AN INTRO TO $\delta_d$-FUZZY GRAPHS

J. Jeromi Jovita$^1$, O. Uma Maheswari, and N. Meenal

ABSTRACT. Graph is a easy way to represent the real life situation. Graph is a combination of Points and Lines. In network analysis, the degree of a point plays a prominent role in Graph Theory. The degree of a point is the number of connections it has with the other points in the point set. Among the degrees of all the points in graph $G^*$, the minimum value is denoted by $\delta(G^*)$. In this article, a new abstraction of fuzzy graph is initiated by combining the parameters, degree of a point and minimum degree of the graph and termed it is as $\delta_d$-fuzzy graphs. Order and Size on $\delta_d$-fuzzy graphs were studied and Handshaking Lemma were explained with illustration. Idea on $\delta_d$-regular fuzzy graph were interpreted using the theorems. Also operations on graphs such as union, intersection, complement, cartesian product, Tensor Product, Corona are extended for $\delta_d$-fuzzy graphs.

1. INTRODUCTION

A graph is a set of points and lines. The lines between points will form a model of relations using graphical representation. [14] Fuzzy set theory was proposed by Prof. Zadeh which has been efficiently used to model many real world problems, which are uncertain. In a crisp graph, the members of the sets takes a single value of 0 or 1. For this reason, the uncertainties of real world problems

$^1$corresponding author
2020 Mathematics Subject Classification. 05C72, 05C07, 03E72, 05C12.
Key words and phrases. Minimum Degree of a Point, Fuzzy Graph, Cartesian Product, Tensor Product.
Submitted: 11.10.2023; Accepted: 26.10.2023; Published: 28.11.2023.
don’t work properly. But the fuzzy set provides its element to have the grade of membership within the range from 0 to 1. A concept of fuzzy graph was introduced by Rosenfeld and some operations are discussed by Mordeson and Nair, they also defined the complement of fuzzy graph. Also, Venugopalam discussed Operations on Fuzzy Graphs. Further these operations are also studied by Sunitha and Vijayakumar. Later Nagoorgani examined the properties of degree, regular, irregular, order and size of fuzzy graphs and also compared the relationship between these parameters. Let $G^* (P, L)$ be a simple graph with point set $P$ and line set $L$. In graph theory, relations can be modeled in to graph for better understanding. The number of points in a graph $G^*$ is known as order and is denoted by $\rho (G^*)$ and the number of lines in $G^*$ is known as size is denoted by $\tau (G^*)$. The sequence of distinct consecutive points followed by distinct consecutive lines is called Path. The closed path is known as Cycle. The degree of a point $p_i$ is defined as the number of lines that are incident to that point and is denoted by $\text{deg}(p_i)$. If the degree of every point is same, then it is a regular graph, otherwise it is irregular. If the degree of all points in a graph are three, then it is cubic. If every point is connected to all other distinct points, then it is called complete graph.

2. $\delta_d$-FUZZY GRAPH

**Definition 2.1.** Let $FG_{\delta_d} : (\sigma_{\delta_d}, \mu_{\delta_d})$ be a $\delta_d$-fuzzy graph with the membership function of point set $\sigma_{\delta_d}$ and membership function of line set $\mu_{\delta_d}$ under the crisp graph $G^* : (P, L)$ were defined by,

$$\sigma_{\delta_d}(p_i) = \frac{\delta(G^*)}{\text{deg}(p_i)}; \quad \forall \ p_i \in P(G^*) \quad \text{and}$$

$$\mu_{\delta_d}(p_ip_j) = \begin{cases} \min\{\sigma_{\delta_d}(p_i), \sigma_{\delta_d}(p_j)\}; & \text{if } p_ip_j \in L(G^*) \\ 0; & \text{Otherwise.} \end{cases}$$

**Definition 2.2.** The Order $O(FG_{\delta_d})$ and Size $S(FG_{\delta_d})$ of $\delta_d$-fuzzy graph under the crisp graph $G^* : (P, L)$ is defined as,

$$\rho_{\delta_d} = O(FG_{\delta_d}) = \sum_{p_i \in P(G^*)} \sigma_{\delta_d}(p_i) \quad \text{and} \quad \tau_{\delta_d} = S(FG_{\delta_d}) = \sum_{p_ip_j \in L(G^*)} \mu_{\delta_d}(p_ip_j).$$
Definition 2.3. The degree of a point in $\delta_d$-fuzzy graph is denoted by $\deg_{\delta_d}(p_i)$ and it is defined by,

$$\deg_{\delta_d}(p_i) = \sum_{p_j \in P(G^*)} \mu_{\delta_d}(p_ip_j); \ \forall \ p_i \in P(G^*).$$

Definition 2.4. The $FG_{\delta_d}$ fuzzy graph is said to be $\delta_d$-complete fuzzy graph, if $\deg_{\delta_d}(p_i) = \rho_{\delta_d} - 1; \ \forall \ p_i \in P(G^*)$.

