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STRUCTURE OF A RING IN WHICH EVERY ELEMENT IS SUM OF 3, 4 OR 5 COMMUTING TRIPOTENTS

Kumar Napoleon Deka[1] and Helen K. Saikia

ABSTRACT. In this paper we show if R be a ring in which every element is sum of three commuting tripotents then for every $k \in R$ we have $(k-3)(k-2)^2(k-1)^2k^2(k+1)^2(k+2)^2(k+3)=0$, if every element of R is sum of four commuting tripotents then for every $k \in R$ we have $(k-4)(k-3)(k-2)^2(k-1)^2k^4(k+1)^2(k+2)^2(k+3)(k+4)=0$, if every element of R is sum of five commuting tripotents then for every $k \in R$ we have $(k-5)(k-4)(k-3)^2(k-2)^3(k-1)^3k^4(k+1)^3(k+2)^3(k+3)^2(k+4)(k+5)=0$. Then we discuss the properties of these type of ring. Finally we find the general structure of a ring in which every element is sum of n commuting tripotents and discuss the properties of it.

1. INTRODUCTION

There are many works done by various authors on the rings which are the sum of idempotents,tripotents, and nilpotents.In the paper [2]the authors discuss the ring in which every element is a sum of two commuting tripotents and their related properties and show that the elements satisfy the identity $x^8 = x^4$. In the paper [2], the authors discuss the rings in which every element is a sum of an idempotent and tripotent that commute. Again in the paper [3] the author discusses the ring

[1]*corresponding author*
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in which every element is the sum of n tripotents and a nilpotent which commute each other, also he showed that in this type of ring $(2n+1)!$ is nilpotent for $1 \leq n \leq 7$ and R is such type of ring then $R \cong (R_1) \times (R_2) \times \ldots \times (R_m)$ where $2, 3, \ldots, p_m$ are primes not greater than $(2n+1), 1 \leq n \leq 7$, each R_i has the property that p_i is nilpotent and $a^{p^i} - a$ is nilpotent for every $a \in R_i$ [3]. Also in each R_i, $J(R_i)$ is nil, and $R_i/J(R_i)$ is subdirect product of fields isomorphic to F_{p_i} [3]. Again in the paper [4] the author shows some properties related to the ring in which every element is a sum of a nilpotent and three tripotents that commute with one another. In the paper [5] the authors discuss the ring in which every element is the sum of a nilpotent and two tripotents(that commute one another) and denoted it by (strong)SNTT-ring. The author [5] shows that If R is strong SNTT- ring then $R \cong A \bigoplus B \bigoplus C$, where A is zero or $A/J(A)$ is Boolean with $J(A)$ is nil, B is zero or $B/J(B)$ is subdirect product of $\mathbb{Z}_3$;s with $J(B)$ nil, C is zero or $C/J(C)$ is a subdirect product of $\mathbb{Z}_5$'s with $J(C)$ nil, $J(R)$ is nil and $R/J(R)$ has the identity $x^5 = x$, $a^5 - a$ is nilpotent for all $a \in R$ [5].

It is very useful for knowing the properties of a ring if we know the structure (especially the equation followed by each element of the ring)of the ring. So in this paper, we are mainly concerned about the structure of the ring. Firstly we find the structure for the ring in which the element is the sum of 3,4,5 commuting tripotents. Then we find the general structure for the ring in which every element is the sum of n commuting tripotents. And finally, we discuss the properties of these types of rings.

2. Preliminaries

All rings consider here are ring with unity. The Jacobson radical is denoted by $J(R)$ for a ring R. Also, all the units of a ring R are denoted by $U(R)$. Again the Chinese Remainder Theorem states "Let R be ring and $I.J$ be ideals in R such that $I + J = R$ then there exists a ring isomorphism $R/(I \cap J) = R/I \times R/J$ ". Also, the Bezout identity states that if a and b be integers with g.c.d d then there exist integers x and y such that $ax + by = d$.

If a ring R is a sum of n commuting tripotents we denote it by ST^n. So the ring in which every element is a sum of 3,4,5 commuting tripotents are ST^3, ST^4, ST^5 respectively.

3. Rings in which every element is difference of a tripotents and an idempotent which commute each other

Proposition 3.1.] *If in a ring R every element is every element sum of an idempotent and tripotent that commute with each other then they satisfy the equation*

$$(x^3 - x)(x - 2) = 0.$$

Proof. Suppose $x \in R$. Then $\exists e, f \in R$ with $e^2 = e, f^3 = f$ and $ef = fe$, such that $x = e + f$. now $(x^3 - x)(x - 2) = \{(e+f)^3 - (e+f)\}(e+f-2) = (3ef + 3ef^2)(e+f) - 6ef^2 - 6ef = 3ef + 3ef^2 + 3ef^2 + 3ef - 6ef^2 - 6ef = 0.$ □

In paper [2] the authors called this type of ring as strong SIT (Sum of idempotent and tripotent) that commute with each other)ring and shows the following properties.

Theorem 3.1. [2]. The following are equivalent for a ring R.

(1) R is a strong SIT-ring.
(2) R has the identity $x^6 = x^4$.
(3) R is one of the following types.
  (a) $R/J(R)$ is Boolean and $U(R)$ is a group of exponent 2.
  (b) R is subdirect product of $\mathbb{Z}_3$.
  (c) $R \cong A \times B$, where $A/J(A)$ is Boolean with $U(A)$ a group of exponent 2, and B is a subdirect product of $\mathbb{Z}_3$'s.

Using the condition $x^6 = x^4$ of the above Theorem 3.1 [2] and using the Proposition 2.1 we can find that every element of a strong SIT ring satisfies some equations which are $2x^5 = 2x^3, 4x^4 = 4x^2$ and finally $8x^3 = 8x$. As SIT ring defined in the paper [2] we can defined **DTI** ring in which every element is difference of a tripotent and an idempotent. It is called strong DTI ring if every element can be expressed as difference of a tripotent and an idempotent that commute each other.

Proposition 3.2. *If in a ring R every element is difference of a tripotent and an idempotent that commute with each other then they satisfy the equation*

$$(x^3 - x)(x + 2) = 0.$$

Proof. Suppose $x \in R$. Then $\exists e, f \in R$ with $e^2 = e, f^3 = f$ and $ef = fe$ such that $x = f - e$. Now $(x^3 - x)(x + 2) = \{(f - e)^3 - (f - e)\}(e + f - 2) = (-3ef^2 + 3ef)(f - e + 2) = -3ef + 3ef^2 - 6ef^2 + 3ef^2 - 3ef + 6ef = 0$. □

Lemma 3.1. *If R be ring with $2 = f - e$, where $e^2 = e, f^3 = f$, then $24 = 0$ in R.*

Proof. From $2 = f - e$ we have $ef = fe$. Then from Proposition 2.2 we have $(2^3 - 2)(2 + 2) = 0 \Rightarrow 24 = 0$ (putting $k = 2$ in the equation). □

Lemma 3.2. *Ring R be ring with unity is (strong) DTI-ring, if, and only if, $R \cong R_1 \times R_2$, where R_1, R_2 are (strong)DTI-rings with $2^3 = 0$ in R_1 and $3 = 0$ in R_2.*

Proof. Using Lemma 2.1 we have 24=0$\Rightarrow$ $2^3 \times 3 = 0$. Then by Chinese Remainder Theorem we get the required result. □

Proposition 3.3. *A ring R with unity is (strong)DIT ring if, and only if, it is a (strong)SIT ring.*

Proof. Let R be a (strong) DTI-ring ring. Let $k \in R$,as R is a ring so $-k \in R$ so $\exists e, f \in R$ with $e^2 = e, f^3 = f$ such that $-k = f - e \Rightarrow k = -f + e$ which is sum of a tripotent and an idempotent as $(-f)^3 = -f, e^2 = e$. So R is (strong) SIT ring.

Conversely suppose R is a (strong) SIT ring. Let $k \in R$. Now $-k \in R$ so $\exists e, f \in R$ with $e^2 = e$, $f^3 = f$ such that $-k = f + e \Rightarrow k = (-f) - e$ which is a difference of a tripotent and an idempotent. Therefore R is (strong)DTI-ring. □

Using the Proposition 3.1 and Proposition 3.2 we get a strong SIT/DTI rings satisfy the equation $x^5 = x^3$.

Proposition 3.4. *In a strong DTI/SIT ring R, we have $n^3 = 0, 2n^2 = 0 \forall n \in Nil(R)$.*

Proof. We get a strong DTI/SIT ring satisfy $x^5 = x^3 \forall x \in R$. Therefore $n^5 - n^3 = 0 \Rightarrow n^3(n^2 - 1) = 0 \Rightarrow n^3 = 0 \forall n \in Nil(R)$ as $n^2 - 1 \in U(R)$. Again we have $(n \pm 2)(n^3 - n) = 0 \Rightarrow (n \pm 2)n(n^2 - 1) = 0 \Rightarrow (n \pm 2)n = 0 \Rightarrow (n \pm 2)n^2 = 0 \Rightarrow 2n^2 = 0 \forall n \in Nil(R)$ as $(n^2 - 1) \in Nil(R)$. □

4. Ring in which every element is sum of 3,4,5 commuting tripotents

Proposition 4.1. *If the ring R is a ST^2 ring then for every $k \in R$ we have $(k-2)(k-1)k(k+1)(k+2) = 0$.*

Proof. For $k \in R, \exists e, f \in R$ with $e^3 = e, f^3 = f, ef = fe$ such that $k = e+f$. Clearly $ke = ek, kf = fk$.

Now, $k^3 = e+f+3ef(e+f) = k+3kef$ which implies $k^3 - k = 3kef$. Therefore $k^3 - k = (k^2-1)(e+f) = 3ef(e+f) = 3(e^2f + ef^2)$. Multiply it by e^2f^2 we have $(k^2-1)(e^3f^2 + e^2f^3) = 3(e^4f^3 + e^3f^4) \Rightarrow (k^2-1)(ef^2 + e^2f) = 3(e^2f + ef^2) \Rightarrow (k^2-4)(ef^2 + e^2f) = 0 \Rightarrow (k^2-4)\{3(ef^2 + e^2f)\} = 0 \Rightarrow (k^2-4)(k^3-k) = 0 \Rightarrow (k-2)(k-1)k(k+1)(k+2) = 0$. □

In the paper [2] the authors show that in the ring R in which every element is sum of two commuting tripotents, satisfy the equation $x^8 = x^4$ and the following

Theorem 5.2. [2]. *The following are equivalent for a ring R.*

(1) *Every element of R is sum of two commuting tripotents.*
(2) *$R \cong R_1 \times R_2 \times R_3$, where R_1 is zero or $R_1/J(R_1)$ is Boolean with $U(R_1)$ is a group of exponent 2, R_2 is zero or subdirect product of Z_3's, and R_3 is zero or a subdirect product of Z_5's.*

Clearly putting $k = 3$ in the equation $(k-2)(k-1)k(k+1)(k+2) = 0$ of the Proposition 2.3 we get $120 = 2^3 \times 3 \times 5 = 0$ and we get the same result as Theorem 5.2 [2].

