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CENTRALIZERS IN THE FIRST WEYL ALGEBRA OVER A 2 OR 3 - CHARACTERISTIC FIELD

Bah S.B. Kouame[1] and Konan M. Kouakou

ABSTRACT. The purpose of this paper is the determination of some centralizers in A_1, the first Weyl Algebra. Some authors have done their studies in the case of zero characteristic field. As far as we're concerned, we have decided to work in 2 or 3 characteristic field. Doing so, we show that if $u \in A_1$ is a minimal element, C-primitive and without constant term, then its centralizer $Z(u) = \mathbb{L}[u] \cap A_1$ where $\mathbb{L}$ is the fractions field of C, the center of A_1. Particularly, when u is ad-invertible, i.e there exists $v \in A_1$ such that $[u, v] = 1$, then we have $Z(u) = C[u]$ which is a result analogous to that of [2].

1. INTRODUCTION

The first Weyl algebra A_1 over a field k is the unital, associative k-algebra generated by two elements x, y with the only commutation relation $[y, x] = 1$. Introduced by Hermann Weyl (1928) in order to study the Heisenberg uncertainty principle in quantum mechanics, a large description of A_1 was done in [1] and [3].

In this paper, we consider A_1 as the first Weyl algebra over a p-characteristic field k and for an element u in A_1, we denote $Z(u)$ its centralizer in A_1, $C = k[x^p, y^p]$ is the center of A_1 and $\mathbb{L}$ the fractions field of C.

[1]*corresponding author*

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In the first section, we give some general and useful results about the centralizers. These results allow us to reduce their studies in the following section. In the second section, we clearly determine the structure of the centralizers in 2 or 3 characteristic field.

It is worth noting that this study opens up the vast area of research on centralizers in nonzero characteristic field in a general context.

2. Preliminaries

In this section, we recall that k is a p-characteristic field and we present some definitions, notations and basic properties which are necessary for the study of centralizers in the first Weyl algebra. $\mathbb{L}$ is the fractions field of C, the center of A_1.

Definition 2.1. *Let u be an element of A_1. The centralizer of u, denoted $Z(u)$, is the subalgebra of A_1 consisting of elements in A_1 that commute with u. Thus, $Z(u) := \{v \in A_1 : [u,v] = uv - vu = 0\}$.*

Definition 2.2. *Let $u \in A_1$.*

- *u is called C-primitive if for all $(a,v) \in C \times A_1$, $u = av$ implies $a = 1$.*
- *u is called without constant term if for all $(a,v) \in C \times A_1$, $u = v + a$ implies $a = 0$.*

The following propositions allow us to consider only the centralizers of C-primitive and non constant term elements.

Proposition 2.1. *Let $u \in A_1$ and $a, b \in C$ with $a \neq 0$. We have $Z(au+b) = Z(u)$.*

Proof. Let $v \in Z(u)$. Then, $[v, au+b] = a[v,u] = 0$ is equivalent to $[v,u] = 0$. Thus, the result follows. □

Example 1. *$Z(a) = A_1$ for any $a \in C$. Let $P \in C[X]$ such that $P \notin C$. Then,*

- $Z(P(x)) = C[x]$
- $Z(P(y)) = C[y]$.

Proposition 2.2. *Let $u \in A_1$ and $\varphi \in End_k(A_1)$. We have:*

(1) $\varphi(Z(u)) \subset Z(\varphi(u))$

(2) *If φ is an automorphism, then we have the equality.*

Proof.

(1) Let $v \in Z(u)$. We have $[\varphi(v), \varphi(u)] = \varphi([v,u]) = \varphi(0) = 0$. This implies that $\varphi(v) \in Z(\varphi(u))$ i.e $\varphi(Z(u)) \subset Z(\varphi(u))$.

(2) Let $v \in Z(\varphi(u))$. Since φ is an automorphism, there exists $v' \in A_1$ such that $v = \varphi(v')$. On a $[v, \varphi(u)] = [\varphi(v'), \varphi(u)] = \varphi([v', u]) = 0$. As φ is injective, we conclude that $[v', u] = 0$ i.e $v' \in Z(u)$. Therefore $v \in \varphi(Z(u))$. Hence, $Z(\varphi(u)) \subset \varphi(Z(u))$. Combining with (1), we obtain the equality $\varphi(Z(u)) = Z(\varphi(u))$.