Definition 2.5. The $FG_{\delta_d}$ fuzzy graph is said to be $\delta_d$-regular fuzzy graph, if $\deg_{\delta_d}(p_i) = k; \ \forall \ p_i \in P(G^*)$ where $k$ is a constant.

Definition 2.6. The total degree of a point on $\delta_d$-fuzzy graph is denoted by $tdeg_{\delta_d}$ and it is defined as,

$$tdeg_{\delta_d}(p_i) = \deg_{\delta_d}(p_i) + \sigma_{\delta_d}(p_i); \ \forall \ p_i \in P(G^*).$$

Illustration 2.1. Consider a $\delta_d$-fuzzy graph given in Figure 1 with Point set $P(G^*) = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8\}$.

![Figure 1](image_url)

The degree of all the points are given by,

$$\deg(p_2) = \deg(p_5) = \deg(p_6) = \deg(p_7) = \deg(p_8) = 3;$$
$$\deg(p_3) = 5; \ \deg(p_1) = 2 = \deg(p_4).$$

Thus $\delta(G^*) = 2$. The membership value of point sets are obtained by, $\sigma_{\delta_d}(p_i) = \frac{\delta(G^*)}{\deg(p_i)}; \ \forall \ p_i \in P(G^*).$ $\sigma_{\delta_d}(p_1) = 1 = \sigma_{\delta_d}(p_4);$ $\sigma_{\delta_d}(p_2) = \sigma_{\delta_d}(p_5) = \sigma_{\delta_d}(p_6) = \sigma_{\delta_d}(p_7) = \sigma_{\delta_d}(p_8) =$...
Theorem 2.2. Equality Theorem of Theorem 2.1

Let \( G \) be a regular graph. Conversely, if \( G \) is a regular graph, then the degree of every point in a graph \( G \) is equal to the degree of every point in a graph \( G \) only if minimum degree is equal to the degree of every point in a graph.

The order and size of \( FG_\delta \)-fuzzy graph are given by, \( \rho_{G_\delta}(FG_\delta) = \sum_{p_i \in P(G^*)} \sigma_{G_\delta} = 5.4 \) and \( \tau_{G_\delta}(FG_\delta) = \sum_{p_i, p_j \in E(G^*)} \mu_{G_\delta}(p_i, p_j) = 0.6(7) + 0.4(5) = 6.2 \) respectively. The degree of \( \delta_\delta \)-fuzzy graph are, \( \deg_{G_\delta}(p_1) = 1.2; \deg_{G_\delta}(p_2) = 1.6 = \deg_{G_\delta}(p_7) = \deg_{G_\delta}(p_8); \deg_{G_\delta}(p_3) = 2; \deg_{G_\delta}(p_4) = 1; \deg_{G_\delta}(p_5) = 1.8 = \deg_{G_\delta}(p_6) \).

Theorem 2.1. Let \( \rho_{G_\delta} \) and \( \tau_{G_\delta} \) be the order and size of \( \delta_\delta \)-fuzzy graph under the crisp graph \( G^* \) with order \( \rho \) and size \( \tau \) respectively, then \( \rho_{G_\delta} \leq \rho \) and \( \tau_{G_\delta} \leq \tau \).

Proof. By Definition 2.1, \( \rho_{G_\delta} = \sum_{p_i \in P(G^*)} \sigma_{G_\delta} = \sum_{p_i \in P(G^*)} \frac{\delta(G^*)}{\deg(p_i)} \). And by Definition 2.4, the complete point has the maximum degree of \( \rho_{G_\delta} - 1 \). Thus, \( \rho_{G_\delta} \leq \sum_{p_i \in P(G^*)} \frac{\rho - 1}{\rho - 1} \leq 1 + 1 + \ldots + 1 (\rho \times \text{times}) \). Therefore \( \rho_{G_\delta} \leq \rho \).