Proposition 4.2. *If the ring R is a ST^3 ring then for every $k \in R$ we have*

$$(k-3)(k-2)^2(k-1)^2k^2(k+1)^2(k+2)^2(k+3) = 0.$$

Proof. For $k \in R, \exists f, g, h \in R$ with $f^3 = f, g^3 = g, h^3 = h; fg = gf, gh = hg, hf = fh$ such that $k = f + g + h$. Clearly $kf = fk, kg = gk, kh = hk$ Now,

$$k^3 - k = 3\{fg(f+g) + gh(g+h) + hf(h+f)\} + 6fgh. \tag{4.1}$$

Therefore, $(k^3-k)fgh = 3\{f^2g^2h(f+g) + f^2gh^2(g+h) + f^2gh^2(h+f)\} + 6f^2g^2h^2 = 3(fg^2h + f^2gh + fg^2h + f^2gh + fgh^2) + 6f^2g^2h^2 = 3fgh.2(f+g+h) + 6f^2g^2h^2 = 6kfgh + 6f^2g^2h^2$, which implies $(k^3-k)fgh = 6kfgh + 6f^2g^2h^2 \Rightarrow (k^3-7k)fgh =$

$gf^2g^2h^2 \Rightarrow (k^3-7k)f^2g^2h^2 = 6fgh \Rightarrow (k^3-7k)^2fgh = 6\{(k^3-7k)f^2g^2h^2\} = 6^2fgh$

$$\Rightarrow \{(k^3-7k)^2-6^2\}fgh = 0 \tag{4.2}$$

Multiplying (4.1) by $\{(k^3-7k)^2-6^2\}$ and using (4.2) we have

$$(k^3-k)\{(k^3-7k)^2-6^2\} = 3\{(k^3-7k)^2-6^2\} \cdot\{fg(f+g)+gh(g+h)+hf(h+f)\}. \tag{4.3}$$

Now,

$$\begin{aligned}&(k^3-k)\{(k^3-7k)^2-6^2\}fg\\ &= 3\{(k^3-7k)^2-6^2\}\{f^3g^2+f^2g^3\} = 3\{(k^3-7k)^2-6^2\}\{fg(f+g)\}.\end{aligned}$$

Similarly, we have

$$(k^3-k)\{(k^3-7k)^2-6^2\}gh = 3\{(k^3-7k)^2-6^2\}\{gh(g+h)\}$$
$$(k^3-k)\{(k^3-7k)^2-6^2\}hf = 3\{(k^3-7k)^2-6^2\}\{hf(h+f)\}$$

Adding and using (4.3) we have

$$\begin{aligned}&(k^3-k)\{(k^3-7k)^2-6^2\}(fg+gh+hf)\\ &= 3\{(k^3-7k)^2-6^2\}\{fg(f+g)+gh(g+h)+hf(h+f)\}\end{aligned}$$

$$\Rightarrow (k^3-k)\{(k^3-7k)^2-6^2\}(fg+gh+hf) = (k^3-k)\{(k^3-7k)^2-6^2\}. \tag{4.4}$$

Multiply (4.4) by f, g, h respectively and using (4.2) we have

$$(k^3-k)\{(k^3-7k)^2-6^2\}(f^2g+hf^2) = (k^3-k)\{(k^3-7k)^2-6^2\}f$$
$$(k^3-k)\{(k^3-7k)^2-6^2\}(fg^2+g^2h) = (k^3-k)\{(k^3-7k)^2-6^2\}g$$
$$(k^3-k)\{(k^3-7k)^2-6^2\}(gh^2+h^2f) = (k^3-k)\{(k^3-7k)^2-6^2\}h.$$

Adding all these and using (4.3) we get

$$\begin{aligned}&(k^3-k)\{(k^3-7k)^2-6^2\}\{fg(f+g)+gh(g+h)+hf(h+f)\}\\ &= (k^3-k)\{(k^3-7k)^2-6^2\}(f+g+h)\\ \Rightarrow\quad &(k^3-k)[3\{(k^3-7k)^2-6^2\}\{fg(f+g)+gh(g+h)+hf(h+f)\}]\\ &= 3(k^3-k)\{(k^3-7k)^2-6^2\}k\end{aligned}$$

$$\Rightarrow \quad (k^3-k)[(k^3-k)\{(k^3-7k)^2-6^2\}] = 3k(k^3-k)\{(k^3-7k)^2-6^2\}$$
$$\Rightarrow \quad k(k^3-k)\{(k^3-7k)^2-6^2\}(k^2-1-3) = 0$$
$$\Rightarrow \quad k(k^2-4)(k^3-k)\{(k^3-7k)^2-6^2\} = 0$$
$$\Rightarrow \quad (k-3)(k-2)^2(k-1)^2k^2(k+1)^2(k+2)^2(k+3) = 0$$

Putting $k = 4$ in the above equation we have $2^8 \times 3^4 \times 5^2 \times 7 = 0$. □

Proposition 4.3. *If the ring R is a ST^4 then for every $k \in R$ we have*

$$(k-4)(k-3)(k-2)^2(k-1)^2k^4(k+1)^2(k+2)^2(k+3)(k+4) = 0.$$

Proof. For $k \in R, \exists e, f, g, h \in R$ with $e^3 = e, f^3 = f, g^3 = g, h^3 = h; e_ie_j = e_je_i; e_i, e_j \in \{e, f, g, h\}$ such that $k = e+f+g+h$.Clearly $ke = ek, kf = fk, kg = gk, kh = hk$. Now,

$$\begin{aligned} k^3 &= e^3+f^3+g^3+h^3+3\{e^2(f+g+h)+f^2(g+h+e)+g^2(h+e+f) \\ &\quad + h^2(e+f+g)\} + 6(efg+fgh+ghe+efg) \end{aligned} \tag{4.5}$$
$$\Rightarrow \quad k^3-k = 3\sum_{cyc} e^2(f+g+h) + 6\sum_{cyc} efg.$$

Multiplying (4.5) by $efgh$ we have

$$(k^3-k)efgh = 3\sum_{cyc} e^3fgh(f+g+h) + 6\sum_{cyc} ef^2g^2h^2$$
$$\Rightarrow \quad (k^3-k)efgh = 3efgh(3e+3f+3g+3h) + 6\sum_{cyc} ef^2g^2h^2$$
$$= 9kefgh + 6\sum_{cyc} ef^2g^2h^2$$
$$\Rightarrow \quad (k^3-10k)efgh = 6\sum_{cyc} ef^2g^2h^2$$
$$\Rightarrow \quad (k^3-k)(efgh)^2 = 6\sum_{cyc} e^2f^3g^3h^3 = 6\sum_{cyc} e^2fgh = (6k)efgh$$
$$\Rightarrow \quad (k^3-10k)(efgh)^3 = (6k)^2(efgh)^2$$

(4.6) $$\Rightarrow \quad (k^3-10k)^2 efgh = 6k\{(k^3-10k)(efgh)^2\} = 6k\{(6k)efgh\}$$

(4.7) $$\Rightarrow \quad \{(k^3-10k)^2-(6k)^2\}efgh = 0.$$

Now multiplying (4.5) by $(k^3-10k)^2-(6k)^2$ we have

$$(k^3-k)\{(k^3-10k)^2-(6k)^2\}$$
$$= 3\{(k^3-10k)^2-(6k)^2\}\sum_{cyc} e^2(f+g+h) + 6\{(k^3-10k)^2-(6k)^2\}\sum_{cyc} efg.$$

Now multiplying the above equation by fgh and using (4.6) we have

$$(k^3-k)\{(k^3-10k)^2-(6k)^2\}fgh$$
$$= 3\{(k^3-10k)^2-(6k)^2\}fgh(3e+2f+2g+2h)$$
$$+ 6\{(k^3-10k)^2-(6k)^2\}(fgh)^2$$

$$\Rightarrow \quad (k^3-k)\{(k^3-10k)^2-(6k)^2\}fgh$$
$$= 6k\{(k^3-10k)^2-(6k)^2\}fgh + 6\{(k^3-10k)^2-(6k)^2\}(fgh)^2$$
$$\Rightarrow \quad (k^3-7k)\{(k^3-10k)^2-(6k)^2\}fgh = 6\{(k^3-10k)^2-(6k)^2\}(fgh)^2$$
$$\Rightarrow \quad (k^3-7k)[6\{(k^3-10k)^2-(6k)^2\}(fgh)^2] = 6^2\{(k^3-10k)^2-(6k)^2\}fgh$$
$$\Rightarrow \quad (k^3-7k)[(k^3-7k)\{(k^3-10k)^2-(6k)^2\}fgh]$$

(4.8) $$= 6^2\{(k^3-10k)^2-(6k)^2\}fgh$$

$$\Rightarrow \quad \{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}fgh = 0.$$

Similarly for others (i.e if we take any three distinct permutation$\{e_i, e_j, e_k\}$ of $\{e, f, g, h\}$). Now multiplying by $\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}$ and using (4.6) and (4.7), we have

$$(k^3-k)\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}$$

(4.9) $$= 3\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\{\sum_{cyc} e^2(f+g+h)\}.$$

Now multiplying (4.8) by fg and using (4.6) and (4.7) we have

$$\begin{aligned}&(k^3-k)\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}fg\\ &=3\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\{fg(f+g)\}.\end{aligned} \tag{4.10}$$

Similarly for the others(i.e if we take any two distinct permutation $\{e_i, e_j\}$ of $\{e, f, g, h\}$). Now multiplying(4.8) by e we have

$$\begin{aligned}&(k^3-k)\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}e\\ =&3\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\{e(f+g+h)+e^2(f^2+g^2+h^2)\}\\ =&3\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\{e(e+f+g+h)\\ &\quad+e^2(f^2+g^2+h^2-e^2)\}\\ =&3k\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}e+3\{(k^3-7k)^2-6^2\}\\ &\quad\cdot\{(k^3-10k)^2-(6k)^2\}\{e^2(f^2+g^2+h^2-e^2)\}\end{aligned}$$

$$\begin{aligned}\Rightarrow\quad&(k^3-4k)\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}e^2\\ &=3\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\{e(f^2+g^2+h^2)-e\}.\end{aligned} \tag{4.11}$$

Similarly we get the results for f, g, h. Adding all these and using (4.8) we get

$$\begin{aligned}&(k^3-4k)\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\sum_{cyc}e^2\\ =&3\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\{\sum_{cyc}e(f^2+g^2+h^2)-\sum_{cyc}e\}\\ =&3\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\sum_{cyc}e(f^2+g^2+h^2)\\ &-3\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}k\\ =&(k^3-k)\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\\ &-3\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}k\\ =&(k^3-4k)\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}.\end{aligned}$$

Now multiplying the above equation by $\sum_{cyc} e$ we have

$$(k^3-4k)\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\{\sum_{cyc}e^2\}\{\sum_{cyc}e\}$$

$$= (k^3-4k)\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}k$$

$$\Rightarrow \quad (k^3-4k)\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\{\sum_{cyc} e^2(f+g+h)\}$$
$$+ (k^3-4k)\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}k$$
$$= (k^3-4k)\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}k$$

$$\Rightarrow \quad (k^3-4k)\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\{\sum_{cyc} e^2(f+g+h)\} = 0$$

$$\Rightarrow \quad (k^3-4k)[3\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\{\sum_{cyc} e^2(f+g+h)\}] = 0.$$

Using (4.8) we have

$$(k^3-4k)(k^3-k)\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\} = 0$$
$$\Rightarrow \quad (k-4)(k-3)(k-2)^2(k-1)^2k^4(k+1)^2(k+2)^2(k+3)(k+4) = 0.$$

Putting $k=5$ in the above equation we have $2^{10}\times 3^7\times 5^4\times 7^3 = 0.$ □

Proposition 4.4. *If the ring R is a ST^5 ring then for every $k\in R$ we have*

$$(k-5)(k-4)(k-3)^2(k-2)^3(k-1)^3k^4(k+1)^3(k+2)^3(k+3)^2(k+4)(k+5) = 0.$$

Proof. For every $k\in R \exists e_i \in R, e_i^3 = e_ie_ie_j = e_je_i, 1\geq i,j\geq 5$ such that $k=\sum_{i=1}^{5} e_i$. Now,

$$k^3-k = 3\sum_{cyc} e_1^2(e_2+e_3+e_4+e_5) + 6\sum_{cyc} e_1e_2e_3. \tag{4.12}$$

