ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **12** (2023), no.11, 963–976
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.12.11.3

AN INTRO TO δ_d-FUZZY GRAPHS

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ABSTRACT. Graph is a easy way to represent the real life situation. Graph is a combination of Points and Lines. In network analysis, the degree of a point plays a prominent role in Graph Theory. The degree of a point is the number of connections it has with the other points in the point set. Among the degrees of all the points in graph G^*, the minimum value is denoted by $\delta(G^*)$. In this article, a new abstraction of fuzzy graph is initiated by combining the parameters, degree of a point and minimum degree of the graph and termed it is as δ_d-fuzzy graphs. Order and Size on δ_d-fuzzy graphs were studied and Handshaking Lemma were explained with illustration. Idea on δ_d-regular fuzzy graph were interpreted using the theorems. Also operations on graphs such as union, intersection, complement, cartesian product, Tensor Product, Corona are extended for δ_d-fuzzy graphs.

1. INTRODUCTION

A graph is a set of points and lines. The lines between points will form a model of relations using graphical representation. [14] Fuzzy set theory was proposed by Prof. Zadeh which has been efficiently used to model many real world problems, which are uncertain. In a crisp graph, the members of the sets takes a single value of 0 or 1. For this reason, the uncertainities of real world problems

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2020 *Mathematics Subject Classification.* 05C72, 05C07, 03E72, 05C12.
Key words and phrases. Minimum Degree of a Point, Fuzzy Graph, Cartesian Product, Tensor Product.
Submitted: 11.10.2023; *Accepted:* 26.10.2023; *Published:* 28.11.2023.

don't work properly. But the fuzzy set provides its element to have the grade of membership within the range from 0 to 1. A concept of fuzzy graph was introduced by [10]Rosenfeld and some operations are discussed by [2] Mordeson and Nair, they also defined the complement of fuzzy graph. Also, Venugopalam [13] discussed Operations on Fuzzy Graphs. Further these operations are also studied by [11, 12] Sunitha and Vijayakumar. Later [7–9] Nagoorgani examined the properties of degree, regular, irregular, order and size of fuzzy graphs and also compared the relationship between these parameters. Let $G^*(P, L)$ be a simple graph with point set P and line set L. [1] In graph theory, relations can be modeled in to graph for better understanding. The number of points in a graph G^* is known as order and is denoted by $\rho(G^*)$ and the number of lines in G^* is known as size is denoted by $\tau(G^*)$. The sequence of distinct consecutive points followed by distinct consecutive lines is called Path. The closed path is known as Cycle. The degree of a point p_i is defined as the number of lines that are incident to that point and is denoted by $deg(p_i)$. If the degree of every point is same, then it is a regular graph, otherwise it is irregular. If the degree of all points in a graph are three, then it is cubic. If every point is connected to all other distinct points, then it is called complete graph.

2. δ_d-FUZZY GRAPH

Definition 2.1. *Let $FG_{\delta_d} : (\sigma_{\delta_d}, \mu_{\delta_d})$ be a δ_d-fuzzy graph with the membership function of point set σ_{δ_d} and membership function of line set μ_{δ_d} under the crisp graph $G^* : (P, L)$ were defined by,*

$$\sigma_{\delta_d}(p_i) = \frac{\delta(G^*)}{\deg(p_i)};\ \forall\ p_i \in P(G^*) \text{ and}$$

$$\mu_{\delta_d}(p_ip_j) = \begin{cases} \min\{\sigma_{\delta_d}(p_i), \sigma_{\delta_d}(p_j)\}; & \text{if } p_ip_j \in L(G^*) \\ 0; \text{ Otherwise.} \end{cases}$$

Definition 2.2. *The Order $O(FG_{\delta_d})$ and Size $S(FG_{\delta_d})$ of δ_d-fuzzy graph under the crisp graph $G^* : (P, L)$ is defined as,*

$$\rho_{\delta_d} = O(FG_{\delta_d}) = \sum_{p_i \in P(G^*)} \sigma_{\delta_d}(p_i) \text{ and } \tau_{\delta_d} = S(FG_{\delta_d}) = \sum_{p_ip_j \in L(G^*)} \mu_{\delta_d}(p_ip_j).$$

Definition 2.3. *The degree of a point in δ_d-fuzzy graph is denoted by $\deg_{\delta_d}(p_i)$ and it is defined by,*

$$\deg_{\delta_d}(p_i) = \sum_{\substack{p_ip_j \in P(G^*) \\ p_i \neq p_j}} \mu_{\delta_d}(p_ip_j); \ \forall \ p_i \in P(G^*).$$

Definition 2.4. *The FG_{δ_d} fuzzy graph is said to be δ_d-complete fuzzy graph, if* $\deg_{\delta_d}(p_i) = \rho_{\delta_d} - 1; \ \forall \ p_i \in P(G^*)$.

Definition 2.5. *The FG_{δ_d} fuzzy graph is said to be δ_d-regular fuzzy graph, if* $\deg_{\delta_d}(p_i) = k; \ \forall \ p_i \in P(G^*)$ *where k is a constant.*

Definition 2.6. *The total degree of a point on δ_d-fuzzy graph is denoted by $tdeg_{\delta_d}$ and it is defined as,*

$$tdeg_{\delta_d}(p_i) = \deg_{\delta_d}(p_i) + \sigma_{\delta_d}(p_i); \ \forall \ p_i \in P(G^*).$$

Illustration 2.1. Consider a δ_d-fuzzy graph given in Figure 1 with Point set $P(G^*) = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8\}$.