Similarly, \( \tau_{G_\delta} = \sum_{p_i, p_j \in L(G^*)} \mu_{G_\delta}(p_i, p_j) = \sum_{p_i, p_j \in L(G^*)} \min\{\sigma_{G_\delta}(p_i), \sigma_{G_\delta}(p_j)\} \). Since the maximum membership value of a point is 1. This implies \( \tau_{G_\delta} \leq 1 + 1 + \ldots + 1 (\tau \times \text{times}) \). Thus, \( \tau_{G_\delta} \leq \tau \).

Theorem 2.2. Equality Theorem of Theorem 2.1

Let \( FG_{\delta_\delta} : (\sigma_{G_\delta}, \mu_{G_\delta}) \) be \( \delta_\delta \)-fuzzy graph with order \( \rho_{G_\delta} \) and size \( \tau_{G_\delta} \) corresponding to a crisp graph of order \( \rho \) and size \( \tau \) respectively, then \( \rho_{G_\delta} = \rho \) and \( \tau_{G_\delta} = \tau \) if and only if \( G^* \) is a regular graph.

Proof. Suppose \( \rho_{G_\delta} = \rho \), then by Definition 2.2, \( \rho_{G_\delta} = \sum_{p_i \in P(G^*)} \frac{\delta(G^*)}{\deg(p_i)} = \rho \). This happens only if \( \deg(p_i) = \delta(G^*) = \min\{\deg(p_i)/p_i \in P(G^*)\} \), which implies that \( G^* \) is a regular graph. Conversely, if \( G^* \) is regular graph, then, the degree of every points are equal to some constant (say \( m \)). Thus, \( \deg(p_i) = \min\{\deg(p_i)/p_i \in P(G^*)\} = \delta(G^*) \). Then, \( \rho_{G_\delta} = \sum_{p_i \in P(G^*)} \frac{\delta(G^*)}{\deg(p_i)} = \sum_{p_i \in P(G^*)} \frac{m}{m} = \sum_{p_i \in P(G^*)} 1 = 1 + 1 + \ldots + 1 (\rho \times \text{times}) = \rho \). Also, let \( \tau_{G_\delta} = \tau \). By Definition 2.2 of \( \delta_\delta \)-fuzzy graph, \( \tau_{G_\delta} = \sum_{p_i, p_j \in L(G^*)} \tau_{G_\delta} = \tau \) only if minimum degree is equal to the degree of every point in a graph \( G^* \), which implies that \( G^* \) is a regular graph. Conversely if \( G^* \) is a regular graph, then \( \deg(p_i) = \min\{\deg(p_i)/p_i \in P(G^*)\} \).
This implies $\tau_{d} = \sum_{p_{i},p_{j} \in L(G^{*})} \mu_{d}(p_{i}p_{j}) = \sum_{p_{i},p_{j} \in L(G^{*})} \min\{\sigma_{d}(p_{i}),\sigma_{d}(p_{j})\} = 1 + 1 + \ldots + 1(\tau \text{ times}) = \tau$. Thus $\tau_{d} = \tau$. \hfill \Box

**Theorem 2.3. Bounds on Order and Size of $\delta_{d}$-fuzzy graph.** Let $FG_{d}$ be a $\delta_{d}$-fuzzy graph of order $\rho_{d}$ and size $\tau_{d}$, corresponding to non-isolated connected simple graph $G^{*}$ with order $\rho$ and size $\tau$, then $2 \leq \rho_{d} \leq \rho$ and $1 \leq \tau_{d} \leq \tau$.

**Proof.** Let us assume minimum degree sequence of non-isolated connected simple graph $G^{*}$ on $\delta_{d}$-fuzzy graph, which is, $K_{2}$ with degree sequence $\{1,1\}$. Thus the order, $\rho_{d} = \sum_{p_{i} \in P(G^{*})} \sigma_{d}(p_{i}) = \frac{1}{4} + \frac{1}{4} = 2$ and size, $\tau_{d} = \sum_{p_{i},p_{j} \in L(G^{*})} \tau_{d}(p_{i}p_{j}) = 1$. The upper bound is obvious from Theorem 2.2. This completes the proof. \hfill \Box

**Proposition 2.1.** Let $FG_{d}$ be $\delta_{d}$-fuzzy graph corresponding to a crisp graph $G^{*} : (P,L)$, then $\deg_{d}(p_{i}) \leq \deg(p_{i}) \ \forall \ p_{i} \in P(G^{*})$.