Multiplying (4.11) by $e_1e_2e_3e_4e_5$ we have

$$(k^3-k)e_1e_2e_3e_4e_5 = 3e_1e_2e_3e_4e_5.4(e_1+e_2+e_3+e_4+e_5) + 6\sum_{cyc} e_1^2e_2^2e_3^2e_4e_5$$
$$=12ke_1e_2e_3e_4e_5 + 6\sum_{cyc} e_1^2e_2^2e_3^2e_4^3e_5^3 = 12ke_1e_2e_3e_4e_5 + 3(e_1e_2e_3e_4e_5)^2(2\sum_{cyc} e_4e_5)$$

$$\Rightarrow \quad (k^3-13k)e_1e_2e_3e_4e_5 = 3(e_1e_2e_3e_4e_5)^2\{k^2-(\sum_{cyc} e_1^2)\}$$

$$\Rightarrow \quad (k^3-13k)(e_1e_2e_3e_4e_5)^2 = (3k^2-15)e_1e_2e_3e_4e_5$$

$$\Rightarrow \quad (k^3-13k)e_1e_2e_3e_4e_5 = (3k^2-15)(e_1e_2e_3e_4e_5)^2$$

$$\Rightarrow \quad (k^3-13k)^2e_1e_2e_3e_4e_5 = (3k^2-15)\{(k^3-13k)(e_1e_2e_3e_4e_5)^2\}$$
$$= (3k^2-15)^2e_1e_2e_3e_4e_5$$

$$\Rightarrow \quad \{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2e_3e_4e_5 = 0. \tag{4.13}$$

Now multiplying (4.11) by $\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2e_3e_4$ and using (4.12) we have

$$(k^3-k)\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2e_3e_4$$
$$=\{(k^3-13k)^2-(3k^2-15)^2\}\{3e_1e_2e_3e_4(3e_1+3e_2+3e_3+3e_4+3e_5)$$
$$+6e_1e_2e_3e_4(\sum_{cyc} e_1e_2e_3)\}$$
$$=\{(k^3-13k)^2-(3k^2-15)^2\}\{9ke_1e_2e_3e_4+6e_1e_2e_3e_4(\sum_{cyc} e_1e_2e_3)\}$$

$$\Rightarrow \quad (k^3-10k)\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2e_3e_4$$
$$=6\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2e_3e_4(\sum_{cyc} e_1e_2e_3)$$

$$\Rightarrow \quad (k^3-10k)\{(k^3-13k)^2-(3k^2-15)^2\}(e_1e_2e_3e_4)^2$$
$$=6k\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2e_3e_4$$

$$\Rightarrow \quad (k^3-10k)^2\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2e_3e_4$$
$$=6k[(k^3-10k)\{(k^3-13k)^2-(3k^2-15)^2\}(e_1e_2e_3e_4)^2]$$
$$=6k.6k\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2e_3e_4$$

$$\Rightarrow \{(k^3-10k)^2-(6k)^2\}\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2e_3e_4 = 0. \tag{4.14}$$

Similarly for the other distinct permutations $e_ie_je_ke_l$ of $\{e_1, e_2, e_3, e_4, e_5\}$ we get the result. Now multiplying (4.11) by $\{(k^3-10k)^2-(6k)^2\}\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2e_3$ and using (4.12) and (4.13) we have

$$\begin{aligned}
&(k^3-k)\{(k^3-10k)^2-(6k)^2\}\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2e_3\\
&= 3\{(k^3-10k)^2-(6k)^2\}\{(k^3-13k)^2-(3k^2-15)^2\}\\
&\quad\cdot\{e_1^3e_2e_3(e_2+e_3+e_4+e_5)+e_1e_2^3e_3(e_1+e_3+e_4+e_5)\\
&\quad+e_1e_2e_3^3(e_1+e_2+e_4+e_5)\}\\
&\quad+6\{(k^3-10k)^2-(6k)^2\}\{(k^3-13k)^2-(3k^2-15)^2\}e_1^2e_2^2e_3^2\\
&= 3\{(k^3-10k)^2-(6k)^2\}\{(k^3-13k)^2-(3k^2-15)^2\}\\
&\quad\cdot e_1e_2e_3(2e_1+2e_2+2e_3+2e_4+2e_5)+6\{(k^3-10k)^2-(6k)^2\}\\
&\quad\cdot\{(k^3-13k)^2-(3k^2-15)^2\}(e_1e_2e_3)^2\\
&= 6k\{(k^3-10k)^2-(6k)^2\}\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2e_3\\
&\quad+6\{(k^3-10k)^2-(6k)^2\}\{(k^3-13k)^2-(3k^2-15)^2\}(e_1e_2e_3)^2
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad &(k^3-7k)\{(k^3-10k)^2-(6k)^2\}\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2e_3\\
&\qquad= 6\{(k^3-10k)^2-(6k)^2\}\{(k^3-13k)^2-(3k^2-15)^2\}(e_1e_2e_3)^2\\
\Rightarrow\quad &(k^3-7k)[6\{(k^3-10k)^2-(6k)^2\}\{(k^3-13k)^2-(3k^2-15)^2\}(e_1e_2e_3)^2]\\
&\qquad= 6^2\{(k^3-10k)^2-(6k)^2\}\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2e_3\\
\Rightarrow\quad &(k^3-7k)[(k^3-7k)\{(k^3-10k)^2-(6k)^2\}\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2e_3]\\
&\qquad= 6^2\{(k^3-10k)^2-(6k)^2\}\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2e_3
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad &\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\\
&\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2e_3=0. \qquad (4.15)
\end{aligned}$$

Similarly for the other distinct permutations $e_ie_je_k$ of $\{e_1, e_2, e_3, e_4, e_5\}$ we get the result. Now we multiply (4.11) by $\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2$ and using (4.12),(4.13),(4.14) we have

$$
\begin{aligned}
&(k^3-k)\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2\\
&=3\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\{(k^3-13k)^2-(3k^2-15)^2\}\\
&\quad\cdot\{\sum_{cyc}e_1^2(e_2+e_3+e_4+e_5)\}e_1e_2\\
&=3\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2k
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad &(k^3-4k)\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\\
&\qquad\{(k^3-13k)^2-(3k^2-15)^2\}e_1e_2=0
\end{aligned}
$$

Similarly for the other distinct permutations e_ie_j of $\{e_1,e_2,e_3,e_4,e_5\}$ we get the result. Now multiplying (4.11) by $(k^3-4k)\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\{(k^3-13k)^2-(3k^2-15)^2\}$ and using (4.14) we have

$$
\begin{aligned}
&(k^3-k)(k^3-4k)\{(k^3-7k)^2-6^2\}\{(k^3-10k)^2-(6k)^2\}\\
&\cdot\{(k^3-13k)^2-(3k^2-15)^2\}=0\\
\Rightarrow\quad &(k-5)(k-4)(k-3)^2(k-2)^3(k-1)^3k^4\\
&\cdot(k+1)^3(k+2)^3(k+3)^2(k+4)(k+5)=0
\end{aligned}
$$

which we required. Now putting $k=6$ in the above we have $2^{21}\times3^{10}\times5^4\times7^3\times11=0$. □

5. RING IN WHICH EVERY ELEMENT IS SUM OF n COMMUTING TRIPOTENTS

Now the simple question arises that "Is it possible to find the structure of a ring in which every element is sum of $n(n\geq 2)$ commuting tripotents?". the answer is clearly affirmative for $n=2,3,4,5$. The answer is also affirmative for n in which $n\geq 5$ but they are multiplied by a arbitrary numeric constant(The structure equations are not monic). Now to find the structure we have to use some lemma which are given below.

Lemma 5.1. *If e is a tripotent which is commutative with $k(ek=ke)$ and $m,n\in\mathbb{N}$ then we get the following conditions.*

(1) $(e^2 \pm e)^2 = 2(e^2 \pm e)$ *and* $(e^2 \pm e)^n = 2^{n-1}(e^2 \pm e)$..

(2) $(e^2 + e)(k - j - e) = (k - j - 1)(e^2 + e)$.

(3) $(e^2 - e)(k - j - e) = (k - j + 1)(e^2 - e)$.

(4) $\{(k-m)-e\}^{i_1}\{(k-m+1)-e\}^{i_2}\dots\{(k-m+j)-e\}^{i_{j+1}}\dots\{(k-1)-e\}^{i_m}(k-e)^{i_{m+1}}\{(k+1)-e\}^{i_{m+2}}\dots\{(k+j)-e\}^{i_{m+j+1}}\dots\{(k+n)-e\}^{i_{m+n+1}}(e^2+e)^{\sum_{j=1}^{m+n+1} i_j} = (k-m-1)^{i_1}(k-m)^{i_2}\dots(k-m+j-1)^{i_{j+1}}\dots(k-2)^{i_m}(k-1)^{i_{m+1}}k^{i_{m+2}}\dots(k+j-1)^{i_{m+j+1}}\dots(k+n-1)^{i_{m+n+1}}2^{\sum_{j=1}^{m+n+1} i_j-1}(e^2+e)$.

(5) $\{(k-m)-e\}^{i_1}\{(k-m+1)-e\}^{i_2}\dots\{(k-m+j)-e\}^{i_{j+1}}\dots\{(k-1)-e\}^{i_m}(k-e)^{i_{m+1}}\{(k+1)-e\}^{i_{m+2}}\dots\{(k+j)-e\}^{i_{m+j+1}}\dots\{(k+n)-e\}^{i_{m+n+1}}(e^2-e)^{\sum_{j=1}^{m+n+1} i_j} = (k-m+1)^{i_1}(k-m+2)^{i_2}\dots(k-m+j+1)^{i_{j+1}}\dots k^{i_m}(k+1)^{i_{m+1}}(k+2)^{i_{m+2}}\dots(k+j+1)^{i_{m+j+1}}\dots(k+n+1)^{i_{m+n+1}}2^{\sum_{j=1}^{m+n+1} i_j-1}(e^2-e)$

Proof.

(1) We have $(e^2 \pm e)^2 = e^4 \pm 2e^3 + e = 2(e^2 \pm e)$. Therefore $(e^2+e)^n = 2^{n-1}(e^2+e)$.

(2) We have $(e^2+e)(k-j-e) = (k-j)(e^2+e) - (e+e^2) = (k-j-1)(e^2+e)$. Therefore $(e^2-e)^n = 2^{n-1}(e^2-e)$.

(3) We have $(e^2-e)(k-j-e) = (k-j)(e^2-e) + (-e+e^2) = (k-j+1)(e^2-e)$.

(4) Using the above conditions (1),(2),(3) clearly we get the result.

(5) Using the above conditions (1),(2),(3) clearly we get the result.

□

Using the above lemma we can get a different structure of the rings in which every element is a sum of 3,4,5 commuting tripotents. And also we can get the structure of the rings in which every element is a sum of $n(n \geq 6)$ commuting tripotents. Firstly, we find the new structure of the rings in which every element is $n = 3, 4, 5, 6, 7, 8$ commuting tripotents, then by induction we find the general structure.

Proposition 5.1. *For a ring R the following conditions are satisfied:*

(1) *If the ring R is ST^3 ring then for every $k \in R$ we have* $(k-3)(k-2)(k-1)k^2(k+1)(k+2)(k+3)2^5 = 0$.

(2) *If the ring R is a ST^4 ring then for every $k \in R$ we have* $(k-4)(k-3)(k-2)(k-1)^2k^2(k+1)^2(k+2)(k+3)(k+4)2^{13} = 0$.

(3) *If the ring R is ST^5 ring then for every $k \in R$ we have* $(k-5)(k-4)(k-3)(k-2)^2(k-1)^2k^3(k+1)^2(k+2)^2(k+3)(k+4)(k+5)2^{25} = 0$.

(4) *If the ring R is a ST^6 ring then for every $k \in R$ we have* $(k-6)(k-5)(k-4)(k-3)^2(k-2)^2(k-1)^3k^3(k+1)^3(k+2)^2(k+3)^2(k+4)(k+5)(k+6)2^{42} = 0$.