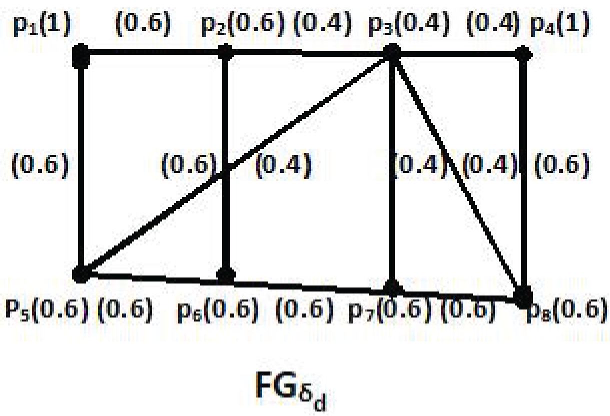

FIGURE 1.

The degree of all the points are given by,

$$\deg(p_2) = \deg(p_5) = \deg(p_6) = \deg(p_7) = \deg(p_8) = 3;$$
$$\deg(p_3) = 5; \deg(p_1) = 2 = \deg(p_4).$$

Thus $\delta(G^*) = 2$. The membership value of point sets are obtained by, $\sigma_{\delta_d}(p_i) = \frac{\delta(G^*)}{\deg(p_i)}; \forall \ p_i \in P(G^*)$. $\sigma_{\delta_d}(p_1) = 1 = \sigma_{\delta_d}(p_4)$; $\sigma_{\delta_d}(p_2) = \sigma_{\delta_d}(p_5) = \sigma_{\delta_d}(p_6) = \sigma_{\delta_d}(p_7) =$

$\sigma_{\delta_d}(p_8) = 0.6$; $\sigma_{\delta_d}(p_3) = 0.4$. The membership value of the Line set are obtained as by Definition 2.1 are, $\mu_{\delta_d}(p_1p_2) = \mu_{\delta_d}(p_1p_5) = \mu_{\delta_d}(p_2p_6) = \mu_{\delta_d}(p_4p_8) = \mu_{\delta_d}(p_5p_6) = \mu_{\delta_d}(p_6p_7) = \mu_{\delta_d}(p_7p_8) = 0.6$; $\mu_{\delta_d}(p_2p_3) = \mu_{\delta_d}(p_3p_5) = \mu_{\delta_d}(p_3p_7) = \mu_{\delta_d}(p_3p_4) = \mu_{\delta_d}(p_3p_8) = 0.4$.

The order and size of $FG\delta_d$ fuzzy graph are given by, $\rho_{\delta_d}(FG_{\delta_d}) = \sum\limits_{p_i \in P(G^*)} \sigma_{\delta_d} = 5.4$ and $\tau_{\delta_d}(FG_{\delta_d}) = \sum\limits_{p_ip_j \in E(G^*)} \mu_{\delta_d}(p_ip_j) = 0.6(7) + 0.4(5) = 6.2$ respectively. The degree of δ_d-fuzzy graph are, $\deg_{\delta_d}(p_1) = 1.2$; $\deg_{\delta_d}(p_2) = 1.6 = \deg_{\delta_d}(p_7) = \deg_{\delta_d}(p_8)$; $\deg_{\delta_d}(p_3) = 2$; $\deg_{\delta_d}(p_4) = 1$; $\deg_{\delta_d}(p_5) = 1.8 = \deg_{\delta_d}(p_6)$.

Theorem 2.1. *Let ρ_{δ_d} and τ_{δ_d} be the order and size of δ_d-fuzzy graph under the crisp graph G^* with order ρ and size τ respectively, then $\rho_{\delta_d} \leq \rho$ and $\tau_{\delta_d} \leq \tau$.*

Proof. By Definition 2.1, $\rho_{\delta_d} = \sum\limits_{p_i \in P(G^*)} \sigma_{\delta_d} = \sum\limits_{p_i \in P(G^*)} \frac{\delta(G^*)}{\deg(p_i)}$. And by Definition 2.4, the complete point has the maximum degree of $\rho_{\delta_d} - 1$. Thus, $\rho_{\delta_d} \leq \sum\limits_{p_i \in P(G^*)} \frac{\rho-1}{\rho-1} \leq 1 + 1 + \ldots + 1$ (ρ times). Therefore $\rho_{\delta_d} \leq \rho$.

Similarly, $\tau_{\delta_d} = \sum\limits_{p_ip_j \in L(G^*)} \mu_{\delta_d}(p_ip_j) = \sum\limits_{p_ip_j \in L(G^*)} \min\{\sigma_{\delta_d}(p_i), \sigma_{\delta_d}(p_j)\}$. Since the maximum membership value of a point is 1. This implies $\tau_{\delta_d} \leq 1 + 1 + \ldots + 1$ (τ times). Thus, $\tau_{\delta_d} \leq \tau$. □

Theorem 2.2. Equality Theorem of Theorem 2.1 *Let $FG_{\delta_d} : (\sigma_{\delta_d}, \mu_{\delta_d})$ be δ_d-fuzzy graph with order ρ_{δ_d} and size τ_{δ_d} corresponding to a crisp graph of order ρ and size τ respectively, then $\rho_{\delta_d} = \rho$ and $\tau_{\delta_d} = \tau$ if and only if G^* is a regular graph.*

Proof. Suppose $\rho_{\delta_d} = \rho$, then by Definition 2.2, $\rho_{\delta_d} = \sum\limits_{p_i \in P(G^*)} \frac{\delta(G^*)}{\deg(p_i)} = \rho$. This happens only if $\deg(p_i) = \delta(G^*) = \min\{\deg(p_i)/p_i \in P(G^*)\}$, which implies that G^* is a regular graph. Conversely, if G^* is regular graph, then, the degree of every points are equal to some constant (say m). Thus, $\deg(p_i) = \min\{\deg(p_i)/p_i \in P(G^*)\} = \delta(G^*)$. Then, $\rho_{\delta_d} = \sum\limits_{p_i \in P(G^*)} \frac{\delta(G^*)}{\deg(p_i)} = \sum\limits_{p_i \in P(G^*)} \frac{m}{m} = \sum\limits_{p_i \in P(G^*)} 1 = 1 + 1 + \ldots + 1$ (ρ times)$= \rho$. Also, let $\tau_{\delta_d} = \tau$, By Definition 2.2 of δ_d-fuzzy graph, $\tau_{\delta_d} = \sum\limits_{p_ip_j \in L(G^*)} = \tau$ only if minimum degree is equal to the degree of every point in a graph G^*, which implies that G^* is a regular graph. Conversely if G^* is a regular graph, then $\deg(p_i) = \min\{\deg(p_i)/p_i \in P(G^*)\}$.