□

Lemma 2.1. *Let $u, v \in A_1$. Set $a = [v,u]$. For any $i \in \mathbb{N}^*$, we have*

$$[v, u^i] = \sum_{s=1}^{i} u^{i-s} a u^{s-1}.$$

Proof. We use induction on $i \geq 1$.

For $i = 1$, the result is immediate.

For $i = 2$, we have $[v, u^2] = ua + au = u^{2-1}au^{1-1} + u^{2-2}au^{2-1}$.

Now, assume that the result is true for an integer $i \geq 2$ and prove it for $i+1$. We have

$$\begin{aligned}[v, u^{i+1}] = u[v, u^i] + au^i &= \left(\sum_{s=1}^{i} u^{i+1-s} a u^{s-1}\right) + u^{i+1-(i+1)} a u^{(i+1)-1} \\ &= \sum_{s=1}^{i+1} u^{i+1-s} a u^{s-1}.\end{aligned}$$

Hence, the result holds for all $i \in \mathbb{N}^*$. □

Proposition 2.3. *Let $u, v \in A_1$, $1 \leq i \leq p-1$ such that $u^i \notin C$. We have $[v, u^i] = 0 \Longleftrightarrow [v,u] = 0$.*

Proof. $[v,u] = 0 \Longrightarrow [v, u^i] = 0$ is immediate. Now, suppose that $[v, u^i] = 0$. Set $a = [v,u]$. According to lemma 2.1, we have

$$[v, u^i] = \sum_{s=1}^{i} u^{i-s} a u^{s-1} = \sum_{s=1}^{i} \left(u^{i-1}a + u^{i-s}[a, u^{s-1}]\right) = iu^{i-1}a + \sum_{s=1}^{i} u^{i-s}[a, u^{s-1}].$$

Let $1 \leq s \leq i$. Notice that $u^{i-1}a = u^{i-s}u^{s-1}a$.

If $a \neq 0$, then $d_x^o(u^{s-1}a) > d_x^o([a, u^{s-1}])$ for all $1 \leq s \leq i$ where $d_x^o(u^{s-1}a)$ and $d_x^o([a, u^{s-1}])$ are respectively the degree in x of $u^{s-1}a$ and the degree in x of $[a, u^{s-1}]$.

Consequently, $d_x^o(iu^{i-1}a) > d_x^o\left(\sum_{s=1}^{i} u^{i-s}[a, u^{s-1}]\right)$. Hence, $[v, u^i] \neq 0$, which completes the proof. $\square$

Definition 2.3. *For $u \in A_1$, we define $u^+ := x^p u$ that we call the positive part of u. Notice with this definition that $u = x^{-p}u^+$ where $x^{-p} \in \mathbb{L}$. Thus, we denote A_1^+, the set of positive parts of A_1's elements. In other words, $A_1^+ = x^p A_1$.*

Lemma 2.2. *Let $j \in \mathbb{N}$ and $f(xy) = a_0 + a_1xy + \ldots + a_n(xy)^n$ where the $a_i \in \mathbb{L}$. Set $f(+j) = f(xy + j) := a_0 + a_1(xy + j) + \ldots + a_n(xy + j)^n$. Then*

$$f(xy)x^j = x^j f(xy + j) \quad \text{and} \quad y^j f(xy) = f(xy + j)y^j.$$

Proof. Let $f(xy) = a_0 + a_1xy + \ldots + a_n(xy)^n \in \mathbb{L}[xy]$.

By induction on $i \geq 0$, we show that $(xy)^i x = x(xy+1)^i$. Then, we use induction on $j \geq 0$ to show that $f(xy)x^j = x^j f(xy + j)$.

Similarly, we prove the second result $y^j f(xy) = f(xy + j)y^j$, using induction too. $\square$

The following theorem is very important in the sequel since it reduces the study of centralizers from A_1 to A_1^+.

Theorem 2.1. *Presentation of elements of A_1^+. All element $u \in A_1^+$ can be uniquely written in the form $u = x^{p-1}u_{p-1} + \ldots + xu + u_0$ where $u_i \in C[xy]$. In other words, A_1^+ is a finitely generated free module over $C[xy]$.*

Proof.

Existence of the form.