**Proof.** By Definition 2.3, $\deg_{d}(p_{i}) = \sum_{p_{i},p_{j} \in L(G^{*})} \mu_{d}(p_{i}p_{j}) = \sum_{p_{i},p_{j} \in L(G^{*})} \min\{\sigma_{d}(p_{i}),\sigma_{d}(p_{j})\} = \sum_{p_{i} \in L(G^{*})} 1 \leq \deg(p_{i})$. \hfill \Box

**Lemma 2.1.** Let $FG_{d}$ be $\delta_{d}$-fuzzy graph under the crisp graph $G^{*} : (P,L)$, then the sum of the degree of every point of $FG_{d}$ is equal to twice the size of $FG_{d}$.

**Illustration 2.2.** From the illustration 2.1, for the $FG_{d}$ graph given in Figure 1.

- Sum of degree of all points $= 1.2 + 1.6 + 2 + 1 + 1.6 + 1.8 + 1.6 + 1.6 = 12.4$.
- Twice the size of $FG_{d}$ fuzzy graph $= 2 \times 6.2 = 12.4$. Hence Lemma 2.1 holds.

**Lemma 2.2.** Let $FG_{d}$ be a $\delta_{d}$-fuzzy graph under the crisp graph $G^{*} : (P,L)$, then the sum of total degree of all points is equal to sum of twice the size of $FG_{d}$ and order of $FG_{d}$.

**Lemma 2.3.** According to Definition 2.6, $tdeg_{d}(p_{i}) = \deg_{d}(p_{i}) + \sigma_{d}(p_{i}); \ \forall \ p_{i} \in P(G^{*})$.

- Sum of all points in $FG_{d}$ fuzzy graph $= \sum_{p_{i},p_{j} \in L(G^{*})} \mu_{d}(p_{i}p_{j}) + \sum_{p_{i} \in P(G^{*})} \sigma_{d}(p_{i})$ and by Lemma 2.1
  - $\sum_{p_{i} \in P(G^{*})} tdeg_{d}(p_{i}) = 2\tau_{d}(FG_{d}) + \rho_{d}(FG_{d})$.

**Illustration 2.3.** For the $FG_{d}$ graph given in Figure 1, the total degree of all points are obtained by, $tdeg_{d}(p_{1}) = 2.2 = tdeg_{d}(p_{2}) = tdeg_{d}(p_{7}) = tdeg_{d}(p_{8})$;
\[ t\deg_{\delta_d}(p_3) = 2.4 = t\deg_{\delta_d}(p_6) = 2.4 = t\deg_{\delta_d}(p_7); \ t\deg_{\delta_d}(p_4) = 2. \] Sum of all total degrees of all points in \( FG_{\delta_d} \) is

\[ 2t\deg_{\delta_d}(FG_{\delta_d}) + \rho_{\delta_d}(FG_{\delta_d}) = 2(6.2) + 5.4 = 12.4 + 5.4 = 17.8. \] Thus the Lemma holds.

3. REGULAR \( \delta_d \)-FUZZY GRAPH

**Definition 3.1.** If degree of every point is equal to some constant \( m \) in \( \delta_d \)-fuzzy graph, under the crisp graph \( G^* \), then it is known to be \( m \)-regular \( \delta_d \)-fuzzy graph.

**Definition 3.2.** If total degree of every point is equal to some constant \( m_1 \) in \( \delta_d \)-fuzzy graph, under the crisp graph \( G^* \), then it is known to be \( m_1 \)-totally regular \( \delta_d \)-fuzzy graph.

**Theorem 3.1.** Let \( FG_{\delta_d} \) be a \( \delta_d \)-fuzzy graph under the crisp graph \( G^* \), if \( \sigma_{\delta_d} \) is constant, then \( FG_{\delta_d} \) is \( \delta_d \)-regular fuzzy graph and \( \delta_d \)-totally regular fuzzy graph.

**Proof.** Suppose \( FG_{\delta_d} \) is not totally fuzzy graph. Then, \( t\deg_{\delta_d}(p_i) \neq t\deg_{\delta_d}(p_j); \forall \ p_i, p_j \in P(G^*) \). This implies \( \deg_{\delta_d}(p_i) + \sigma_{\delta_d}(p_i) \neq \deg_{\delta_d}(p_j) + \sigma_{\delta_d}(p_j) \). Since \( \sigma_{\delta_d} \) is constant. \( \Rightarrow \) \( \deg_{\delta_d}(p_i) \neq \deg_{\delta_d}(p_j) \) which is contradiction. Thus we conclude that \( FG_{\delta_d} \) is \( \delta_d \)-regular fuzzy graph.