This implies $\tau_{\delta_d} = \sum_{p_ip_j \in L(G^*)} \mu_{\delta_d}(p_ip_j) = \sum_{p_ip_j \in L(G^*)} \min\{\sigma_{\delta_d}(p_i), \sigma_{\delta_d}(p_j)\} = 1 + 1 + \ldots + 1(\tau \text{ times}) = \tau$. Thus $\tau_{\delta_d} = \tau$. □

Theorem 2.3. Bounds on Order and Size of δ_d-fuzzy graph. *Let FG_{δ_d} be a δ_d-fuzzy graph of order ρ_{δ_d} and size τ_{δ_d}, corresponding to non-isolated connected simple graph G^* with order ρ and size τ, then $2 \leq \rho_{\delta_d} \leq \rho$ and $1 \leq \tau_{\delta_d} \leq \tau$.*

Proof. Let us assume minimum degree sequence of non-isolated connected simple graph G^* on δ_d-fuzzy graph, which is, K_2 with deg ree sequence $\{1, 1\}$. Thus the order, $\rho_{\delta_d} = \sum_{p_i \in P(G^*)} \sigma_{\delta_d}(p_i) = \frac{1}{1} + \frac{1}{1} = 2$ and size, $\tau_{\delta_d} = \sum_{p_ip_j \in L(G^*)} \tau_{\delta_d}(p_ip_j) = 1$. The upper bound is obvious from Theorem 2.2. This completes the proof. □

Proposition 2.1. *Let FG_{δ_d} be δ_d-fuzzy graph corresponding to a crisp graph $G^* : (P, L)$, then $\deg_{\delta_d}(p_i) \leq \deg(p_i)\ \forall\ p_i \in P(G^*)$.*

Proof. By Definition 2.3, $\deg_{\delta_d}(p_i) = \sum_{p_ip_j \in L(G^*)} \mu_{\delta_d}(p_ip_j) = \sum \min\{\sigma_{\delta_d}(p_i), \sigma_{\delta_d}(p_j)\} = \sum_{p_ip_j \in L(G^*)} 1 \leq \deg(p_i)$. □

Lemma 2.1. *Let FG_{δ_d} be δ_d-fuzzy graph under the crisp graph $G^* : (P, L)$, then the sum of the degree of every point of FG_{δ_d} is equal to twice the size of $FG\delta_d$.*

Illustration 2.2. From the illustration 2.1, for the FG_{δ_d} graph given in Figure 1.

Sum of degree of all points=1.2+1.6+2+1+1.6+1.8+1.6+1.6=12.4.

Twice the size of $FG\delta_d$ fuzzy graph= $2 \times 6.2 = 12.4$. Hence Lemma 2.1, holds.

Lemma 2.2. *Let FG_{δ_d} be a δ_d-fuzzy graph under the crisp graph $G^* : (P, L)$, then the sum of total degree of all points is equal to sum of twice the size of FG_{δ_d} and order of FG_{δ_d}.*

Lemma 2.3. *According to Definition2.6, $tdeg_{\delta_d}(p_i) = \deg_{\delta_d}(p_i) + \sigma_{\delta_d}(p_i);\ \forall\ p_i \in P(G^*)$.*

Sum of all points in $FG\delta_d$ fuzzy graph= $\sum_{p_ip_j \in L(G^*)} \mu_{\delta_d}(p_ip_j) + \sum_{p_i \in P(G^*)} \sigma_{\delta_d}(p_i)$ and by Lemma 2.1,

$\sum_{p_i \in P(G^*)} tdeg_{\delta_d}(p_i) = 2\tau_{\delta_d}(FG_{\delta_d}) + \rho_{\delta_d}(FG\delta_d)$.

Illustration 2.3. For the FG_{δ_d} graph given in Figure 1, the total degree of all points are obtained by, $tdeg_{\delta_d}(p_1) = 2.2 = tdeg_{\delta_d}(p_2) = tdeg_{\delta_d}(p_7) = tdeg_{\delta_d}(p_8)$;

$tdeg_{\delta_d}(p_3) = 2.4 = tdeg_{\delta_d}(p_5) = 2.4 = tdeg_{\delta_d}(p_6)$; $tdeg_{\delta_d}(p_4) = 2$. Sum of all total degrees of all points in $FG\delta_d$=2.2+2.2+2.4+2+2.2+2.4+2.2+2.2=17.8.

$2\tau_{\delta_d}(FG\delta_d) + \rho_{\delta_d}(FG_{\delta_d})$=2(6.2)+5.4=12.4+5.4=17.8. Thus the Lemma holds.

