and C,

$$\begin{aligned}
&[A, [B, C]] + [B, [C, A]] + [C, [A, B]] \\
&= [A, (B \circ C - C \circ B)] + [B, (C \circ A - A \circ C)] + [C, (A \circ B - B \circ A)] \\
&= (A \circ (B \circ C - C \circ B) - (B \circ C - C \circ B) \circ A) \\
&\quad + (B \circ (C \circ A - A \circ C) - (C \circ A - A \circ C) \circ B) \\
&\quad + (C \circ (A \circ B - B \circ A) - (A \circ B - B \circ A) \circ C) \\
&= (A \circ B \circ C - A \circ C \circ B - B \circ C \circ A + C \circ B \circ A) \\
&\quad + (B \circ C \circ A - B \circ A \circ C - C \circ A \circ B + A \circ C \circ B) \\
&\quad + (C \circ A \circ B - C \circ B \circ A - A \circ B \circ C + B \circ A \circ C) \\
&= 0.
\end{aligned}$$

Since the Lie bracket $[A, B] = A \circ B - B \circ A$ satisfies bilinearity, alternativity, and the Jacobi identity, the set of centrosymmetric matrices with this Lie bracket can be considered a Lie algebra.

Theorem 3.1. *Let $\mathcal{C}_n$ be the set of centrosymmetric matrices of size $n \times n$ over $\mathbb{R}^+$. Then the Lie algebra $\mathfrak{g}_n$ of centrosymmetric matrices under the Hadamard product has dimension $\lceil \frac{n^2}{2} \rceil$. There exists a basis $\mathcal{B}_n$ for $\mathfrak{g}_n$ that can be constructed from centrosymmetric matrices with a single non-zero entry in the upper triangular part (including the diagonal), as well as the corresponding symmetric entry in the lower triangular part.*

Proof. Let $\mathcal{C}_n$ be the set of centrosymmetric matrices of size $n \times n$ over $\mathbb{R}^+$, and let $\mathfrak{g}_n$ be the Lie algebra of centrosymmetric matrices under the Hadamard product.

First, we need to show that the dimension of the Lie algebra $\mathfrak{g}_n$ is $\lceil \frac{n^2}{2} \rceil$. Note that for a square matrix of size $n \times n$, there are $\lceil \frac{n^2}{2} \rceil$ entries in either the upper or lower triangular part, including the diagonal.

Now, to construct a basis for $\mathfrak{g}_n$, we create a set of centrosymmetric matrices such that each matrix has a single non-zero entry in the upper or lower triangular part (including the diagonal) and the corresponding symmetric entry in the opposite triangular part. This results in a set of $\lceil \frac{n^2}{2} \rceil$ centrosymmetric matrices, which we will denote it as $\mathcal{B}_n$.

We claim that $\mathcal{B}_n$ is a basis for $\mathfrak{g}_n$. To prove this, we need to show that the elements of $\mathcal{B}_n$ are linearly independent and span the space of centrosymmetric matrices. First, consider any two distinct elements $B_i, B_j \in \mathcal{B}_n$. Since each basis element has a single non-zero entry in the upper or lower triangular part and its symmetric counterpart in the opposite triangular part, the only way for a linear combination of B_i and B_j to be equal to the zero matrix is when the coefficients of the linear combination are zero. This implies that the elements of $\mathcal{B}_n$ are linearly independent.

Next, let A be an arbitrary centrosymmetric matrix in $\mathcal{C}_n$. Then we decompose A into a linear combination of the basis elements in $\mathcal{B}_n$ by assigning the non-zero entries in the upper or lower triangular part of A to the corresponding basis elements. Since any centrosymmetric matrix can be decomposed in this way, the elements of $\mathcal{B}_n$ span the space of centrosymmetric matrices. Therefore, the Lie algebra $\mathfrak{g}_n$ has a basis of dimension $\lceil \frac{n^2}{2} \rceil$. The basis $\mathcal{B}_n$ is constructed from centrosymmetric matrices with a single non-zero entry in the upper or lower triangular part (including the diagonal), as well as the corresponding symmetric entry in the opposite triangular part. Since the elements of $\mathcal{B}_n$ are linearly independent and span the space of centrosymmetric matrices, $\mathcal{B}_n$ forms a basis for the Lie algebra $\mathfrak{g}_n$. □