(5) *If the ring R is a ST^7 then for every $k \in R$ we have* $(k-7)(k-6)(k-5)(k-4)^2(k-3)^2(k-2)^3(k-1)^3k^4(k+1)^3(k+2)^3(k+3)^2(k+4)^2(k+5)(k+6)(k+7)2^{65} = 0$.

(6) *If the ring R is a ST^8 then for every $k \in R$ we have* $(k-8)(k-7)(k-6)(k-5)^2(k-4)^2(k-3)^3(k-2)^3(k-1)^4k^4(k+1)^4(k+2)^3(k+3)^3(k+4)^2(k+5)^2(k+6)(k+7)(k+8)2^{95} = 0$.

Proof. Suppose $k = \sum_{i=1}^{n} e_i$ where $e_i^3 = e_i$ is sum of n commuting tripotents. Then $k - e_j = \sum_{i=1}^{n} e_i - e_j$ is sum of $n-1$ commuting tripotents. We consider this e_j as e. Therefore if k is sum of n commuting tripotents then $k - e$ is sum of $n-1$ commuting tripotents. Also we denote $\sum_{i=1}^{n} e_i$ by $\sum e$.

(1) We get from Proposition 4.1 that if k is sum of 2 commuting tripotents then it satisfies $(k-2)(k-1)k(k+1)(k+2) = 0$. Now if k is sum of 3 commuting tripotents then $k - e$ is sum of 2 commuting tripotents. Then using Proposition 4.1 we have

$$(k-e-2)(k-e-1)(k-e)(k-e+1)(k-e+2) = 0$$
$$\Rightarrow (k-2-e)(k-1-e)(k-e)(k+1-e)(k+2-e) = 0.$$

Now multiplying the above equation by $(e^2+e)^5$ and $(e^2-e)^5$ respectively and using the Lemma 5.1 we have

$$(k-2-e)(k-1-e)(k-e)(k+1-e)(k+2-e)(e^2+e)^5$$
$$= (k-3)(k-2)(k-1)k(k+1)2^{5-1}(e^2+e) = 0.$$

And $(k-2-e)(k-1-e)(k-e)(k+1-e)(k+2-e)(e^2-e)^5 = (k-1)k(k+1)(k+2)(k+3)2^{5-1}(e^2-e) = 0$. Using the above equations we get

$$(k-3)(k-2)(k-1)k(k+1)(k+2)(k+3)2^{5-1}(e^2+e) = 0$$
$$(k-3)(k-2)(k-1)k(k+1)(k+2)(k+3)2^{5-1}(e^2-e) = 0.$$

Therefore taking difference of the above 2 equations , then taking sum all over the elements we have

$$(k-3)(k-2)(k-1)k(k+1)(k+2)(k+3)2^5e = 0$$
$$(k-3)(k-2)(k-1)k(k+1)(k+2)(k+3)2^5(\sum e) = 0$$
$$(k-3)(k-2)(k-1)k^2(k+1)(k+2)(k+3)2^5 = 0.$$

The above proof we can write in compact form and in the later proof(for $n = 4, 5, 6, 7, 8$), we proof like this.
$n = 3; k = \sum_{i=1}^{3} e_i; e_i^3 = e_i$
$(k-3)(k-2)(k-1)k(k+1)2^{5-1}(e^2+e) = 0$
$(k-1)k(k+1)(k+2)(k+3)2^{5-1}(e^2-e) = 0$
$\therefore (k-3)(k-2)(k-1)k(k+1)(k+2)(k+3)2^5e = 0$
$\Rightarrow (k-3)(k-2)(k-1)k^2(k+1)(k+2)(k+3)2^5 = 0$

(2) $n = 4; k = \sum_{i=1}^{4} e_i; e_i^3 = e_i$
$(k-4)(k-3)(k-2)(k-1)^2k(k+1)(k+2)(k+3)2^{5+8-1}(e^2+e) = 0$
$(k-2)(k-1)k(k+1)^2(k+2)(k+3)(k+4)2^{5+8-1}(e^2-e) = 0$
$\therefore (k-4)(k-3)(k-2)(k-1)^2k(k+1)^2(k+2)(k+3)(k+4)2^{5+8}e = 0$
$\Rightarrow (k-4)(k-3)(k-2)(k-1)^2k^2(k+1)^2(k+2)(k+3)(k+4)2^{13}e = 0$

(3) $n = 5; k = \sum_{i=1}^{5} e_i; e_i^3 = e_i$
$(k-5)(k-4)(k-3)(k-2)^2(k-1)^2k^2(k+1)(k+2)(k+3)2^{13+12-1}(e^2+e) = 0$
$(k-3)(k-2)(k-1)k^2(k+1)^2(k+2)^2(k+3)(k+4)2^{13+12-1}(e^2-e) = 0$
$\therefore (k-5)(k-4)(k-3)(k-2)^2(k-1)^2k^2(k+1)^2(k+2)^2(k+3)(k+4)(k+5)2^{13+12}e = 0$
$\Rightarrow (k-5)(k-4)(k-3)(k-2)^2(k-1)^2k^3(k+1)^2(k+2)^2(k+3)(k+4)(k+5)2^{25} = 0.$

(4) $n = 6; k = \sum_{i=1}^{6} e_i; e_i^3 = e_i$
$(k-6)(k-5)(k-4)(k-3)^2(k-2)^2(k-1)^3k^2(k+1)^2(k+2)(k+3)(k+4)2^{25+17-1}(e^2+e) = 0$
$(k-4)(k-3)(k-2)(k-1)^2k^2(k+1)^3(k+2)^2(k+3)^2(k+4)(k+5)(k+6)2^{25+17-1}(e^2-e) = 0$
$\therefore (k-6)(k-5)(k-4)(k-3)^2(k-2)^2(k-1)^3k^2(k+1)^3(k+2)^2(k+3)^2(k+4)(k+5)(k+6)2^{25+17}e = 0$

$\Rightarrow (k-6)(k-5)(k-4)(k-3)^2(k-2)^2(k-1)^3k^3(k+1)^3(k+2)^2(k+3)^2(k+4)(k+5)(k+6)2^{42} = 0$

(5) $n = 7; k = \sum_{i=1}^{7} e_i; e_i^3 = e_i$

$(k-7)(k-6)(k-5)(k-4)^2(k-3)^2(k-2)^3(k-1)^3k^3(k+1)^2(k+2)^2(k+3)(k+4)(k+5)(k+6)2^{42+23-1}(e^2+e) = 0$

$(k-5)(k-4)(k-3)(k-2)^2(k-1)^2k^3(k+1)^3(k+2)^3(k+3)^2(k+4)^2(k+5)(k+6)(k+7)2^{42+23-1}(e^2-e) = 0$

$\therefore (k-7)(k-6)(k-5)(k-4)^2(k-3)^2(k-2)^3(k-1)^3k^3(k+1)^3(k+2)^3(k+3)^2(k+4)^2(k+5)(k+6)(k+7)2^{42+23}e = 0$

$\Rightarrow (k-7)(k-6)(k-5)(k-4)^2(k-3)^2(k-2)^3(k-1)^3k^4(k+1)^3(k+2)^3(k+3)^2(k+4)^2(k+5)(k+6)(k+7)2^{65} = 0$

(6) $n = 8; k = \sum_{i=1}^{8} e_i; e_i^3 = e_i$

$(k-8)(k-7)(k-6)(k-5)^2(k-4)^2(k-3)^3(k-2)^3(k-1)^4k^3(k+1)^3(k+2)^3(k+3)^2(k+4)(k+5)(k+6)2^{65+30-1}(e^2+e) = 0$

$(k-6)(k-5)(k-4)(k-3)^2(k-2)^2(k-1)^3k^3(k+1)^4(k+2)^3(k+3)^3(k+4)^2(k+5)^2(k+6)(k+7)(k+8)2^{65+30-1}(e^2-e) = 0$

$\therefore (k-8)(k-7)(k-6)(k-5)^2(k-4)^2(k-3)^3(k-2)^3(k-1)^4k^3(k+1)^4(k+2)^3(k+3)^3(k+4)^2(k+5)^2(k+6)(k+7)(k+8)2^{65+30}e = 0$

$\Rightarrow (k-8)(k-7)(k-6)(k-5)^2(k-4)^2(k-3)^3(k-2)^3(k-1)^4k^4(k+1)^4(k+2)^3(k+3)^3(k+4)^2(k+5)^2(k+6)(k+7)(k+8)2^{95} = 0$

□

By viewing the above 6 forms $(n = 3, 4, 5, 6, 7, 8)$ we are going to find the general form.

Proposition 5.2. *The following conditions are true for a ring* R

(1) *If every element of a ring* R *is sum of* $n(= 2m+1), m \geq 1$ *commuting tripotents then for every* $k \in R$ *we have* $\{k-(2m+1)\}\{k-2m\}\{k-(2m-1)\}\{k-(2m-2)\}^2\{k-(2m-3)\}^2\{k-(2m-4)\}^3\{k-(2m-5)\}^3\{k-(2m-6)\}^4\{k-(2m-7)\}^4 \dots \{k-(2m-2j)\}^{j+1}\{k-(2m-2j-1)\}^{j+1} \dots \{k-\{2m-(2m-2)\}\}^m\{k-\{2m-(2m-1)\}\}^m\{k\pm(2m-2m)\}^{m+1}\{k+\{2m-2(m-1)-1\}\}^m\{k+\{2m-2(m-1)\}\}^m\{k+\{2m-2(m-2)-1\}\}^{m-1}\{k+\{2m-2(m-2)\}\}^{m-1} \dots \{k+(2m-2j-1)\}^{j+1}\{k+(2m-2j)\}^{j+1} \dots \{k+(2m-3)\}^2\{k+(2m-2)\}^2\{k+(2m-1)\}\{k+2m\}\{k+(2m+1)\}2^{3(2m-1)+\frac{2m(m+2)(2m-1)}{3}} = 0.$

(2) *If every element of a ring R is sum of $n(=2m+2), m \geq 1$ commuting tripotents then for every $k \in R$ we have* $\{k-(2m+2)\}\{k-(2m+1)\}\{k-2m\}\{k-(2m-1)\}^2\{k-(2m-2)\}^2\{k-(2m-3)\}^3\{k-(2m-4)\}^3\{k-(2m-5)\}^4\{k-(2m-6)\}^4\{k-(2m-7)\}^5\ldots\{k-(2m-2j)\}^{j+1}\{k-(2m-2j-1)\}^{j+2}\ldots\{k-\{2m-(2m-2)\}\}^m\{k-\{2m-(2m-1)\}\}^{m+1}\{k\pm(2m-2m)\}^{m+1}\{k+\{2m-2(m-1)-1\}\}^{m+1}\{k+\{2m-2(m-1)\}\}^m\{k+\{2m-2(m-2)-1\}\}^m\{k+\{2m-2(m-2)\}\}^{m-1}\ldots\{k+(2m-2j-1)\}^{j+2}\{k+(2m-2j)\}^{j+1}\ldots\{k+(2m-3)\}^3\{k+(2m-2)\}^2\{k+(2m-1)\}^2\{k+2m\}\{k+(2m+1)\}\{k+(2m+2)\}2^{6m+\frac{m(2m+1)(2m+5)}{3}}=0.$