3. Regular δ_d-fuzzy graph

Definition 3.1. *If degree of every point is equal to some constant(m) in δ_d-fuzzy graph, under the crisp graph G^*, then it is known to be m-regular δ_d-fuzzy graph.*

Definition 3.2. *If total degree of every point is equal to some constant (m_1) in δ_d-fuzzy graph, under the crisp graph G^*, then it is known to be m_1-totally regular δ_d-fuzzy graph.*

Theorem 3.1. *Let FG_{δ_d} be a δ_d-fuzzy graph under the crisp graph G^*, if σ_{δ_d} is constant, then FG_{δ_d} is δ_d-regular fuzzy graph and δ_d-totally regular fuzzy graph.*

Proof. Suppose FG_{δ_d} is not totally fuzzy graph. Then, $tdeg_{\delta_d}(p_i) \neq tdeg_{\delta_d}(p_j)$; $\forall$ $p_i, p_j \in P(G^*)$. This implies $\deg_{\delta_d}(p_i) + \sigma_{\delta_d}(p_i) \neq \deg_{\delta_d}(p_j) + \sigma_{\delta_d}(p_j)$. Since σ_{δ_d} is constant. $\implies$ $\deg_{\delta_d}(p_i) \neq \deg_{\delta_d}(p_j)$ which is contradiction. Thus we conclude that FG_{δ_d} is δ_d-regular fuzzy graph.

Conversely, let FG_{δ_d} be a δ_d-regular fuzzy graph. Then $\deg_{\delta_d}(p_i) = \deg_{\delta_d}(p_j)$; $\forall$ $p_i, p_j \in P(G^*)$. This implies, $\deg_{\delta_d}(p_i) + \sigma_{\delta_d}(p_i) = \deg_{\delta_d}(p_j) + \sigma_{\delta_d}(p_j)$. Since σ_{δ_d} is constant. This becomes $tdeg_{\delta_d}(p_i) = tdeg_{\delta_d}(p_j)$, which proves that FG_{δ_d} is δ_d-totally regular fuzzy graph. □

Illustration 3.1. Consider a δ_d fuzzy graph in Figure 2, with Point set $P(G^*) = \{p_1, p_2, p_3, p_4, p_5\}$.

The degree of each points are $\deg(p_i) = 3$ $\forall$ $p_i \in P(G^*)$. Thus $\delta(G^*) = 3$. The membership value of point sets are obtained by, $\sigma_{\delta_d}(p_i) = \frac{\delta(G^*)}{\deg(p_i)}$; $\forall$ $p_i \in P(G^*)$. Also $\sigma_{\delta_d}(p_i) = 1$; $\forall$ $p_i \in P(G^*)$. The membership value of Line set are obtained as by Definition 2.1 are, $\mu_{\delta_d}(p_i p_j) = 1$; $\forall$ $p_i p_j \in L(G^*)$. The degree of δ_d-fuzzy graph are, $\deg_{\delta_d}(p_1) = 3$; $\forall$ $p_i \in P(G^*)$. Since the degree of every points in δ_d-fuzzy graph are equal. Thus FG_{δ_d} is 3-regular δ_d-fuzzy graph.

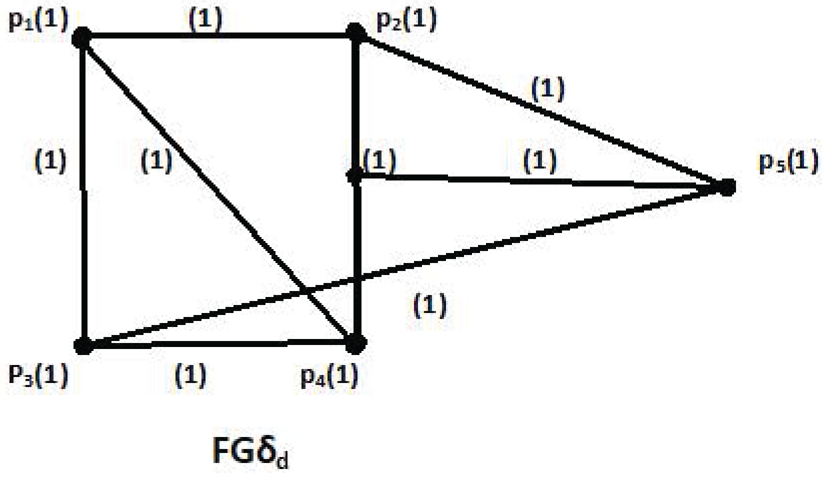

FIGURE 2.

4. OPERATIONS ON δ_d-FUZZY GRAPHS

Definition 4.1. *Let* $FG_{\delta_{d_1}} : (\sigma_{\delta_{d_1}}, \mu_{\delta_{d_1}})$ *and* $FG_{\delta_{d_2}} : (\sigma_{\delta_{d_2}}, \mu_{\delta_{d_2}})$ *be two* δ_d*-fuzzy graph under the crisp graph* $G_1^* : (P_1, L_1)$ *and* $G_2^* : (P_2, L_2)$ *respectively. The union of two* δ_d*-fuzzy graphs* $FG_{\delta_{d_3}}$ *are defined as,*

$$\sigma_{\delta_d}(p_i) = (\sigma_{\delta_{d_1}} \cup \sigma_{\delta_{d_2}})(p_i) = \begin{cases} \sigma_{\delta_{d_1}}(p_i); p_i \in P(G_1^*) - P(G_2*) \\ \sigma_{\delta_{d_2}}(p_i); p_i \in P(G_2^*) - P(G_1*) \\ \max\{\sigma_{\delta_{d_1}}(p_i), \sigma_{\delta_{d_2}}(p_i)\}; p_i \in P(G_1^*) \cap P(G_2*) \end{cases}$$

$$\mu_{\delta_d}(p_ip_j) = (\mu_{\delta_{d_1}} \cup \mu_{\delta_{d_2}})(p_ip_j) = \begin{cases} \mu_{\delta_{d_1}}(p_ip_j); p_ip_j \in L(G_1^*) - L(G_2*) \\ \mu_{\delta_{d_2}}(p_ip_j); p_ip_j \in L(G_2^*) - L(G_1*) \\ \max\{\mu_{\delta_{d_1}}(p_ip_j), \mu_{\delta_{d_2}}(p_ip_j)\}; p_ip_j \in L(G_1^*) \cap L(G_2*) \end{cases}$$

Illustration 4.1. Consider the two δ_d-fuzzy graphs $FG_{\delta_{d_1}}$ and $FG_{\delta_{d_2}}$ given in Figure 3(a) and 3(b). Then the union of these two δ_d-fuzzy graphs is given in Figure 3(c).