Let $u = x^p \sum a_{ij}x^i y^j \in A_1^+$ with $a_{ij} \in C$ for all $0 \leq i, j \leq p - 1$. First, note that $y = x^{-1}(xy)$. Using lemma 2.2, we have $y^2 = x^{-1}(xy)y = x^{-1}y(xy - 1) = x^{-2}(xy)(xy - 1)$. Assume that $y^j = x^{-j}(xy)(xy - 1) \ldots (xy - (j - 1))$ for $j \geq 2$. We have

$$\begin{aligned} y^{j+1} &= x^{-j}(xy)(xy - 1) \ldots (xy - (j - 1))y \\ &= x^{-j}y(xy - 1) \ldots (xy - j) \\ &= x^{-(j+1)}(xy)(xy - 1) \ldots (xy - j). \end{aligned}$$

Hence, $x^i y^j = x^{i-j} H_j$ where $H_j = (xy)(xy-1)\ldots(xy-(j-1))$.

Now, consider $x^p x^i y^j = x^{p+i-j} H_j$ for $i-j \in \{-(p-1), \ldots, 0\}$ and $x^p x^i y^j = x^{i-j} x^p H_j$ for $i-j \in \{0, \ldots, p-1\}$. This justifies the existence of the form.

Uniqueness of the form.

Indeed, assume that $u = x^{p-1}u_{p-1} + \ldots + xu_1 + u_0 = 0$ and demonstrate that the u_i are zeros. Taking the commutator with xy, we obtain $[xy, x^{p-1}]u_{p-1} + \ldots + [xy, x]u_1 = 0$. We have $[xy, x^n] = nx^n$ for any whole number n. Thus, $(p-1)x^{p-1}u_{p-1} + \ldots + xu_1 = 0$. Factoring out x, we get $(p-1)x^{p-2}u_{p-1} + \ldots + 2xu_2 + u_1 = 0$. Continuing this process iteratively, we have $u_{p-1} = 0$, and then $u_i = 0$ for all $i \in \{0, \ldots, p-1\}$. Hence, the result follows. $\square$

Corollary 2.1. *Let $u, v \in A_1$. Then:*

$$[v, u] = 0 \Longleftrightarrow [v^+, u^+] = 0.$$

Proof. We have $[v^+, u^+] = [x^p v, x^p u] = x^{2p}[v, u]$. Since $x^{2p} \neq 0$ and A_1 is a domain, then, the result is immediate. $\square$

Remark 2.1. *Determining the centralizer of an element $u \in A_1$ is equivalent to determining the centralizer of $u^+ \in A_1^+$.*

Definition 2.4. *Let $u = x^{p-1}u_{p-1} + \ldots + xu_1 + u_0 \in A_1^+$ with the $u_i \in C[xy]$. We define the p-degree of u as the greatest integer $i \in \{0, \ldots, p-1\}$ such that $u_i \neq 0$. And this degree will be denoted $d_p^0(u)$. Any element u is said to be minimal if $u \notin C$ and there is no non-central element in its centralizer with a p-degree lower than the p-degree of u.*

Proposition 2.4. *Let $u \in A_1$ such that $u \notin C$. If $d_p^o(u) = 0$, then $Z(u) = C[xy]$.*

Proof. It is sufficient to search $Z(xy)$. Let $v = x^{p-1}v_{p-1} + \ldots + xv_1 + v_0 \in Z(xy)$ where $v_i \in C[xy]$. We have $[v, xy] = [x^{p-1}, xy]\, v_{p-1} + \ldots + [x, xy]\, v_1 = 0$. That implies $v_i = 0$ for all $1 \leq i \leq p-1$. Then, $v = v_0 \in C[xy]$. And, we have the final result. $\square$

Proposition 2.5. *Let $Q \in \mathbb{L}[xy]$. Then*

$$Q(xy+1) = Q(xy) \Longrightarrow Q \in \mathbb{L}.$$

Proof. Set $Q = \alpha_0 + \alpha_1 xy + \ldots + \alpha_n (xy)^n \in \mathbb{L}[xy]$. We can rewrite Q uniquely as $Q = a_0 + a_1 xy + \ldots + a_{p-1}(xy)^{p-1}$ with the $a_i \in \mathbb{L}$.