Proof. From Proposition 5.1 clearly we can see the result is true for $n=3,4,5,6,7,8,(n=2m+1/2m+2, m\geq 1)$. Suppose the result is true for $n=2m+1$. We have to prove the result for $n=2m=2.2m+3$(If we take the result is true for $n=2m+2$, then we have to prove the result for $n=2m+3, 2m+4$;the procedure is same, so we prove the result for 1st case). Then by induction the result is true $\forall n\in\mathbb{N}$. Let $k=\sum_{i=1}^{2m+2}e_i$ where $e_i^3=e_i$. Then k is sum of $(2m+2)$ tripotents. Therefore $k-e$ is sum of $(2m+1)$ commuting tripotents(assumations as used in the proof of the Proposition 5.1). Therefore we have $\{k-(2m+1)-e\}\{k-2m-e\}\{k-(2m-1)-e\}\{k-(2m-2)-e\}^2\{k-(2m-3)-e\}^2\{k-(2m-4)-e\}^3\{k-(2m-5)-e\}^3\{k-(2m-6)-e\}^4\{k-(2m-7)-e\}^4\ldots\{k-(2m-2j)-e\}^{j+1}\{k-(2m-2j-1)-e\}^{j+1}\ldots\{k-\{2m-(2m-2)\}-e\}^m\{k-\{2m-(2m-1)\}-e\}^m\{k\pm(2m-2m)-e\}^{m+1}\{k+\{2m-2(m-1)-1\}-e\}^m\{k+\{2m-2(m-1)\}-e\}^m\{k+\{2m-2(m-2)-1\}-e\}^{m-1}\{k+\{2m-2(m-2)\}-e\}^{m-1}\ldots\{k+(2m-2j-1)-e\}^{j+1}\{k+(2m-2j)-e\}^{j+1}\ldots\{k+(2m-3)-e\}^2\{k+(2m-2)-e\}^2\{k+(2m-1)-e\}\{k+2m-e\}\{k+(2m+1)-e\}2^{3(2m-1)+\frac{2m(m+2)(2m-1)}{3}}=0.$

Now multiply the above equation by

$$(e^2+e)^{2+4(1+2+\ldots+m)+(m+1)}=(e^2+e)^{2m^2+3m+3}$$

and $(e^2-e)^{2m^2+3m+3}$ and using the Lemma 5.1 we get $\{k-(2m+2)\}\{k-(2m+1)\}\{k-2m\}\{k-(2m-1)\}^2\{k-(2m-2)\}^2\{k-(2m-3)\}^3\{k-(2m-4)\}^3\{k-(2m-5)\}^4\{k-(2m-6)\}^4\ldots\{k-(2m-2j+1)\}^{j+1}\{k-(2m-2j)\}^{j+1}\ldots\{k-3\}^m\{k-2\}^m\{k-1\}^{m+1}k^m\{k+1\}^m\{k+2\}^{m-1}\{k+3\}^{m-1}\ldots\{k+(2m-2j-2)\}^{j+1}\{k+(2m-2j-1)\}^{j+1}\ldots\{k+(2m-4)\}^2\{k+(2m-3)\}^2\{k+(2m-2)\}\{k+2m-1\}\{k+2m+\}2^{(2m^2+3m+3)+3(2m-1)+\frac{2m(m+2)(2m-1)}{3}-1}(e^2+e)=0.$

And, $\{k-2m\}\{k-(2m-1)\}\{k-(2m-2)\}\{k-(2m-3)\}^2\{k-(2m-4)\}^2\{k-(2m-5)\}^3\{k-(2m-6)\}^3\{k-(2m-7)\}^4\{k-(2m-8)\}^4\dots\{k-(2m-2j-1)\}^{j+1}\{k-(2m-2j))\}^{j+1}\dots\{k-1\}^m k^m\{k+1\}^{m+1}\{k+2\}^m\{k+3\}^m\{k+4\}^{m-1}\{k+5\}^{m-1}\dots\{k+(2m-2j)\}^{j+1}\{k+(2m-2j+1)\}^{j+1}\dots\{k+(2m-2)\}^2\{k+(2m-1)\}^2\{k+2m\}\{k+(2m+1)\}\{k+(2m+2)\}2^{(2m^2+3m+3)+3(2m-1)+\frac{2m(m+2)(2m-1)}{3}-1}(e^2-e)=0.$

Using above 2 equations we have $\{k-(2m+2)\}\{k-(2m+1)\}\{k-2m\}\{k-(2m-1)\}^2\{k-(2m-2)\}^2\{k-(2m-3)\}^3\{k-(2m-4)\}^3\{k-(2m-5)\}^4\{k-(2m-6)\}^4\dots\{k-(2m-2j+1)\}^{j+1}\{k-(2m-2j))\}^{j+1}\dots\{k-3\}^m\{k-2\}^m\{k-1\}^{m+1}k^m\{k+1\}^{m+1}\{k+2\}^m\{k+3\}^m\dots\{k+(2m-2j-2)\}^{j+1}\{k+(2m-2j-1)\}^{j+1}\dots\{k+(2m-4)\}^3\{k+(2m-3)\}^3\{k+(2m-2)\}^2\{k+2m-1\}^2\{k+2m+\}\{k+(2m+1)\}\{k+(2m+2)\}2^{(2m^2+3m+3)+3(2m-1)+\frac{2m(m+2)(2m-1)}{3}-1}(e^2+e)=0.$

And, $\{k-(2m+2)\}\{k-(2m+1)\}\{k-2m\}\{k-(2m-1)\}^2\{k-(2m-2)\}^2\{k-(2m-3)\}^3\{k-(2m-4)\}^3\{k-(2m-5)\}^4\{k-(2m-6)\}^4\dots\{k-(2m-2j+1)\}^{j+1}\{k-(2m-2j))\}^{j+1}\dots\{k-3\}^m\{k-2\}^m\{k-1\}^{m+1}k^m\{k+1\}^{m+1}\{k+2\}^m\{k+3\}^m\dots\{k+(2m-2j-2)\}^{j+1}\{k+(2m-2j-1)\}^{j+1}\dots\{k+(2m-4)\}^3\{k+(2m-3)\}^3\{k+(2m-2)\}^2\{k+2m-1\}^2\{k+2m+\}\{k+(2m+1)\}\{k+(2m+2)\}2^{(2m^2+3m+3)+3(2m-1)+\frac{2m(m+2)(2m-1)}{3}-1}(e^2-e)=0.$

Subtracting the above 2 equations we have $\{k-(2m+2)\}\{k-(2m+1)\}\{k-2m\}\{k-(2m-1)\}^2\{k-(2m-2)\}^2\{k-(2m-3)\}^3\{k-(2m-4)\}^3\{k-(2m-5)\}^4\{k-(2m-6)\}^4\dots\{k-(2m-2j+1)\}^{j+1}\{k-(2m-2j))\}^{j+1}\dots\{k-3\}^m\{k-2\}^m\{k-1\}^{m+1}k^m\{k+1\}^{m+1}\{k+2\}^m\{k+3\}^m\dots\{k+(2m-2j-2)\}^{j+1}\{k+(2m-2j-1)\}^{j+1}\dots\{k+(2m-4)\}^3\{k+(2m-3)\}^3\{k+(2m-2)\}^2\{k+2m-1\}^2\{k+2m+\}\{k+(2m+1)\}\{k+(2m+2)\}2^{(2m^2+3m+3)+3(2m-1)+\frac{2m(m+2)(2m-1)}{3}}e=0.$

Taking the summation over e we get $\{k-(2m+2)\}\{k-(2m+1)\}\{k-2m\}\{k-(2m-1)\}^2\{k-(2m-2)\}^2\{k-(2m-3)\}^3\{k-(2m-4)\}^3\{k-(2m-5)\}^4\{k-(2m-6)\}^4\dots\{k-(2m-2j+1)\}^{j+1}\{k-(2m-2j))\}^{j+1}\dots\{k-3\}^m\{k-2\}^m\{k-1\}^{m+1}k^{m+1}\{k+1\}^{m+1}\{k+2\}^m\{k+3\}^m\dots\{k+(2m-2j-2)\}^{j+1}\{k+(2m-2j-1)\}^{j+1}\dots\{k+(2m-4)\}^3\{k+(2m-3)\}^3\{k+(2m-2)\}^2\{k+2m-1\}^2\{k+2m+\}\{k+(2m+1)\}\{k+(2m+2)\}2^{6m+\frac{m(2m+1)(2m+5)}{3}}=0.$

Hence the result is true for $n=2m+2$. For $n=2m+3$, as we get the structure for the ring R in which each element is sum of $(n=2m+2)$ commuting tripotents. So by the same procedure as above we get the result. So by induction we get the result. □

Clearly there are another possible forms for a ring in which every element is sum of n commuting tripotents. Considering $x^8 = x^4$ for the ring in which every element is sum of two commuting tripotents, we can get another form of a ring in which every element is sum of n commuting tripotents. Combining all the forms we get a better equivalent for the ring. Let us 1st take the form for the ring in which every element is a sum of three commuting tripotents. For this we use some lemma which are given below.

Lemma 5.2. [6]. Let p be a prime. The following are equivalent for a ring R:

(1) $p \in Nil(R)$ and $a^p - a$ is nilpotent for all $a \in R$.
(2) $J(R)$ is nil and $R/J(R)$ is a subdirect product of Z_p's.

Lemma 5.3. [2]. Let $a \in R$. If $a^2 - a$ is nilpotent, then there exists a monic polynomial $\theta(t) \in Z(t)$ such that $\theta(a)^2 = \theta(a)$ and $a - \theta(a)$ is nilpotent.

Proposition 5.3. *If the ring R is a ST^3 ring with $3^2 = 0$ then the following conditions are satisfied.*

(1) *For every $x \in R$ we have $x^9 = x^3, (x^3 - x)^2 = 0, 3(x^3 - x) = 0$ and $3 \in J(R)$.*
(2) *For every $n \in Nil(R)$ we have $n^2 = 0, 3n = 0$.*
(3) *$R/J(R)$ is subdirect product of Z_3's. $J(R)$ is nil with $j^2 = 0; 3j = 0 \forall j \in J(R)$ and $ij = -ji \forall i, j \in J(R)$.*
(4) *$U(R)$ is a group of exponent 6 and $3u^2 = 3 \forall u \in U(R)$.*

Proof.

(1) Here $3^2 = 0$. Let $x \in R$ so $x = f + g + h$ such that $f^3 = f, g^3 = g, h^3 = h, fg = gf, gh = hg, hf = fh$. Now $x^3 - x = 3[\sum_{cyc} fg(f + g) + 2fgh]$. Therefore $(x^3 - x)^2 = 0, 3(x^3 - x) = 0$. Again $(x^3)^3 = [x + 3\{\sum_{cyc} fg(f + g) + 2fgh\}]^3 \Rightarrow x^9 = x^3$ as $9 = 0$.
(2) Let $n \in Nil(R)$. Therefore $(1 - n^2) \in U(R)$. Now $(n^3 - n)^2 = 0 \Rightarrow n^2(n^2 - 1)^2 = 0$. As $n^2 - 1 \in U(R)$ so $n^2 = 0$. Again $3(n^3 - n) = 0 \Rightarrow 3n(n^2 - 1) = 0 \Rightarrow 3n = 0$.
(3) Using Lemma 3.5 we have $J(R)$ is nil and $R/J(R)$ is subdirect product of Z_3's. Let $j \in J(R_2)$ so $-j \in J(R)$ (As $J(R)$ is the intersection of all right/left maximal ideal). Therefore $(1 - j) \in U(R)$ and $\{1 - (-j)\} \in U(R) \Rightarrow 1 + j \in U(R)$ as $J(R)$ is nil. Now $(j^3 - j)^2 = 0 \Rightarrow j^2(j - 1)^2(j +$

$1)^2 = 0$. As $j-1, j+1 \in U(R)$ so $j^2 = 0$ which imply $J(R)$ is nil of order 2. Again $3j(j^2-1) = 0 \Rightarrow 3j = 0$. Again for $i, j \in J(R)$ we have $(i+j)^2 = 0 \Rightarrow i^2 + ij + ji + j^2 = 0 \Rightarrow ij = -ji$.

(4) Let $u \in U(R)$. Now $u^9 = u^3 \Rightarrow u^3(u^6-1) = 0 \Rightarrow u^6 - 1 = 0 \Rightarrow u^6 = 1$ as $u^3 \in U(R)$. So $U(R)$ is group of exponent 6. Also $3(u^3-u) = 0 \Rightarrow 3u(u^2-1) = 0 \Rightarrow 3u^2 = 3$ as $u \in U(R)$.