Remark 4.1. The graph FG_{δ_d} given in Figure 3(c). is not a δ_d-fuzzy graph. Since according to the definition of the node membership function σ_{δ_d} of δ_d-Fuzzy Graph, $\sigma_{\delta_d}(p_2) = \frac{\delta_d(G^*)}{\deg(p_2)} = \frac{2}{4} = 0.5 \neq 1$. Thus the union of two δ_d-fuzzy graphs $FG_{\delta_{d_1}}$ and

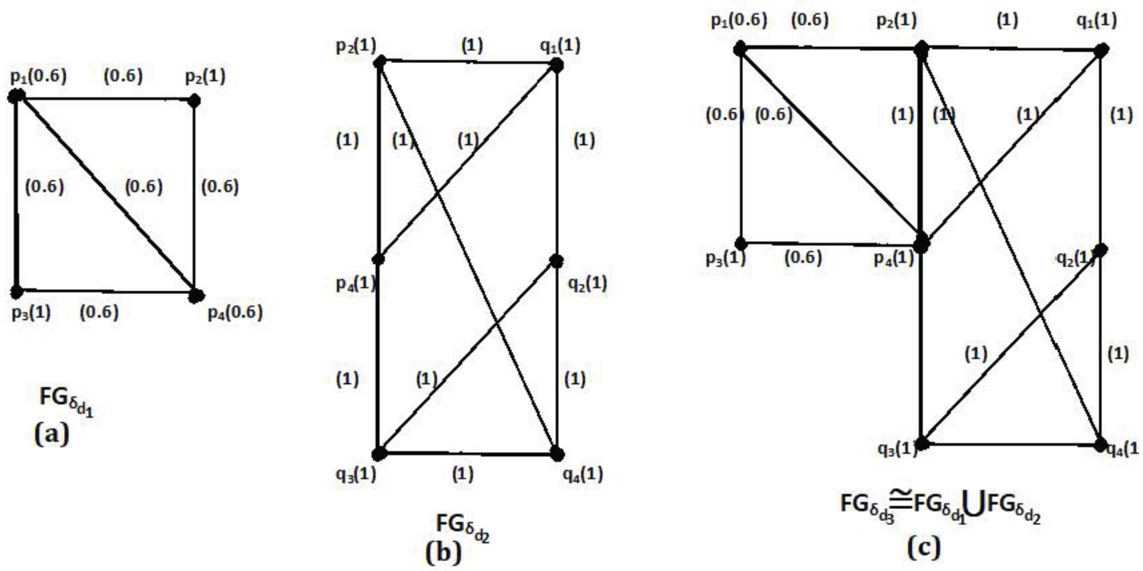

FIGURE 3.

$FG_{\delta_{d_2}}$ need not be a δ_d-fuzzy graph, but it is a fuzzy graph and call it as pseudo δ_d-fuzzy graph.

Definition 4.2. *Let* $FG_{\delta_{d_1}} : (\sigma_{\delta_{d_1}}, \mu_{\delta_{d_1}})$ *and* $FG_{\delta_{d_2}} : (\sigma_{\delta_{d_2}}, \mu_{\delta_{d_2}})$ *be two* δ_d*-fuzzy graph under the crisp graph* $G_1^* : (P_1, L_1)$ *and* $G_2^* : (P_2, L_2)$ *respectively. The intersection of two* δ_d*-fuzzy graphs* $FG_{\delta_{d_4}}$ *are defined as,*

$$\sigma_{\delta_d}(p_i) = (\sigma_{\delta_{d_1}} \cap \sigma_{\delta_{d_2}})(p_i) = \begin{cases} \min\{\sigma_{\delta_{d_1}}(p_i), \sigma_{\delta_{d_2}}(p_i)\}; p_i \in P(G_1^*) \cap P(G_2*) \\ 0; \textit{otherwise} \end{cases}$$

$$\mu_{\delta_d}(p_ip_j) = (\mu_{\delta_{d_1}} \cap \mu_{\delta_{d_2}})(p_ip_j) = \begin{cases} \min\{\mu_{\delta_{d_1}}(p_ip_j), \mu_{\delta_{d_2}}(p_ip_j)\}; p_ip_j \in L(G_1^*) \cap L(G_2*) \\ 0; \textit{otherwise} \end{cases}$$

Illustration 4.2. For the two δ_d-fuzzy graphs $FG_{\delta_{d_1}}$ and $FG_{\delta_{d_2}}$ given in Figure 3(a) and 3(b), the intersection of these two δ_d-fuzzy graph $FG_{\delta_{d_4}}$ is given in Figure 4.

Remark 4.2. The graph $FG_{\delta_{d_4}}$ given in Figure 4, is not a δ_d-fuzzy graph. Since according to the Definition of the node membership function σ_{δ_d} of δ_d-fuzzy graph, $\sigma_{\delta_d}(p_2) = \frac{\delta_d(G^*)}{\deg(p_2)} = \frac{1}{1} = 1$. Thus the intersection of two δ_d-fuzzy graphs $FG\delta_{d_1}$ and $FG\delta_{d_2}$ is not a δ_d fuzzy graph, but it is a fuzzy graph called pseudo δ_d-fuzzy graph.

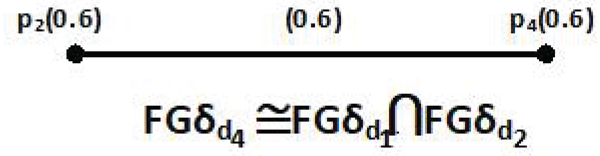

FIGURE 4.