Conversely, let \( FG_{\delta_d} \) be a \( \delta_d \)-regular fuzzy graph. Then \( \deg_{\delta_d}(p_i) = \deg_{\delta_d}(p_j); \forall \ p_i, p_j \in P(G^*) \). This implies, \( \deg_{\delta_d}(p_i) + \sigma_{\delta_d}(p_i) = \deg_{\delta_d}(p_j) + \sigma_{\delta_d}(p_j) \). Since \( \sigma_{\delta_d} \) is constant. This becomes \( t\deg_{\delta_d}(p_i) = t\deg_{\delta_d}(p_j) \), which proves that \( FG_{\delta_d} \) is \( \delta_d \)-totally regular fuzzy graph. \( \square \)

**Illustration 3.1.** Consider a \( \delta_d \) fuzzy graph in Figure 2, with Point set \( P(G^*) = \{ p_1, p_2, p_3, p_4, p_5 \} \).

The degree of each points are \( \deg(p_i) = 3 \forall \ p_i \in P(G^*) \). Thus \( \delta(G^*) = 3 \). The membership value of point sets are obtained by, \( \sigma_{\delta_d}(p_i) = \frac{\delta(G^*)}{\deg(p_i)}; \forall \ p_i \in P(G^*) \). Also \( \sigma_{\delta_d}(p_i) = 1; \forall \ p_i \in P(G^*) \). The membership value of Line set are obtained as by Definition 2.1 are, \( \mu_{\delta_d}(p_ip_j) = 1; \forall \ p_ip_j \in L(G^*) \). The degree of \( \delta_d \)-fuzzy graph are, \( \deg_{\delta_d}(p_i) = 3; \forall \ p_i \in P(G^*) \). Since the degree of every points in \( \delta_d \)-fuzzy graph are equal. Thus \( FG_{\delta_d} \) is 3-regular \( \delta_d \)-fuzzy graph.
4. OPERATIONS ON $\delta_d$-FUZZY GRAPHS

**Definition 4.1.** Let $FG_{\delta_{d_1}} : (\sigma_{\delta_{d_1}}, \mu_{\delta_{d_1}})$ and $FG_{\delta_{d_2}} : (\sigma_{\delta_{d_2}}, \mu_{\delta_{d_2}})$ be two $\delta_d$-fuzzy graphs under the crisp graph $G^*_1 : (P_1, L_1)$ and $G^*_2 : (P_2, L_2)$ respectively. The union of two $\delta_d$-fuzzy graphs $FG_{\delta_{d_3}}$ are defined as,

$$
\sigma_{\delta_d}(p_i) = (\sigma_{\delta_{d_1}} \cup \sigma_{\delta_{d_2}})(p_i) = \begin{cases} 
\sigma_{\delta_{d_1}}(p_i); & p_i \in P(G^*_1) - P(G^*_2) \\
\sigma_{\delta_{d_2}}(p_i); & p_i \in P(G^*_2) - P(G^*_1) \\
\max\{\sigma_{\delta_{d_1}}(p_i), \sigma_{\delta_{d_2}}(p_i)\}; & p_i \in P(G^*_1) \cap P(G^*_2)
\end{cases}
$$

$$
\mu_{\delta_d}(p_ip_j) = (\mu_{\delta_{d_1}} \cup \mu_{\delta_{d_2}})(p_ip_j) = \begin{cases} 
\mu_{\delta_{d_1}}(p_ip_j); & p_ip_j \in L(G^*_1) - L(G^*_2) \\
\mu_{\delta_{d_2}}(p_ip_j); & p_ip_j \in L(G^*_2) - L(G^*_1) \\
\max\{\mu_{\delta_{d_1}}(p_ip_j), \mu_{\delta_{d_2}}(p_ip_j)\}; & p_ip_j \in L(G^*_1) \cap L(G^*_2)
\end{cases}
$$

**Illustration 4.1.** Consider the two $\delta_d$-fuzzy graphs $FG_{\delta_{d_1}}$ and $FG_{\delta_{d_2}}$ given in Figure 3(a) and 3(b). Then the union of these two $\delta_d$-fuzzy graphs is given in Figure 3(c).