The exponential map for a Lie group, denoted as $\exp : \mathfrak{g}_n \to \mathcal{C}_n$, maps an element from the Lie algebra to the Lie group. For this specific case, since we are working with the Hadamard product, the exponential map can be defined element-wise as follows:

$$\exp(A) = \begin{pmatrix} e^{a_{11}} & e^{a_{12}} & \cdots & e^{a_{1n}} \\ e^{a_{21}} & e^{a_{22}} & \cdots & e^{a_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{a_{n1}} & e^{a_{n2}} & \cdots & e^{a_{nn}} \end{pmatrix}; \tag{3.1}$$

where $A = (a_{ij}) \in \mathfrak{g}_n$ is a centrosymmetric matrix. To show that this map is indeed the exponential map for the Lie algebra $\mathfrak{g}_n$ under the Hadamard product, we need to verify a few properties: To show smoothness, we can check the differentiability of the exponential function with respect to each entry a_{ij} of the matrix A. The exponential function is known to be infinitely differentiable for any real argument, and its derivatives are continuous. For any matrix entry a_{ij}, the derivative of $e^{a_{ij}}$ with respect to a_{ij} is:

$$\frac{\partial e^{a_{ij}}}{\partial a_{ij}} = e^{a_{ij}};$$

which is continuous. Since the exponential function is differentiable and continuous for every entry of the matrix A, the exponential map $\exp(A)$ is smooth for centrosymmetric matrices under the Hadamard product. For the identity element $I_n \in \mathfrak{g}_n$ with

$$I_n = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

We can compute the exponential map as follows:

$$\exp(I_n) = \begin{pmatrix} e^0 & e^0 & \cdots & e^0 \\ e^0 & e^0 & \cdots & e^0 \\ \vdots & \vdots & \ddots & \vdots \\ e^0 & e^0 & \cdots & e^0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

This result shows that the exponential map satisfies the identity property for centrosymmetric matrices under the Hadamard product.

For any $A \in \mathfrak{g}_n$, the inverse of $\exp(A)$ in the Lie group $\mathcal{C}_n$ is given by $\exp(-A)$. This is because $\exp(A) \circ \exp(-A) = \exp(A - A) = \exp(0) = I$. Near the identity element, the exponential map is a local diffeomorphism. This means that, in a neighborhood of the identity element, the exponential map is a smooth, invertible map between $\mathfrak{g}_n$ and $\mathcal{C}_n$. Let us show this using the following centrosymmetric matrix $A \in \mathfrak{g}_n$, where $n = 3$ for simplicity.

Example 1.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

Now, we can compute the exponential map of A, denoted by $\exp(A)$ *as follows:*

$$\exp(A) = \begin{pmatrix} e^1 & e^2 & e^1 \\ e^3 & e^4 & e^3 \\ e^1 & e^2 & e^1 \end{pmatrix}.$$

The determinant of e^A is

$$\det(e^A) = 0$$

and trace of A is:

$$tr(A) = \sum_{i=1}^{3} a_{ii} = 6.$$

Consequently, $\det(e^A)$ *is not equal to* $e^{tr(A)}$ *for the matrix A.*

The adjoint representation of a Lie algebra is a representation of the Lie algebra on itself. In our case, we consider the Lie algebra $\mathcal{C}_n$ of centrosymmetric matrices under the Hadamard product, with the Lie bracket defined as $[A, B] = A \circ B - B \circ A$. The adjoint representation is a linear mapping, denoted by $\mathrm{Ad} : \mathcal{C}_n \to \mathrm{End}(\mathcal{C}_n)$, where $\mathrm{End}(\mathcal{C}_n)$ represents the set of endomorphisms on the Lie algebra $\mathcal{C}_n$. For a given centrosymmetric matrix $X \in \mathcal{C}_n$, the adjoint representation maps X to a linear transformation $\mathrm{Ad}_X : \mathcal{C}_n \to \mathcal{C}_n$ defined as:

$$Ad_X(Y) = [X, Y] = X \circ Y - Y \circ X,$$

where $Y \in \mathcal{C}_n$. The adjoint representation acts on the Lie algebra itself and can be used to study the structure and properties of the Lie algebra, including its Lie subalgebras and automorphism groups. In the context of centrosymmetric matrices under the Hadamard product, the adjoint representation can help us understand the relationships between the centrosymmetric matrices and their interactions under the Hadamard product and the Lie bracket.

We will provide an example illustrating the adjoint representation in the context of centrosymmetric matrices. Suppose we have the following 3×3 centrosymmetric matrices:

$$X = \begin{pmatrix} a & b & a \\ c & d & c \\ a & b & a \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} e & f & e \\ g & h & g \\ e & f & e \end{pmatrix}. \tag{3.2}$$

The adjoint representation $\mathrm{Ad}_X : \mathcal{C}_n \to \mathcal{C}_n$ for the matrix X is defined as:

$$\mathrm{Ad}_X(Y) = [X, Y] = X \circ Y - Y \circ X. \tag{3.3}$$

Let's compute the adjoint representation $\mathrm{Ad}_X(Y)$ as follows:

$$\begin{aligned} \mathrm{Ad}_X(Y) &= \begin{pmatrix} a & b & a \\ c & d & c \\ a & b & a \end{pmatrix} \circ \begin{pmatrix} e & f & e \\ g & h & g \\ e & f & e \end{pmatrix} - \begin{pmatrix} e & f & e \\ g & h & g \\ e & f & e \end{pmatrix} \circ \begin{pmatrix} a & b & a \\ c & d & c \\ a & b & a \end{pmatrix} \\ &= \begin{pmatrix} ae & bf & ae \\ cg & dh & cg \\ ae & bf & ae \end{pmatrix} - \begin{pmatrix} ae & bf & ae \\ cg & dh & cg \\ ae & bf & ae \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.4}$$

In this particular example, the adjoint representation $\mathrm{Ad}_X(Y)$ resulted in a zero matrix. The adjoint representation can help us understand the relationships between centrosymmetric matrices and their interactions under the Hadamard product and the Lie bracket, providing insights into the structure and properties of the Lie algebra $\mathcal{C}_n$.