Definition 4.3. *Let $FG_{\delta_{d_1}} : (\sigma_{\delta_{d_1}}, \mu_{\delta_{d_1}})$ and $FG_{\delta_{d_2}} : (\sigma_{\delta_{d_2}}, \mu_{\delta_{d_2}})$ be two δ_d-fuzzy graph under the crisp graph $G_1^* : (P_1, L_1)$ and $G_2^* : (P_2, L_2)$ respectively. Assume $P(G_1^*) \cap P(G*_2) = \phi$. The join of two δ_d-fuzzy graphs with points and lines are given by $P(G^*) = P(G_1^*) \cup P(G_2^*)$ and $L(G^*) = L(G_1^*) \cup L(G_2^*) \cup L(G')$, where $L(G')$ is the set of all lines joining points of points $P(G_1^*)$ with points of $P(G_2^*)$ is defined as,*

$$\sigma_{\delta_d}(p_i) = (\sigma_{\delta_{d_1}} + \sigma_{\delta_{d_2}})(p_i) = \sigma_{\delta_{d_1}}(p_i) \cup \sigma_{\delta_{d_2}}(p_i), \ \forall \ p_i \in P(G_1^*) \text{ and } p_j \in P(G_2^*)$$

$$\begin{aligned} \mu_{\delta_d}(p_ip_j) &= (\mu_{\delta_{d_1}} + \mu_{\delta_{d_2}})(p_ip_j) \\ &= \begin{cases} (\mu_{\delta_{d_1}}(p_ip_j) \cup \mu_{\delta_{d_2}})(p_ip_j); \ \forall \ p_ip_j \in L(G_1^*) \cup L(G_2*) \\ \max\{\sigma_{\delta_d}(p_i), \sigma_{\delta_d}(p_j); if(p_i, p_j) \in L(G') \end{cases}. \end{aligned}$$

Illustration 4.3. Consider the two δ_d-fuzzy graphs, given in Figure 5(a) and 5(b). Then the join of two δ_d-fuzzy graphs. $FG_{\delta_{d_7}}$ is given in Figure 5(c).

Remark 4.3. The graph $FG_{\delta_{d_7}}$ given in Figure 5(c) is not a δ_d-fuzzy graph. Since according to the definition of the node membership function σ_{δ_d} of δ_d- fuzzy graph, $\sigma_{\delta_d}(q_1) = \frac{\delta_d(G^*)}{\deg(q_1)} = \frac{3}{6} = 0.5 \neq 1$. Thus the join of two δ_d-fuzzy graphs $FG_{\delta_{d_5}}$ and $FG_{\delta_{d_6}}$ need not be a δ_d-fuzzy graph, but it is a fuzzy graph known as pseudo δ_d-fuzzy graph.

Definition 4.4. *Let $FG_{\delta_{d_1}} : (\sigma_{\delta_{d_1}}, \mu_{\delta_{d_1}})$ and $FG_{\delta_{d_2}} : (\sigma_{\delta_{d_2}}, \mu_{\delta_{d_2}})$ be two δ_d-fuzzy graph under the crisp graph $G_1^* : (P_1, L_1)$ and $G_2^* : (Q_2, L_2)$ respectively. Then the cartesian product '$\times$' of two δ_d-fuzzy graphs with points and lines are given by $P(G^*) = P(G_1^*), Q(G_2^*)$ and $L(G^*) = \{(p_1, q_1)(p_2, q_2)/p_1 = p_2, and(q_1, q_2) \in L(G_2^*) \ or \ q_1 = q_2, \ and \ (p_1, p_2) \in L(G_1^*)\}$ is defined as,*

$$\sigma_{\delta_d}(p_iq_j) = (\sigma_{\delta_{d_1}}(p_i), \sigma_{\delta_{d_2}}(q_j)) = \min\{\sigma_{\delta_{d_1}}(p_i), \sigma_{\delta_{d_2}}(q_j)\}$$

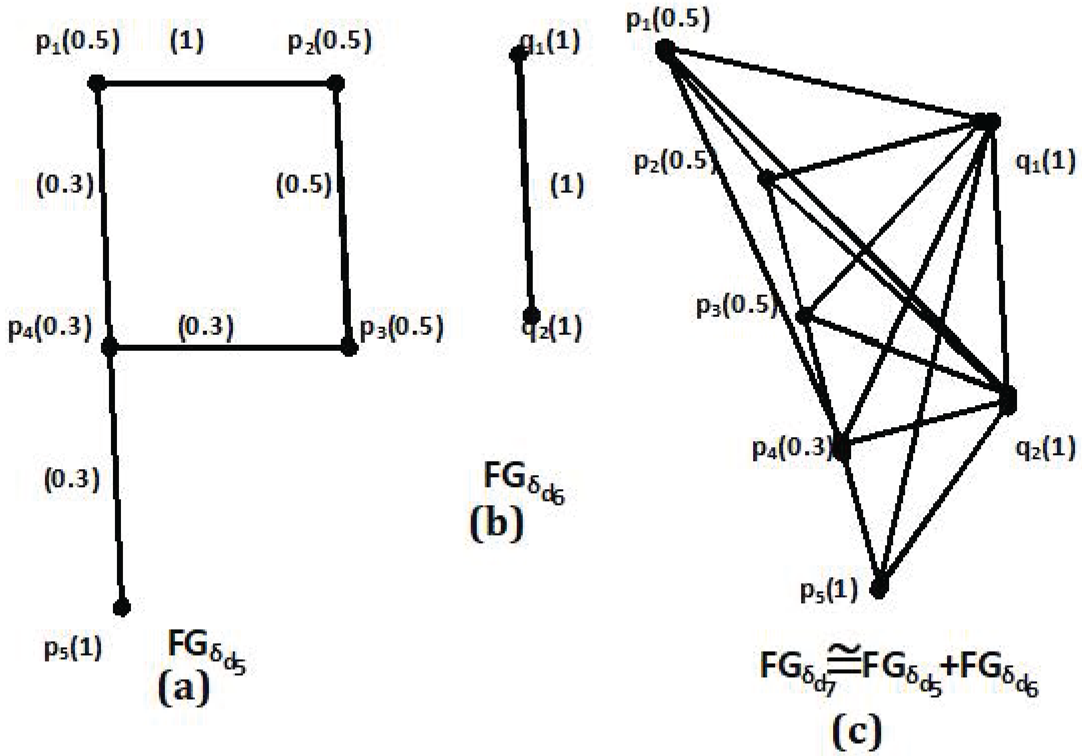

FIGURE 5.