□

Proposition 5.4. *If the ring R is a ST^3 ring then $R \cong R_1 \times R_2 \times R_3 \times R_4$ where*

- *R_1 is zero or ST^3 ring with $2^8 = 0$.*
- *R_2 is zero or ST^3 ring with $3^2 = 0$. R_2 has the identity $x^9 = x^3, (x^3-x)^2 = 0, 3(x^3-x) = 0 \forall x \in R_2$. For every $n \in Nil(R_2)$ we have $n^2 = 0, 3n = 0$. $R_2/J(R_2)$ is a subdirect product of Z_3's. For every $j \in J(R_2)$ we have $j^2 = 0, 3j = 0$ and $ij = -ji$ for $i, j \in J(R_2)$. $U(R_2)$ is a group of exponent 6 and $3u^2 = 3 \forall u \in U(R_2)$.*
- *R_3 is zero or a subdirect product of Z_5's.*
- *R_4 is zero or a subdirect product of Z_7's.*

Proof. Putting $k = 4$ in $(k-3)(k-2)(k-1)k^2(k+1)(k+2)(k+3)2^5 = 0$ in the Propositon 5.1 for the ST^3 ring we have $2^{11} \times 3^2 \times 5 \times 7 = 0$. Again from Proposition 4.2 we have $2^8 \times 3^4 \times 5^2 \times 7 = 0$. Taking g.c.d of $2^{11} \times 3^2 \times 5 \times 7$, $2^8 \times 3^4 \times 5^2 \times 7$ we get $2^8 \times 3^2 \times 5 \times 7 = 0$. So using Chinese Remainder Theorem we get a more reduced form which is $R \cong (\frac{R}{2^8 R}) \times (\frac{R}{3^2 R}) \times (\frac{R}{5R}) \times (\frac{R}{7R})$.

Now clearly $R \cong R_1 \times R_2 \times R_3 \times R_4$ where $R_1 \cong \frac{R}{2^8 R}, R_2 \cong \frac{R}{3^2 R}, R_3 \cong \frac{R}{5R}, R_4 \cong \frac{R}{7R}$. Clearly R_1 is ST^3 ring with $2^8 = 0$.

Suppose $R_2 \neq 0$. As R_2 is ST^3 ring with $3^2 = 0$ so by Proposition 4.3 we get the result. Suppose $R_3 \neq 0$.Here $5 = 0$ in R_3. If $k^2 = 0$ in R_2, then putting $k = f+g+h$ where f, g, h are commuting tripotents. Then $0 = k^5 = f^5 + g^5 + h^5 + 5P(f,g,h) = f + g + h = k$ where $P(f,g,h)$ is a function of f, g, h. Hence R_3 is a reduced ring, so R_3 is a subdirect product of the domains $\{R_\alpha\}$. But R_α has only the trivial tripotents $0, 1, -1$, we infer that $R_\alpha = \{-2, -1, 0, 1, 2\}$. As $5 = 0$ in R_α, $R_\alpha \cong \mathbb{Z}_5$. Hence, R_3 is a subdirect product of $\mathbb{Z}_5$'s. Suppose $R_4 \neq 0$. Here $7 = 0$ in R_3. If $k^2 = 0$ in R_2, then putting $k = f + g + h$ where f, g, h are commuting tripotents. Then $0 = k^7 = f^7 + g^7 + h^7 + 7P(f,g,h) = f + g + h = k$.Hence R_4 is a reduced ring, So R_4 is a subdirect product of the domains $\{R_\alpha\}$. But R_α has only the trivial tripotents $0, 1, -1$, we infer that $R_\alpha = \{-3, -2. -1, 0, 1, 2, 3\}$. But $7 = 0$ in R_α, so $R_\alpha \cong \mathbb{Z}_7$. Therefore ,$R_4$ is a subdirect product of $\mathbb{Z}_7$'s. □

Proposition 5.5. *If R is a ST^4 ring then $R \cong R_1 \times R_2 \times R_3 \times R_4$ where*

- *R_1 is zero or ST^4 ring with $2^{10} = 0$.*
- *R_2 is zero or ST^4 ring with $3^5 = 0$; R_2 has the identity $(x^3 - x)^5 = 0, 3^4(x^3 - x) = 0 \forall x \in R_2$. For every $n \in Nil(R_2)$ we have $n^5 = 0; 3^4 n = 0$. $R_2/J(R_2)$ is subdirect product of Z_3's. For every $j \in J(R_2)$ we have $j^5 = 0, 3^4 j = 0$. For $U(R_2)$ is group of exponent 2×3^5 and $3^4 u^2 = 3^4$ for every $u \in U(R_2)$.*
- *R_3 is zero or ST^4 ring with $5^2 = 0$; R_3 has the identity $(k^5 - k)^2 = 0; 5(k^5 - k) = 0$. For every $n \in Nil(R_2)$ we have $n^2 = 0; 5n = 0$. $R_3/J(R_3)$ is subdirect product of Z_5's. For every $i, j \in J(R_3)$ we have $j^2 = 0, 5j = 0, ij = -ji$. $U(R_3)$ is group of exponent $2 \times 5 = 10$ and $5u^4 = 5$ for every $u \in U(R_3)$.*
- *R_4 is zero or R_4 is subdirect product of $\mathbb{Z}_7$.*

Proof. From Proposition 4.3, For a ST^4 ring we have $2^{10} \times 3^7 \times 5^4 \times 7^3 = 0$. Again from Proposition 5.1, For a ST^4 ring R we have $(k-4)(k-3)(k-2)(k-1)^2 k^2 (k+1)^2 (k+2)(k+3)(k+4)2^{13} = 0 \forall k \in R$, Putting $k = 5$ we get $2^{23} \times 3^5 \times 5^2 \times 7 = 0$. Now taking g.c.d of $2^{10} \times 3^7 \times 5^4 \times 7^3$ and $2^{23} \times 3^5 \times 5^2 \times 7$ we get $2^{10} \times 3^5 \times 5^2 \times 7 = 0$. So using Chinese Remainder Theorem we get a more reduced form which is $R \cong (\frac{R}{2^{10}R}) \times (\frac{R}{3^5 R}) \times (\frac{R}{5^2 R}) \times (\frac{R}{7R})$.

Now clearly $R \cong R_1 \times R_2 \times R_3 \times R_4$ where $R_1 \cong \frac{R}{2^{10}R}$, $R_2 \cong \frac{R}{3^5 R}$, $R_3 \cong \frac{R}{5^2 R}$, $R_4 \cong \frac{R}{7R}$. Clearly R_1 is a ST^4 ring with $2^{10} = 0$.

Suppose $R_2 \neq 0$. Here $3^5 = 0$ in R_2. Let $k \in R_2$ so we can write $k = \sum_{i=1}^4 e_i; e_i^3 = e_i; e_i e_j = e_j e_i; i, j \in \{1, 2, 3, 4\}$. Now $k^3 - k = 3P$ where $P = P(e_1, e_2, e_3, e_4)$ is a function of e_1, e_2, e_3, e_4. Therefore $(k^3 - k)^5 = 0, 3^4(k^3 - k) = 0 \ \forall k \in R_2$.

Now for every $n \in Nil(R_2)$ we have $1 - n^\alpha \in U(R_2)$ where $\alpha \in \mathbb{N}$. Now for $n \in Nil(R_2)$, $(n^3 - n)^5 = 0 \Rightarrow n^5(n^2 - 1)^5 = 0 \Rightarrow n^5 = 0$. Also $3^4(n^3 - n) = 0 \Rightarrow 3^4 n(n^2 - 1) = 0 \Rightarrow 3^4 n = 0$.

Now $k^3 - k$ is nilpotent for all $k \in R_2$. So by using Lemma 5.3 we have $R_2/J(R_2)$ is subdirect product of Z_3's and $J(R_2)$ is nil.

Now for $j \in J(R_2)$ we have $-j \in J(R_2)$. Therefore $1-j, 1+j \in U(R_2)$ as $J(R_2)$ is nil. Now $(j^3-j)^5 = 0 \Rightarrow j^5(j-1)^5(j+1)^5 = 0 \Rightarrow j^5 = 0$ as $(j-1)^5, (j+1)^5 \in U(R_2)$. Again $3^4(j^3-j) = 0 \Rightarrow 3^4 j(j+1)(j-1) = 0 \Rightarrow 3^4 j = 0$ as $(j-1), (j+1) \in U(R_2)$.

Let $u \in U(R_2)$. Now $(u^3-u)^5 = 0 \Rightarrow u^5(u^2-1)^5 = 0 \Rightarrow (u^2-1)^5 = 0 \Rightarrow u^2 - 1 = n \Rightarrow u^2 = 1+n$ where $n \in Nil(R_2)$.Now $(u^2)^{3^5} = (1+n)^{3^5} = 1 + 3^5 n + \frac{3^5(3^5-1)}{2}n^2 + \frac{3^5(3^5-1)(3^5-2)}{3.2}n^3 + \frac{3^5(3^5-1)(3^5-2)(3^5-3)}{4.3.2}n^4 = 1$ as $3^4 n = 0, n^5 = 0$ for $n \in Nil(R_2)$. Therefore $U(R_2)$ is a group of exponent 2×3^5. Again for $u \in U(R_2)$ we have $3^4(u^3-u) = 0 \Rightarrow 3^4 u(u^2-1) = 0 \Rightarrow 3^4 u^2 = 3^4$.

Suppose $R_3 \neq 0$. Here $5^2 = 0$ in R_3.Let $k \in R_3$ so we can write $k = \sum_{i=1}^{4} e_i; e_i^3 = e_i; e_i e_j = e_j e_i; i, j \in \{1,2,3,4\}$. Now $k^5 - k = 5P$ where $P = P(e_1, e_2, e_3, e_4)$ is a function of e_1, e_2, e_3, e_4. Therefore $(k^5-k)^2 = 0; 5(k^5-k) = 0$.

Now for every $n \in Nil(R_2)$ we have $1 - n^\alpha \in U(R_2)$ where $\alpha \in \mathbb{N}$. Now $n^2(n^4-1)^2 = 0 \Rightarrow n^2 = 0$. Also $5n(n^4-1) = 0 \Rightarrow 5n = 0$.

Now $k^5 - k$ is nilpotent for all $k \in R_3$. So by using Lemma 5.3 we have $R_3/J(R_3)$ is subdirect product of Z_5's and $J(R_3)$ is nil.

Now for $j \in J(R_3)$ we have $\pm j^2 \in J(R_3)$. So $1-j^2, 1+j^2 \in U(R_3)$ as $J(R_3)$ is nil. Now $(j^5-j)^2 = 0 \Rightarrow j^2(1-j^2)^2(1+j^2)^2 = 0 \Rightarrow j^2 = 0$. Again for $i, j \in J(R_3)$ we have $(i+j)^2 = 0 \Rightarrow ij = -ji$.Again $5(j^5-j) = 0 \Rightarrow 5j(j^4-1) = 0 \Rightarrow 5j = 0$.

Let $u \in U(R_3)$. We have $(u^5-u)^2 = 0 \Rightarrow u^2(u^4-1)^2 = 0 \Rightarrow (u^4-1)^2 = 0 \Rightarrow u^4 - 1 = n \in Nil(R_3)$. Now $u^{20} = (u^4)^5 = (1+n)^5 = 1 + 5n = 1$ as $n^2 = 0; 5n = 0$ for every $n \in Nil(R)$. So $U(R_3)$ is group of exponent $4 \times 5 = 20$.