**Remark 4.1.** The graph $FG_{\delta_d}$ given in Figure 3(c) is not a $\delta_d$-fuzzy graph. Since according to the definition of the node membership function $\sigma_{\delta_d}$ of $\delta_d$-Fuzzy Graph, $\sigma_{\delta_d}(p_2) = \frac{\delta_d(G^*_1)}{\deg(p_2)} = \frac{2}{4} = 0.5 \neq 1$. Thus the union of two $\delta_d$-fuzzy graphs $FG_{\delta_{d_1}}$ and
$FG_{\delta d_2}$ need not be a $\delta_d$-fuzzy graph, but it is a fuzzy graph and call it as pseudo $\delta_d$-fuzzy graph.

**Definition 4.2.** Let $FG_{\delta d_1} : (\sigma_{\delta d_1}, \mu_{\delta d_1})$ and $FG_{\delta d_2} : (\sigma_{\delta d_2}, \mu_{\delta d_2})$ be two $\delta_d$-fuzzy graph under the crisp graph $G_1^* : (P_1, L_1)$ and $G_2^* : (P_2, L_2)$ respectively. The intersection of two $\delta_d$-fuzzy graphs $FG_{\delta d_4}$ are defined as,

\[
\sigma_{\delta d}(p_i) = (\sigma_{\delta d_1} \cap \sigma_{\delta d_2})(p_i) = \begin{cases} 
\min\{\sigma_{\delta d_1}(p_i), \sigma_{\delta d_2}(p_i)\}; p_i \in P(G_1^*) \cap P(G_2^*) \\
0; \text{otherwise}
\end{cases}
\]

\[
\mu_{\delta d}(p_ip_j) = (\mu_{\delta d_1} \cap \mu_{\delta d_2})(p_ip_j) = \begin{cases} 
\min\{\mu_{\delta d_1}(p_ip_j), \mu_{\delta d_2}(p_ip_j)\}; p_ip_j \in L(G_1^*) \cap L(G_2^*) \\
0; \text{otherwise}
\end{cases}
\]

**Illustration 4.2.** For the two $\delta_d$-fuzzy graphs $FG_{\delta d_1}$ and $FG_{\delta d_2}$ given in Figure 3(a) and 3(b), the intersection of these two $\delta_d$-fuzzy graph $FG_{\delta d_4}$ is given in Figure 4.

**Remark 4.2.** The graph $FG_{\delta d_4}$ given in Figure 4, is not a $\delta_d$-fuzzy graph. Since according to the Definition of the node membership function $\sigma_{\delta d}$ of $\delta_d$-fuzzy graph, $\sigma_{\delta d}(p_2) = \frac{\delta_d(G_1^*)}{\deg(p_2)} = \frac{1}{1} = 1$. Thus the intersection of two $\delta_d$-fuzzy graphs $FG_{\delta d_1}$ and $FG_{\delta d_2}$ is not a $\delta_d$ fuzzy graph, but it is a fuzzy graph called pseudo $\delta_d$-fuzzy graph.
Definition 4.3. Let $FG_{\delta_{d_1}} : (\sigma_{\delta_{d_1}}, \mu_{\delta_{d_1}})$ and $FG_{\delta_{d_2}} : (\sigma_{\delta_{d_2}}, \mu_{\delta_{d_2}})$ be two $\delta_d$-fuzzy graphs under the crisp graph $G_1^* : (P_1, L_1)$ and $G_2^* : (P_2, L_2)$ respectively. Assume $P(G_1^*) \cap P(G_2^*) = \phi$. The join of two $\delta_d$-fuzzy graphs with points and lines are given by $P(G^*) = P(G_1^*) \cup P(G_2^*)$ and $L(G^*) = L(G_1^*) \cup L(G_2^*) \cup L(G')$, where $L(G')$ is the set of all lines joining points of points $P(G_1^*)$ with points of $P(G_2^*)$ is defined as,

$$
\begin{align*}
\sigma_{\delta_d}(p_i) &= (\sigma_{\delta_{d_1}} + \sigma_{\delta_{d_2}})(p_i) = \sigma_{\delta_{d_1}}(p_i) \cup \sigma_{\delta_{d_2}}(p_i), \quad \forall \ p_i \in P(G_1^*) \text{ and } p_j \in P(G_2^*) \\
\mu_{\delta_d}(p_ip_j) &= (\mu_{\delta_{d_1}} + \mu_{\delta_{d_2}})(p_ip_j) \\
&= \begin{cases} 
(\mu_{\delta_{d_1}}(p_ip_j) \cup \mu_{\delta_{d_2}}(p_ip_j)); \quad \forall \ p_i p_j \in L(G_1^*) \cup L(G_2^*) \\
\max\{\sigma_{\delta_d}(p_i), \sigma_{\delta_d}(p_j)\}; \quad \text{if} \ (p_i, p_j) \in L(G') 
\end{cases}
\end{align*}
$$