$$\mu_{\delta_d}(p_ip_j) = (\mu_{\delta_{d_1}} \times \mu_{\delta_{d_2}})(p_ip_j)$$
$$= \begin{cases} \max\{\sigma_{\delta_{d_1}}(p_1), \mu_{\delta_{d_2}}(q_1q_2)\}; if p_1 = p_2 \;\; and \;\; (q_1q_2) \in L(G_2^*) \\ \max\{\mu_{\delta_{d_1}}(p_1p_2), \sigma_{\delta_{d_2}}(q_1)\}; if q_1 = q_2 \;\; and \;\; (p_1p_2) \in L(G_1^*) \end{cases}.$$

Illustration 4.4. Consider the two δ_d-fuzzy graphs $FG_{\delta_{d_8}}$ and $FG_{\delta_{d_9}}$ given in Figure 6(a) and 6(b). Then the cartesian product of these two δ_d-fuzzy graphs $FG_{\delta_{d_{10}}}$ is given in Figure 6(c).

Remark 4.4. The graph FG_{δ_d} given in Figure 6(c). is not a δ_d-fuzzy graph. Since according to the definition of the node membership function σ_{δ_d} of δ_d-fuzzy graph, $\sigma_{\delta_d}(p_1, q_1) = \frac{\delta_d(G^*)}{\deg(p_1,q_1)} = \frac{2}{3} = 0.6 \neq 0.5$. Thus the cartesian product of two δ_d-fuzzy graphs $FG_{\delta_{d_1}}$ and $FG_{\delta_{d_2}}$ need not be a δ_d- fuzzy graph, but it is a fuzzy graph known as pseudo δ_d-fuzzy graph.

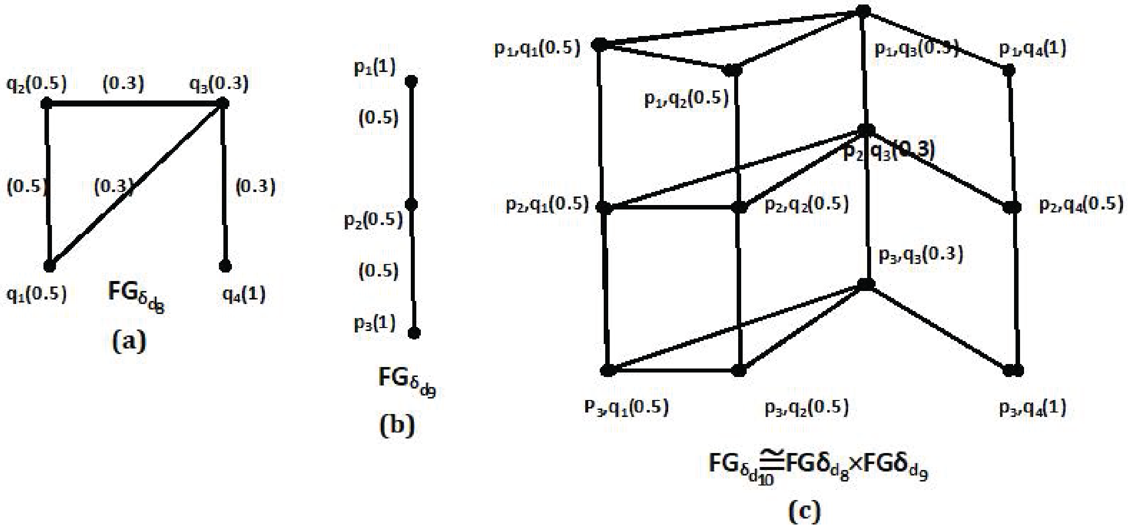

FIGURE 6.

Definition 4.5. *Let $FG_{\delta_{d_1}} : (\sigma_{\delta_{d_1}}, \mu_{\delta_{d_1}})$ and $FG_{\delta_{d_2}} : (\sigma_{\delta_{d_2}}, \mu_{\delta_{d_2}})$ be two δ_d-fuzzy graphs under the crisp graph $G_1^* : (P_1, L_1)$ and $G_2^* : (Q_2, L_2)$ respectively. Then the tensor product '$\otimes$' of two δ_d-fuzzy graphs are defined as,*

$$(\sigma_{\delta_{d_1}} \otimes \sigma_{\delta_{d_2}})(p_i q_j) = \max\{\sigma_{\delta_{d_1}}(p_i), \sigma_{\delta_{d_2}}(q_j)\}; (p_i, q_j) \in P(G_1^*) \times Q(G_2*)$$

and

$$(\mu_{\delta_{d_1}} \otimes \mu_{\delta_{d_2}})((p_1 q_1), (p_2 q_2)) = \max\{\mu_{\delta_{d_1}}(p_1 p_2), \mu_{\delta_{d_2}}(q_1 q_2)\},$$

for all $p_1 = p_2$, $q_1 q_2 \in L(G_2^*)$, $q_1 = q_2$ and $p_1 p_2 \in L(G_1^*)$.

Illustration 4.5. Consider the two δ_d-fuzzy graphs $FG_{\delta_{d_8}}$ and $FG_{\delta_{d_9}}$ given in Figure 6(a) and 6(b). Then the tensor product of these two δ_d-fuzzy graphs $FG_{\delta_{d_10}}$ is given in Figure 6(c).