Suppose $R_4 \neq 0$.Here $7 = 0$ in R_3.If $k^2 = 0$ in R_2, then putting $k = \sum_{i=1}^{4} e_i$ where $e_i, i = 1,2,3,4$ are commuting tripotents. Then $0 = k^7 = e_1^7 + e_2^7 + e_3^7 + e_4^7 + 7P(e_1, e_2, e_3, e_4) = e_1 + e_2 + e_3 + e_4 = k$. Hence R_4 is a reduced ring, so R_4 is a subdirect product of the domains $\{R_\alpha\}$. But R_α has only the trivial tripotents $0, 1, -1$, we infer that $R_\alpha = \{-3, -2. -1, 0, 1, 2, 3\}$. But $7 = 0$ in R_α, so $R_\alpha \cong \mathbb{Z}_7$. Therefore ,$R_4$ is a subdirect product of $\mathbb{Z}_7$'s. □

Proposition 5.6. *If R is a ST^5 ring then $R \cong R_1 \times R_2 \times R_3 \times R_4 \times R_5$ where*

- *R_1 is zero or ST^5 with $2^{21} = 0$.*
- *R_2 is zero or ST^5 with $3^6 = 0$; R_2 has the identity $(k^3-k)^6 = 0; 3^5(k^3-k) = 0 \forall k \in R_2$. For every $n \in Nil(R)$ we have $n^6 = 0, 3^5 n = 0$. $R_2/J(R_2)$ is subdirect product of Z_3's. For every $j \in J(R_2)$ we have $j^6 = 0, 3^5 j = 0$.. $U(R)$ is a group of exponent 2×3^6 and $3^5 u^2 = 3^5$ for every $u \in R_2$*

- *R_3 is zero or ST^5 ring with $5^3 = 0$. R_3 has identity $(k^5 - k)^3 = 0; 5^2(k^5 - k) = 0 \forall k \in R_3$. For every $n \in Nil(R)$ we have $n^3 = 0, 5^2 n = 0$. $R_2/J(R_2)$ is subdirect product of Z_5's. For every $j \in J(R_3)$ we have $j^3 = 0, 5^2 j = 0$. $U(R_3)$ is a group of exponent $4 \times 5^2 = 100$ and $5^2 u^4 = 5^2$ for every $u \in U(R_3)$.*
- *R_4 is zero or ST^5 ring with $7^2 = 0$. R_4 has identity $(k^7 - k)^2 = 0; 7(k^7 - k) = 0 \forall k \in R_4$. $R_4/J(R_4)$ is subdirect product of Z_7's. For every $n \in Nil(R_4)$ we have $n^2 = 0, 7n = 0$. For every $j \in J(R_4)$ we have $j^2 = 0, 7j = 0$ and $ij = -ji$ for all $i, j \in J(R_4)$. $R_4/J(R_4)$ is subdirect product of Z_7's. $U(R_4)$ is a group of exponent $6 \times 7 = 42$ and $7u^6 = 7$ for every $u \in U(R_4)$*
- *R_5 is zero or subdirect product of $\mathbb{Z}_{11}$.*

Proof. From Proposition 4.4 we have $2^{21} \times 3^{10} \times 5^4 \times 7^3 \times 11 = 0$. Again from Proposition 5.1 , For a ST^5 ring we have $(k-5)(k-4)(k-3)(k-2)^2(k-1)^2 k^3 (k+1)^2(k+2)^2(k+3)(k+4)(k+5)2^{25} = 0 \forall k \in R$. Putting $k = 6$ we get $2^{40} \times 3^6 \times 5^3 \times 7^2 \times 11 = 0$. Now taking g.c.d of $2^{21} \times 3^{10} \times 5^4 \times 7^3 \times 11$ and $2^{40} \times 3^6 \times 5^3 \times 7^2 \times 11$ we get $2^{21} \times 3^6 \times 5^3 \times 7^2 \times 11 = 0$. So by using Chinese Remainder theorem we get a reduced form which is $R \cong (\frac{R}{2^{21}R}) \times (\frac{R}{3^6 R}) \times (\frac{R}{5^3 R}) \times (\frac{R}{7^2 R}) \times (\frac{R}{11R})$.

Now clearly $R \cong R_1 \times R_2 \times R_3 \times R_4 \times R_5$ where $R_1 \cong \frac{R}{2^{21}R}$, $R_2 \cong \frac{R}{3^5 R}$, $R_3 \cong \frac{R}{5^2 R}$, $R_4 \cong \frac{R}{7R}$. Clearly R_1 is a ST^5 ring with $2^{21} = 0$.

Suppose $R_2 \neq 0$. Here $3^6 = 0$ in R_2. Let $k \in R_2$ so we can write $k = \sum_{i=1}^{5} e_i; e_i^3 = e_i; e_i e_j = e_j e_i; i, j \in \{1, 2, 3, 4, 5\}$. Now $k^3 - k = 3P$ where $P = P(e_1, e_2, e_3, e_4, e_5)$ is a function of e_1, e_2, e_3, e_4, e_5. Therefore $(k^3 - k)^6 = 0, 3^5(k^3 - k) = 0 \ \forall k \in R_2$.

Now for every $n \in Nil(R_2)$ we have $1 - n^\alpha \in U(R_2)$ where $\alpha \in \mathbb{N}$. Now $(n^3 - n)^6 = 0 \Rightarrow n^6(n^2 - 1)^2 = 0 \Rightarrow n^6 = 0$. Again $3^5 n(n^2 - 1) = 0 \Rightarrow 3^5 n = 0$. Using the Lemma 5.2 we have $R_2/J(R_2)$ is subdirect of Z_3's and $J(R_2)$ is nil.

Now for $j \in J(R_2)$ we have $-j \in J(R_2)$. Therefore $1-j, 1+j \in U(R_2)$ as $J(R_2)$ is nil. Now $(j^3 - j)^6 = 0 \Rightarrow j^6(j-1)^6(j+1)^6 = 0 \Rightarrow j^6 = 0$ as $(j-1)^6, (j+1)^6 \in U(R_2)$. Again $3^5(j^3 - j) = 0 \Rightarrow 3^5 j(j+1)(j-1) = 0 \Rightarrow 3^5 j = 0$ as $(j-1), (j+1) \in U(R_2)$.

Let $u \in U(R_2)$. Now $(u^3 - u)^6 = 0 \Rightarrow u^6(u^2 - 1)^6 = 0 \Rightarrow (u^2 - 1)^6 = 0 \Rightarrow u^2 - 1 = n \in Nil(R_2)$. $(u^2)^{3^6} = (1+n)^{3^6} = 1 + 3^6 n + \frac{3^6(3^6-1)}{2} n^2 + \frac{3^6(3^6-1)(3^6-2)}{3.2} n^3 + \frac{3^6(3^6-1)(3^6-2)(3^6-3)}{4.3.2} n^4 + \frac{3^6(3^6-1)(3^6-2)(3^6-3)(3^6-4)}{5.4.3.2} n^5 = 1$ as $3^5 n = 0, n^6 = 0$. Therefore $U(R_2)$ is a group of exponent 2×3^6. Again $3^5 u(u^2 - 1) = 0 \Rightarrow 3^5 u^2 = 3^5$.

Suppose $R_3 \neq 0$. Here $5^3 = 0$ in R_3.Let $k \in R_3$ so we can write $k = \sum_{i=1}^{5} e_i; e_i^3 = e_i; e_ie_j = e_je_i; i, j \in \{1, 2, 3, 4, 5\}$. Now $k^5 - k = 5P$ where $P = P(e_1, e_2, e_3, e_4, e_5)$ is a function of e_1, e_2, e_3, e_4, e_5. Therefore $(k^5 - k)^3 = 0; 5^2(k^5 - k) = 0 \forall k \in R_3$.

Now for every $n \in Nil(R_4)$ we have $1 - n^\alpha \in U(R_2)$ where $\alpha \in \mathbb{N}$. Now $(n^5 - n)^3 = 0 \Rightarrow n^3(n^2 - 1)^3 = 0 \Rightarrow n^3 = 0$. Again $5^2n(n^4 - 1) = 0 \Rightarrow 5^2n = 0$.

Using Lemma 5.2 we have $R_3/J(R_3)$ is subdirect product of Z_5's and $J(R_3)$ is nil.

Now for $j \in J(R_3)$ we have $\pm j^2 \in J(R_3)$. So $1 - j^2, 1 + j^2 \in U(R_3)$ as $J(R_3)$ is nil. Now $(j^5 - j)^3 = 0 \Rightarrow j^3(1 - j^2)^3(1 + j^2)^3 = 0 \Rightarrow j^3 = 0$.Again $5^2(j^5 - j) = 0 \Rightarrow 5^2j = 0$.

Let $u \in U(R_3)$. Now $(u^5 - u)^3 = 0 \Rightarrow u^3(u^4 - 1)^3 = 0 \Rightarrow (u^4 - 1)^3 = 0 \Rightarrow u^4 - 1 = n \in Nil(R_3)$. So $u^{4\times5^2} = (u^4)^{5^2} = (1 + n)^{5^2} = 1 + 5^2n + \frac{5^2(5^2-1)}{2}n^2 = 1$ as $n^3 = 0, 5^2n = 0$. So $U(R_3)$ is group of exponent of $4 \times 5^2 = 100$. Also $5^2u(u^4 - 1) = 0 \Rightarrow 5u^4 = 5$.

Suppose $R_4 \neq 0$. Here $7^2 = 0$ in R_4.Let $k \in R_4$ so we can write $k = \sum_{i=1}^{5} e_i; e_i^3 = e_i; e_ie_j = e_je_i; i, j \in \{1, 2, 3, 4, 5\}$. Now $k^7 - k = 7P$ where $P = P(e_1, e_2, e_3, e_4, e_5)$ is a function of e_1, e_2, e_3, e_4, e_5. Therefore $(k^7 - k)^2 = 0; 7(k^7 - k) = 0 \forall k \in R_4$.

Now for every $n \in Nil(R_4)$ we have $1 - n^\alpha \in U(R_2)$ where $\alpha \in \mathbb{N}$. Now $(n^7 - n)^2 = 0 \Rightarrow n^2(n^6 - 1)^2 = 0 \Rightarrow n^2 = 0$. Again $7n(n^6 - 1) = 0 \Rightarrow 7n = 0$

Now for $j \in J(R_4)$ we have $\pm j^3 \in J(R_4)$. So $1 - j^3, 1 + j^3 \in U(R_4)$. Now $(j^7 - j)^2 = 0 \Rightarrow j^2(1 - j^3)^2(1 + j^3)^2 = 0 \Rightarrow j^2 = 0$ as $1 - j^3, 1 + j^3 \in U(R_4)$.Again $7(j^5 - j) = 0 \Rightarrow 7j = 0$. Again for $i, j \in J(R_4)$ we have $(i + j)^2 = 0 \Rightarrow ij = -ji$.

Let $u \in U(R_4)$. Now $(u^7 - u)^2 = 0 \Rightarrow u^2(u^6 - 1)^2 = 0 \Rightarrow (u^6 - 1)^2 = 0 \Rightarrow u^6 - 1 = n \in Nil(R_4)$. Again now $(u^6)^7 = (1 + n)^7 = 1 + 7n = 1$.So $U(R_4)$ is group of exponent $6 \times 7 = 42$.

Suppose $R_5 \neq 0$.Here $11 = 0$ in R_5.If $k^2 = 0$ in R_5, then putting $k = \sum_{i=1}^{5} e_i$ where $e_i, i = 1, 2, 3, 4, 5$ are commuting tripotents. Then $0 = k^{11} = e_1^{11} + e_2^{11} + e_3^{11} + e_4^{11} + 11P(e_1, e_2, e_3, e_4, e_5) = e_1 + e_2 + e_3 + e_4 + e_5 = k$. Hence R_5 is a reduced ring, so R_4 is a subdirect product of the domains $\{R_\alpha\}$. But R_α has only the trivial tripotents $0, 1, -1$, we infer that $R_\alpha = \{-5, -4, -3, -2. - 1, 0, 1, 2, 3, 4, 5\}$. But $11 = 0$ in R_α, so $R_\alpha \cong \mathbb{Z}_{11}$. Therefore ,$R_4$ is a subdirect product of $\mathbb{Z}_{11}$'s. □

Now using the Proposition 5.2 we have the following Corollaries.

