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THE CONSECUTIVE MATRIX THEOREM - A PROPERTY OF DETERMINANTS

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ABSTRACT. The purpose of this paper is to present the proof that the determinant of any matrix of order higher than 2 ($k > 2$) where elements are consecutive numbers or numbers following an arithmetic progression will always be zero.

1. INTRODUCTION

Consider a 4 by 4 matrix with consecutive elements

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 14 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

The determinant of such matrix will always be zero. Here, this example will be expanded to a more generic format and proven for any matrix of order higher than 2 ($k > 2$).

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2. Theorem description

Let A be a k by k matrix such that

$$A = \begin{bmatrix} a_{11} & a_{12}+x & a_{13}+y & \cdots & a_{1k}+z \\ a_{21} & a_{22}+x & a_{23}+y & \cdots & a_{2k}+z \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k-1,1} & a_{k-1,2}+x & a_{k-1,3}+y & \cdots & a_{k-1,k}+z \\ a_{k1} & a_{k2}+x & a_{k3}+y & \cdots & a_{kk}+z \end{bmatrix},$$

where a_{ij} is an element a in row i and column j.

Now consider a special case of matrix A, in which all elements ï£¡a in the same row are equal such that $a_{11} = a_{12} = \cdots = a_{1k}$, $a_{21} = a_{22} = \cdots = a_{2k}$, $a_{k1} = a_{k2} = \cdots = a_{kk}$, et. c. Such matrix could be written as

$$B = \begin{bmatrix} a & a+x & a+y & \cdots & a+z \\ b & b+x & b+y & \cdots & b+z \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k & k+x & k+y & \cdots & k+z \end{bmatrix}.$$

Here, it is shown that the determinant of such matrix, $\det(B)$, is zero for any matrix of order greater than 2 ($k > 2$), and for $a_{ij}, x, y, z, \ldots \in \mathbb{R}$.

3. Lemmas

The following lemmas will be used:

Lemma 3.1. *Element Sum Property. The determinant can be expressed as the sum of two other determinants if the determinant in each element in any row or column consists of two terms.*

Lemma 3.2. *Repetition Property. The value of a determinant is zero if two rows or columns of a determinant are identical.*

Lemma 3.3. *Multiplying a row or column by a scalar. If a given matrix D is obtained from matrix C by multiplying a row or column of C by a real number r, then* $\det(C) = r \cdot \det(D)$.

Proofs for all lemmas used can be found in references [1–3].

4. Theorem Proof

Consider the k by k matrix B introduced in the theorem description. B is of order greater than 2 ($k > 2$) and $a, b, \ldots, k, x, y, \ldots, z \in \mathbb{R}$ such that

$$B = \begin{bmatrix} a & a+x & a+y & \cdots & a+z \\ b & b+x & b+y & \cdots & b+z \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k & k+x & k+y & \cdots & k+z \end{bmatrix}.$$

The determinant of B can be calculated as

$$\det(B) = \begin{vmatrix} a & a+x & a+y & \cdots & a+z \\ b & b+x & b+y & \cdots & b+z \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k & k+x & k+y & \cdots & k+z \end{vmatrix}.$$

According to the Lemma 3.1, $\det(B)$ can be decomposed into two terms

$$\therefore \det(B) = \begin{vmatrix} a & a & a+y & \cdots & a+z \\ b & b & b+y & \cdots & b+z \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k & k & k+y & \cdots & k+z \end{vmatrix} + \begin{vmatrix} a & x & a+y & \cdots & a+z \\ b & x & b+y & \cdots & b+z \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k & x & k+y & \cdots & k+z \end{vmatrix}.$$

The first term has two identical columns; thus, such determinant is zero according to Lemma 3.2, and

$$\therefore \det(B) = 0 + \begin{vmatrix} a & x & a+y & \cdots & a+z \\ b & x & b+y & \cdots & b+z \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k & x & k+y & \cdots & k+z \end{vmatrix}.$$

According to the lemma 3.1, $\det(B)$ can be decomposed again into two terms

$$\therefore \det(B) = \begin{vmatrix} a & x & a & \cdots & a+z \\ b & x & b & \cdots & b+z \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k & x & k & \cdots & k+z \end{vmatrix} + \begin{vmatrix} a & x & y & \cdots & a+z \\ b & x & y & \cdots & b+z \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k & x & y & \cdots & k+z \end{vmatrix}.$$

The first term has two identical columns; thus such determinant is zero according to Lemma 3.2, and

$$\therefore \det(B) = 0 + \begin{vmatrix} a & x & y & \cdots & a+z \\ b & x & y & \cdots & b+z \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k & x & y & \cdots & k+z \end{vmatrix}.$$

The second term has a column in which all elements are identical and equal to x, and one in which all elements are identical and equal to y. According to Lemma 3.1, the $\det(B)$ can be written as

$$\therefore \det(B) = xy \begin{vmatrix} a & 1 & 1 & \cdots & a+z \\ b & 1 & 1 & \cdots & b+z \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ k & 1 & 1 & \cdots & k+z \end{vmatrix}.$$

The determinant now has two identical columns, thus the determinant is zero according to Lemma 3.2, and

$$\therefore \det(B) = xy \cdot 0 = 0.$$

Note that a matrix of consecutive numbers falls into the generic case proposed, as does the case of any matrix of arithmetic progression. E.g.,

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

can be written in the generic form with $x = 1$ and $y = 2$ as

$$\begin{vmatrix} a & a+x & a+y \\ b & b+x & b+y \\ c & c+x & c+y \end{vmatrix} = \begin{vmatrix} 1 & 1+1 & 1+2 \\ 4 & 4+1 & 4+2 \\ 7 & 7+1 & 7+2 \end{vmatrix}.$$

This was proven to be equal zero.

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