Remark 4.5. The graph FG_{δ_d} given in Figure 7(c). is not a δ_d-fuzzy graph. Since according to the definition of the node membership function σ_{δ_d} of δ_d-fuzzy graph, $\sigma_{\delta_d}(p_2, q_1) = \frac{\delta_d(G^*)}{\deg(p_2, q_1)} = \frac{1}{2} = 0.5 \neq 1$. Thus the tensor product of two δ_d-fuzzy graphs $FG_{\delta_{d_{11}}}$ and $FG_{\delta_{d_{12}}}$ need not be a δ_d- fuzzy graph, but it is a fuzzy graph known as pseudo δ_d-fuzzy graph.

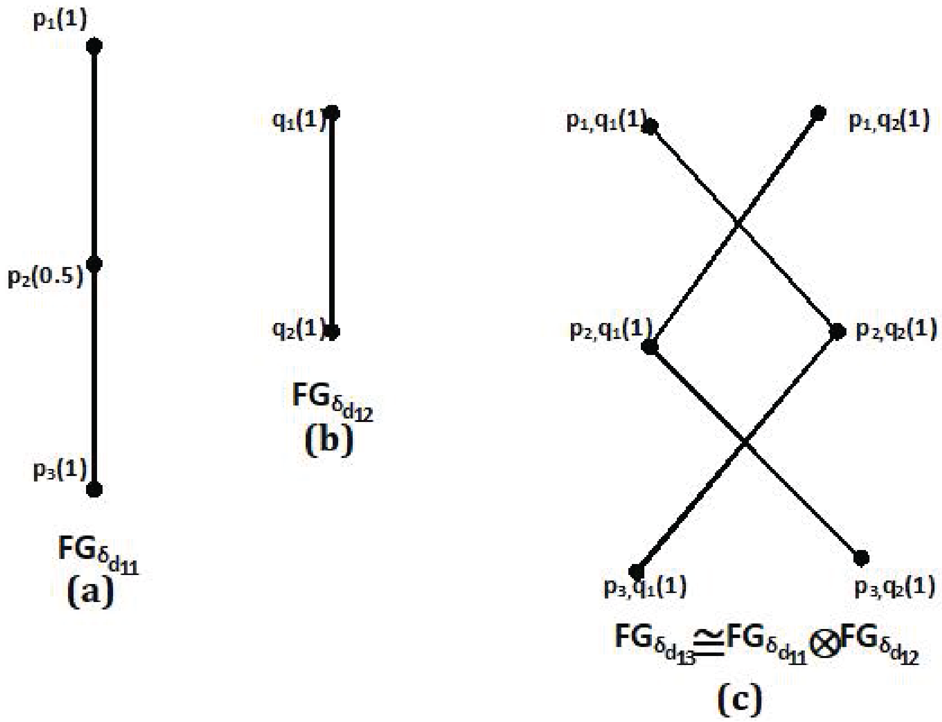

FIGURE 7.

Definition 4.6. *Let $FG_{\delta_d} : (\sigma_{\delta_d}, \mu_{\delta_d})$ be δ_d-fuzzy graph corresponding to a crisp graph $G^* : (P, L)$. The complement of δ_d-fuzzy graph $FG\delta_d^c$ is defined as, $\sigma_{\delta_d}^c(p_i) = \sigma_{\delta_d}(p_i)$, $\forall\ p_i \in P(G^*)$ and* $\mu_{\delta_d}^c(p_ip_j) = \begin{cases} \min\{\sigma_{\delta_d}(p_i), \sigma_{\delta_d}(p_j)\}; if p_ip_j \notin L(G^*) \\ 0; p_ip_j \in L(G^*) \end{cases}$

Illustration 4.6. *For the δ_d-fuzzy graph $FG_{\delta_{d_{14}}}$ given in Figure 8(a), its complement is given in Figure 8(b).*

Remark 4.6. The complement of $FG\delta_{d_{14}}$ fuzzy graph is not $\delta_{d_{14}}$-fuzzy graph but, it is a fuzzy graph called pseudo δ_d-fuzzy graph.

5. CONCLUSION

A new concept of δ_d-fuzzy graph is defined using the degree of the point set and minimum degree of a crisp graph. Also the Handshaking lemma is confined with illustrations is confined in this article. Theorems based on regular δ_d-fuzzy graph

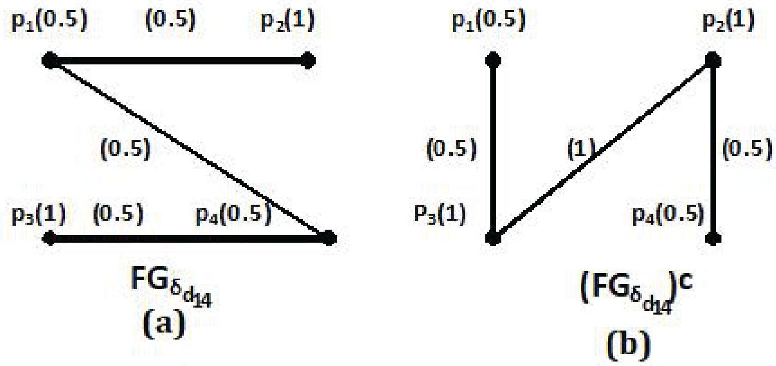

FIGURE 8.

and totally regular δ_d-fuzzy graph were explained with some examples. Also, some operations like union, intersection, join, cartesian product, complement, tensor product were defined and it is also shown that the δ_d-fuzzy graph obtained using these operations need not be a δ_d-fuzzy graph and it is known as pseudo δ_d-fuzzy graph. To find the domination number for δ_d-fuzzy graphs are considered to be the future work.

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