Corollary 5.2. *In a* ST^{2m+1} *ring, we have* $1.2.3.4^2.5^2.6^3.7^3.8^4.9^4\dots(2j+2)^{j+1}(2j+3)^{j+1}\dots(2m)^m(2m+1)^m(2m+2)^{m+1}(2m+3)^m(2m+4)^m(2m+5)^{m-1}(2m+6)^{m-1}\dots(4m+2-2j-1)^{j+1}(4m+2-2j)^{j+1}\dots(4m-1)^2(4m)^2(4m+1)(4m+2)(4m+3)$ $2^{3(2m-1)+\frac{2m(m+2)(2m-1)}{3}}=0$

Corollary 5.3. *In a* ST^{2m+2} *ring, we have* $1.2.3.4^2.5^2.6^3.7^3.8^4.9^4.10^5\dots(2j+3)^{j+1}(2j+4)^{j+2}\dots(2m+1)^m(2m+2)^{m+1}(2m+3)^{m+1}(2m+4)^{m+1}(2m+5)^m(2m+6)^m(2m+7)^{m-1}\dots(4m-2j+2)^{j+2}(4m-2j+3)^{j+1}\dots4m^3(4m+1)^2(4m+2)^2(4m+3)(4m+4)(4m+5)2^{6m+\frac{m(2m+1)(2m+5)}{3}}=0.$

Putting $k=2m+2$ *and* $k=2m+3$ *in the equations of Proposition 5.2 we get the result.*

Lemma 5.4. *If* e *is a tripotent then* $e^{2m+1}=e$ *where* $m\geq 1$.

Proof. We have $e^3=e$,$e^5=e^3e^2=e^3=e$. So the result is true for small number. Suppose $e^{2m+1}=e$. Now $e^{2m+3}=e^{2m+1}e^2=ee^2=e^3=e$. Hence by induction the result is for all $n\in N$.

In general for every prime p other than 2 we have $e^p=e$.

We can generalize the property for ST^n ring. Just in general case we have to obtain the exponent of $U(R_i)$ manually. Excluding this case, we establish the general form of ST^n ring. □

Proposition 5.7. *Suppose* R *is* ST^n *ring. Then we have the following properties*

(1) *If* $n=2m+1$ *where* $m\geq 1$. *Suppose* $1.2.3.4^2.5^2.6^3.7^3.8^4.9^4\dots(2j+2)^{j+1}(2j+3)^{j+1}\dots(2m)^m(2m+1)^m(2m+2)^{m+1}(2m+3)^m(2m+4)^m(2m+5)^{m-1}(2m+6)^{m-1}\dots(4m+2-2j-1)^{j+1}(4m+2-2j)^{j+1}\dots(4m-1)^2(4m)^2(4m+1)(4m+2)(4m+3)2^{3(2m-1)+\frac{2m(m+2)(2m-1)}{3}}=2^{a_1}3^{a_2}5^{a_3}\dots{p_i}^{a_i}\dots{p_l}^{a_l}$.

(2) *Or if* $n=2m+2$ *where* $m\geq 1$. *Suppose* $1.2.3.4^2.5^2.6^3.7^3.8^4.9^4.10^5\dots(2j+3)^{j+1}(2j+4)^{j+2}\dots(2m+1)^m(2m+2)^{m+1}(2m+3)^{m+1}(2m+4)^{m+1}(2m+5)^m(2m+6)^m(2m+7)^{m-1}\dots(4m-2j+2)^{j+2}(4m-2j+3)^{j+1}\dots4m^3(4m+1)^2(4m+2)^2(4m+3)(4m+4)(4m+5)2^{6m+\frac{m(2m+1)(2m+5)}{3}}=2^{a_1}3^{a_2}5^{a_3}\dots{p_i}^{a_i}\dots{p_l}^{a_l}$.

Then $R\cong R_1\times R_2\times R_3\times\dots\times R_i\times\dots\times R_l$ *where*

(1) R_1 *is zero or* ST^n *ring with* $2^{a_1}=0$.

(2) R_2 *is zero or* ST^n *ring with* $3^{a_2}=0$. R_2 *has the identity* $(k^3-k)^{a_2}=0;3^{a_2-1}(k^3-k)=0\forall k\in R_2$. *For every* $n\in Nil(R_2)$ *we have* $n^{a_2}=$

$0; 3^{a_2-1}n = 0$. $R_2/J(R_2)$ is subdirect product of Z_3's. For every $j \in J(R_2)$ we have $j^{a_2} = 0, 3^{a_2-1}j = 0$. For every $u \in U(R_2)$ we have $3^{a_2-1}u^2 = 3^{a_2-1}$.

(3) *R_3 is zero or ST^n ring with $5^{a_3} = 0$. R_3 has the identity $(k^5 - k)^{a_3} = 0; 5^{a_3-1}(k^5 - k) = 0 \forall k \in R_2$. For every $n \in Nil(R_3)$ we have $n^{a_3} = 0; 5^{a_2-1}n = 0$. $R_2/J(R_2)$ is subdirect product of Z_5's. For every $j \in J(R_2)$ we have $j^{a_3} = 0, 5^{a_3-1}j = 0$. For every $u \in U(R_2)$ we have $5^{a_3-1}u^4 = 5^{a_3-1}$. If $a_3 = 1$ then R_3 is subdirect product of Z_5's.*

... .

(i) *R_i is zero or ST^n ring with ${p_i}^{a_i} = 0$. R_3 has the identity $(k^{p_i} - k)^{a_i} = 0; p_i^{a_i-1}(k^p - k) = 0 \forall k \in R_i$. For every $n \in Nil(R_i)$ we have $n^{a_i} = 0; p_i^{a_i-1}n = 0$. $R_i/J(R_i)$ is subdirect product of Z_{p_i}'s. For every $j \in J(R_i)$ we have $j^{a_i} = 0, p_i^{a_i-1}j = 0$. For every $u \in U(R_i)$ we have $p_i^{a_i-1}u^{p_i-1} = p_i^{a_i-1}$. If $a_i = 1$ then R_i is subdirect product of Z_{p_i}'s.*

... .

(k) *R_l is zero or ST^n ring with ${p_l}^{a_l} = 0$. R_k has the identity $(k^{p_l} - k)^{a_l} = 0; p_i^{a_l-1}(k^p - k) = 0 \forall k \in R_l$. For every $n \in Nil(R_l)$ we have $n^{a_l} = 0; p_l^{a_l-1}n = 0$. $R_l/J(R_l)$ is subdirect product of Z_{p_l}'s. For every $j \in J(R_l)$ we have $j^{a_l} = 0, p_l^{a_l-1}j = 0$. For every $u \in U(R_l)$ we have $p_l^{a_l-1}u^{p_l-1} = p_l^{a_l-1}$. If $a_l = 1$ then R_l is subdirect product of Z_{p_l}'s.*

Proof. We prove the result only for $n = 2m + 1$, $m \geq 1$. Now if $n = 2m + 1$ and R is a ST^n ring then using Corollary 5.2 we have $1.2.3.4^2.5^2.6^3.7^3.8^4.9^4 \ldots (2j + 2)^{j+1}(2j+3)^{j+1} \ldots (2m)^m(2m+1)^m(2m+2)^{m+1}(2m+3)^m(2m+4)^m(2m+5)^{m-1}(2m+6)^{m-1} \ldots (4m + 2 - 2j - 1)^{j+1}(4m + 2 - 2j)^{j+1} \ldots (4m - 1)^2(4m)^2(4m + 1)(4m + 2)(4m + 3)2^{3(2m-1)+\frac{2m(m+2)(2m-1)}{3}} = 0$, and so $2^{a_1}3^{a_2}5^{a_3} \ldots {p_i}^{a_i} \ldots {p_l}^{a_l} = 0$.

So by Chinese Remainder Theorem we have

$$R \cong R_1 \times R_2 \times R_3 \times \ldots \times R_i \times \ldots \times R_l.$$

Here $R_1 \cong \frac{R}{2^{a_1}R}$, $R_2 \cong \frac{R}{3^{a_2}R}$,. . . ,$R_i \cong \frac{R}{p_i^{a_i}R}$,. . . ,$R_l \cong \frac{R}{p_l^{a_l}R}$. Now R_2 is zero or ST^{2m+1} ring with 2^{a_1}.

Considering R_i. Now R_i is zero or ST^{2m+1} ring with ${p_i}^{a_i} = 0$. Let $k \in R_i$ so we can write $k = \sum_{i=1}^{2m+1} e_i; e_ie_j = e_je_i; i, j \in \{1, 2, \ldots, 2m + 1\}$. Now using Lemma 5.4 we have $k^{p_i} - k = p_iP(\{p_i|1 \geq i \geq (2m + 1)\})$ where $P(\{p_i|1 \geq i \geq 2m + 1\})$ is a function of $e_1, e_2, \ldots, e_{2m+1}$. Therefore $(k^{p_i} - k)^{a_i} = 0$ and ${p_i}^{a_i-1}(k^{p_i} - k) = 0$.

Now for $n \in Nil(R_i)$ we have $1 - n^{\alpha} \in U(R_i)$ for $\alpha \in \mathbb{N}$. Again for $n \in Nil(R_i)$ we have $(n^{p_i} - n)^{a_i} = 0 \Rightarrow n^{a_i}(n^{p_i - 1} - 1)^{a_i} = 0 \Rightarrow n^{a_i} = 0$. Again ${p_i}^{a_i - 1}(n^{p_i} - n) = 0 \Rightarrow {p_i}^{a_i - 1}n(n^{p_i - 1} - 1) = 0 \Rightarrow {p_i}^{a_i - 1}n = 0$. Using Lemma 5.2 we have $R_i/J(R_i)$ is subdirect product of Z_p's and $J(R_i)$ is nil. So if $j \in J(R_i)$ then $j \in Nil(R_i)$ so $j^{a_i} = 0$, ${p_i}^{a_i - 1}j = 0$. Now let $u \in U(R_i)$ we have ${p_i}^{a_i - 1}(u^{p_i} - u) = 0 \Rightarrow {p_i}^{a_i - 1}u(u^{p_i - 1} - 1) = 0 \Rightarrow {p_i}^{a_i - 1}u^{p_i - 1} = {p_i}^{a_i - 1}$.

Now if $a_i = 1$ then $p_i = 0$. Let $k \in R_i$ with$k^2 = 0$.Let $k \in R_i$ so we can write $k = \sum_{i=1}^{2m+1} e_i; e_i e_j = e_j e_i; i, j \in \{1, 2, \ldots, 2m + 1\}$. Now using Lemma 5.4 we have $k^{p_i} - k = p_i P(\{p_i | 1 \geq i \geq (2m + 1)\})$ where $P(\{p_i | 1 \geq i \geq 2m + 1\})$ is a function of $e_1, e_2, \ldots, e_{2m+1}$. Therefore $k^{p_i} - k = 0 \Rightarrow k = 0$ as $k^2 = 0$. So R_i is subdirect product of domains $\{R_\alpha\}$. But R_α has only trivial tripotents $0, 1, -1$. We infer that $R_\alpha = \{0, \pm 1, \pm 2, \ldots, \pm \frac{p_i - 1}{2}\}$. But $p_i = 0$ in R_α. So $R_\alpha \cong Z_{p_i}$. So R_i is subdirect product of Z_{p_i}'s.

Similarly we can prove for $R_2, R_3, \ldots, R_l$.

If $n = 2m + 2$ we can prove the result same as above. □

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Research Scholar
Department of Mathematics
Gauhati University
Jalukbari-781014, Guwahati, Assam
India.
Email address: kumarnapoleondeka@gmail.com

Professor
Department of Mathematics
Gauhati University
Jalukbari-781014, Guwahati, Assam,
India.
Email address: hsaikia@yahoo.com