Illustration 4.3. Consider the two $\delta_d$-fuzzy graphs, given in Figure 5(a) and 5(b). Then the join of two $\delta_d$-fuzzy graphs $FG_{\delta_{d_2}}$ is given in Figure 5(c).

Remark 4.3. The graph $FG_{\delta_{d_2}}$ given in Figure 5(c) is not a $\delta_d$-fuzzy graph. Since according to the definition of the node membership function $\sigma_{\delta_d}$ of $\delta_d$-fuzzy graph, $\sigma_{\delta_d}(q_1) = \frac{\delta_d(G^*)}{\deg(q_1)} = \frac{3}{6} = 0.5 \neq 1$. Thus the join of two $\delta_d$-fuzzy graphs $FG_{\delta_{d_2}}$ and $FG_{\delta_{d_4}}$ need not be a $\delta_d$-fuzzy graph, but it is a fuzzy graph known as pseudo $\delta_d$-fuzzy graph.

Definition 4.4. Let $FG_{\delta_{d_1}} : (\sigma_{\delta_{d_1}}, \mu_{\delta_{d_1}})$ and $FG_{\delta_{d_2}} : (\sigma_{\delta_{d_2}}, \mu_{\delta_{d_2}})$ be two $\delta_d$-fuzzy graph under the crisp graph $G_1^* : (P_1, L_1)$ and $G_2^* : (P_2, L_2)$ respectively. Then the cartesian product ‘$\times$’ of two $\delta_d$-fuzzy graphs with points and lines are given by $P(G^*) = P(G_1^*) \times P(G_2^*)$ and $L(G^*) = \{(p_1, q_1)(p_2, q_2)/p_1 = p_2, \text{and} \ (q_1, q_2) \in L(G_2^*) \text{ or } q_1 = q_2, \text{ and} \ (p_1, p_2) \in L(G_1^*)\}$ is defined as,

$$
\sigma_{\delta_d}(p_jq_j) = (\sigma_{\delta_{d_1}}(p_i), \sigma_{\delta_{d_2}}(q_j)) = \min\{\sigma_{\delta_{d_1}}(p_i), \sigma_{\delta_{d_2}}(q_j)\}
$$
\[ \mu_{\delta_d}(p_ip_j) = (\mu_{\delta_{d_1}} \times \mu_{\delta_{d_2}})(p_ip_j) \]

\[ = \begin{cases} 
\max\{\sigma_{\delta_{d_1}}(p_1), \mu_{\delta_{d_2}}(q_1q_2)\}; & \text{if } p_1 = p_2 \text{ and } (q_1q_2) \in L(G^*_2) \\
\max\{\mu_{\delta_{d_1}}(p_1p_2), \sigma_{\delta_{d_2}}(q_1)\}; & \text{if } q_1 = q_2 \text{ and } (p_1p_2) \in L(G^*_1) 
\end{cases} \]

**Illustration 4.4.** Consider the two \(\delta_d\)-fuzzy graphs \(FG_{\delta_{d_8}}\) and \(FG_{\delta_{d_9}}\) given in Figure 6(a) and 6(b). Then the cartesian product of these two \(\delta_d\)-fuzzy graphs \(FG_{\delta_{d10}}\) is given in Figure 6(c).