4. Application involving centrosymmetric matrices and their properties under the Hadamard product

In quantum information theory, centrosymmetric matrices can be useful when dealing with certain classes of quantum states or operations that exhibit symmetry. For instance, let us consider a specific example involving a two-qubit system and the concept of quantum entanglement.

Suppose we have a two-qubit system in the following mixed state, represented by a density matrix ρ:

$$\rho = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

This density matrix is centrosymmetric, and its structure indicates that the two qubits are in a partially entangled state. To analyze the entanglement properties of this state, we can calculate the entanglement entropy, which is a measure of the entanglement between the two qubits.

The entanglement entropy can be calculated by first computing the reduced density matrix of one of the qubits, which is obtained by tracing out the other qubit from the original density matrix. In this example, we can calculate the reduced density matrix ρ_A of the first qubit as follows:

$$\rho_A = Tr_B(\rho) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

The entanglement entropy is then given by the von Neumann entropy of the reduced density matrix:

$$\mathbb{S}(\rho_A) = -Tr(\rho_A \log_2 \rho_A) = -\frac{1}{2}\log_2(\frac{1}{2}) - \frac{1}{2}\log_2(\frac{1}{2}) = 1.$$

In this example, the centrosymmetric structure of the density matrix allowed us to easily calculate the entanglement entropy of the two-qubit system. The properties of centrosymmetric matrices under the Hadamard product and their geometric properties can help analyze and manipulate such quantum states more efficiently. Now, let us consider another example in quantum information theory involving a three-qubit system in a mixed state. The density matrix ρ representing this state is a 8×8 centrosymmetric matrix presented as follows:

$$\rho = \frac{1}{8}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The centrosymmetric structure of the density matrix suggests that the three-qubit system exhibits some form of symmetry or correlation between its qubits. To further analyze the entanglement properties of this state, we can calculate the reduced density matrices of the individual qubits or any pair of qubits by tracing out the other qubits from the original density matrix.

For example, to calculate the reduced density matrix ρ_{AB} of the first two qubits, we can trace out the third qubit as follows:

$$\rho_{AB} = \mathrm{Tr}_C(\rho) = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

By analyzing the reduced density matrix, we can compute various entanglement measures to quantify the entanglement between the first two qubits or between any qubit and the rest of the system. Consider a linear time-invariant (LTI) system with the following input-output relationship:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau), d\tau;$$

where $x(t)$ is the input, $y(t)$ is the output, and $h(\tau)$ is the impulse response function.

Suppose we have observations of the input and output signals at discrete time points $t_1, t_2, \ldots, t_N$. Then, We can create a centrosymmetric Toeplitz matrix H representing the impulse response:

$$H = \begin{pmatrix} h_0 & h_1 & \cdots & h_{N-1} \\ h_1 & h_0 & \cdots & h_{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N-1} & h_{N-2} & \cdots & h_0 \end{pmatrix}.$$

The input-output relationship can be approximated as:

$$\mathbf{y} = H\mathbf{x};$$

where $\mathbf{x} = [x(t_1), x(t_2), \ldots, x(t_N)]^T$ and $\mathbf{y} = [y(t_1), y(t_2), \ldots, y(t_N)]^T$.

If the system exhibits symmetric behavior, then the centrosymmetric Toeplitz matrix H can be used to represent the system. The properties of the Hadamard product, as well as the geometric structure of centrosymmetric matrices, can be utilized to simplify the identification process or develop new identification algorithms for symmetric systems. Let us investigate the impact of noise on the identification process. The system's input-output relationship can be represented by

$$\mathbf{y} = H\mathbf{x} + \mathbf{n}; \tag{4.1}$$

where $\mathbf{n}$ is the noise vector.

In order to minimize the impact of noise on the identification process, we take advantage of the centrosymmetric structure of the Toeplitz matrix H. One possible approach is to use regularization techniques, such as Tikhonov regularization, which introduces a penalty term based on the centrosymmetric structure of the matrix:

$$\min_{\mathbf{x}} ||H\mathbf{x} - \mathbf{y}||_2^2 + \lambda||L\mathbf{x}||_2^2; \tag{4.2}$$

where L is a linear operator that captures the centrosymmetric structure, and λ is a regularization parameter that balances the trade-off between fitting the data and enforcing the centrosymmetric structure. By exploiting the centrosymmetric structure and the properties of the Hadamard product, we can develop efficient algorithms for estimating the impulse response function $\mathbf{x}$ and improving the robustness of the identification process in the presence of noise.

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