**Remark 4.4.** The graph \(FG_{\delta_d}\) given in Figure 6(c) is not a \(\delta_d\)-fuzzy graph. Since according to the definition of the node membership function \(\sigma_{\delta_d}\) of \(\delta_d\)-fuzzy graph,

\[ \sigma_{\delta_d}(p_1, q_1) = \frac{\delta_d(G^*_1)}{\deg(p_1, q_1)} = \frac{2}{3} = 0.6 \neq 0.5. \]

Thus the cartesian product of two \(\delta_d\)-fuzzy graphs \(FG_{\delta_{d_1}}\) and \(FG_{\delta_{d_2}}\) need not be a \(\delta_d\)-fuzzy graph, but it is a fuzzy graph known as pseudo \(\delta_d\)-fuzzy graph.
Definition 4.5. Let $FG_{\delta_d_1} : (\sigma_{\delta_d_1}, \mu_{\delta_d_1})$ and $FG_{\delta_d_2} : (\sigma_{\delta_d_2}, \mu_{\delta_d_2})$ be two $\delta_d$-fuzzy graphs under the crisp graph $G_1^* : (P_1, L_1)$ and $G_2^* : (Q_2, L_2)$ respectively. Then the tensor product $\otimes'$ of two $\delta_d$-fuzzy graphs are defined as,

$$(\sigma_{\delta_d_1} \otimes \sigma_{\delta_d_2})(p_i q_j) = \max\{\sigma_{\delta_d_1}(p_i), \sigma_{\delta_d_2}(q_j)\}; (p_i, q_j) \in P(G_1^*) \times Q(G_2^*)$$

and

$$(\mu_{\delta_d_1} \otimes \mu_{\delta_d_2})(p_1 q_1, p_2 q_2) = \max\{\mu_{\delta_d_1}(p_1 p_2), \mu_{\delta_d_2}(q_1 q_2)\},$$

for all $p_1 = p_2, q_1 q_2 \in L(G_2^*), q_1 = q_2$ and $p_1 p_2 \in L(G_1^*)$.

Illustration 4.5. Consider the two $\delta_d$-fuzzy graphs $FG_{\delta_d_8}$ and $FG_{\delta_d_9}$ given in Figure 6(a) and 6(b). Then the tensor product of these two $\delta_d$-fuzzy graphs $FG_{\delta_d_10}$ is given in Figure 6(c).

Remark 4.5. The graph $FG_{\delta_d}$ given in Figure 7(c). is not a $\delta_d$-fuzzy graph. Since according to the definition of the node membership function $\sigma_{\delta_d}$ of $\delta_d$-fuzzy graph, $\sigma_{\delta_d}(p_2, q_1) = \frac{\delta_d(G^*)}{deg(p_2 q_1)} = \frac{1}{2} = 0.5 \neq 1$. Thus the tensor product of two $\delta_d$-fuzzy graphs $FG_{\delta_d_{11}}$ and $FG_{\delta_d_{12}}$ need not be a $\delta_d$-fuzzy graph, but it is a fuzzy graph known as pseudo $\delta_d$-fuzzy graph.
Definition 4.6. Let $FG_{\delta_d} : (\sigma_{\delta_d}, \mu_{\delta_d})$ be $\delta_d$-fuzzy graph corresponding to a crisp graph $G^* : (P, L)$. The complement of $\delta_d$-fuzzy graph $FG_{\delta_d}$ is defined as, $\sigma_{\delta_d}^c(p_i) = \sigma_{\delta_d}(p_i)$, $\forall p_i \in P(G^*)$ and $\mu_{\delta_d}^c(p_i p_j) = \begin{cases} \min\{\sigma_{\delta_d}(p_i), \sigma_{\delta_d}(p_j)\}; & \text{if } p_i p_j \notin L(G^*) \\ 0; & p_i p_j \in L(G^*) \end{cases}$

Illustration 4.6. For the $\delta_d$-fuzzy graph $FG_{\delta_{d1}}$ given in Figure 8(a), its complement is given in Figure 8(b).

Remark 4.6. The complement of $FG_{\delta_{d14}}$ fuzzy graph is not $\delta_{d14}$-fuzzy graph but, it is a fuzzy graph called pseudo $\delta_d$-fuzzy graph.

5. Conclusion

A new concept of $\delta_d$-fuzzy graph is defined using the degree of the point set and minimum degree of a crisp graph. Also the Handshaking lemma is confined with illustrations is confined in this article. Theorems based on regular $\delta_d$-fuzzy graph
and totally regular $\delta_d$-fuzzy graph were explained with some examples. Also, some operations like union, intersection, join, cartesian product, complement, tensor product were defined and it is also shown that the $\delta_d$-fuzzy graph obtained using these operations need not be a $\delta_d$-fuzzy graph and it is known as pseudo $\delta_d$-fuzzy graph. To find the domination number for $\delta_d$-fuzzy graphs are considered to be the future work.

REFERENCES