We also have, $Q(X+1) = a_0 + a_1(xy+1) + \ldots + a_{p-1}(xy+1)^{p-1}$. Then, $Q(X+1) = Q(X)$ implies $a_{p-1} = 0$. By rewriting Q and proceeding in the same way, we obtain $Q = a_0 \in \mathbb{L}$. $\Box$ $\Box$

Remark 2.2. *For all* $Q(xy) \in \mathbb{L}[xy]$*,* $Q(xy) + Q(xy+1) + \ldots + Q(xy+p-1)$ *and* $Q(xy)Q(xy+1)\ldots Q(xy+p-1)$ *are elements of* $\mathbb{L}$*.*

Proposition 2.6. *We have:*

(1) $\mathbb{L}[x] \cap \mathbb{L}[y] = \mathbb{L}$
(2) $\mathbb{L}[x] \cap \mathbb{L}[xy] = \mathbb{L}$
(3) $\mathbb{L}[y] \cap \mathbb{L}[xy] = \mathbb{L}$

Proof. (1) Let $Q \in \mathbb{L}[x] \cap \mathbb{L}[y]$ such that $Q = \alpha_0 + \alpha_1 x + \ldots + \alpha_{p-1}x^{p-1} = \beta_0 + \beta_1 y + \ldots + \beta_{p-1}y^{p-1}$ with $\alpha_i, \beta_i \in \mathbb{L}$. We have $[y, Q] = \alpha_1 + \ldots + (p-1)\alpha_{p-1}x^{p-2} = 0$. Then, for all $0 \leq i \leq p-1$, $\alpha_i = 0$. Hence, $Q = \alpha_0 \in \mathbb{L}$. That ends the proof.

For (2) and (3), the demonstrations are analogous to (1). $\Box$

3. Structure of the centralizers when the characteristic $carac(k) \in \{2, 3\}$

In this section, we provide a deep description of the centralizer of an element in 2 or 3 characteristic field.

Lemma 3.1. *Let* $u, v \in A_1$*,* $a, b \in C$ *such that* $gcd(a, b) = 1$*. If* $au = bv$ *then* b *divides* u *and* a *divides* v *.*

Proof. Considering A_1 as a C-module, set

$$u = \sum_{0 \leq i,j \leq p-1} a_{ij} x^i y^j$$

and

$$v = \sum_{0 \leq i,j \leq p-1} b_{ij} x^i y^j,$$

with $a_{ij}, b_{ij} \in C = k[x^p, y^p]$. We have

$$au = \sum_{0 \leq i,j \leq p-1} a a_{ij} x^i y^j$$

and

$$bv = \sum_{0 \le i,j \le p-1} bb_{ij} x^i y^j.$$

Thus, $aa_{ij} = bb_{ij}$ for all i, j. Since C is a factorial domain, b divides a_{ij} for all i, j. Consequently, b divides u. Similarly, we also have a divides v. □

Case of characteristic 2

Remark 3.1. *If $u = ax + u_0$ where $a \in C$ and $u_0 \in C[xy]$, then $Z(u) = \mathbb{L}[u] \cap A_1$. In fact, for $v = xv_1 + v_0 \in Z(u)$, we obtain $v_1 \in C$. That implies the requested result.*

Proposition 3.1. *Let $u \in A_1$ be a minimal element. Then, $Z(u) = \mathbb{L}[u] \cap A_1$.*

Proof. $C[xy]$ is a free C-module of rank 2, with $\{1, xy\}$ as a basis. If $d_2^o(u) = 0$, then $Z(u) = C[xy]$ acoording to Proposition 2.4. Let $d_2^o(u) = 1$. On one hand, if $u = ax + u_0$ where $a \in C$, then we have immediately the result according to Remark 3.1. Otherwise, set $u = xu_1 + u_0$. Let $v = xv_1 + v_0 \in Z(u)$ where $v_i \in C[xy]$ for all $i = 0, 1$. Then, $v_1, u_1, 1$ are linearly dependent, i.e there exist $\alpha, \beta \in \mathbb{L}$ such that $v_1 = \alpha u_1 + \beta$. Therefore, $v = \alpha u + \lambda$ where $\lambda \in \mathbb{L}$. This completes the proof. □

Remark 3.2. *If $u = xu_1 + u_0 \in A_1$ with the $u_i \in C[xy]$ such that $u \notin C[xy]$, then we have $u^2 = au + b$ where $a, b \in C$.*

Corollary 3.1. *Let $u \in A_1$ such that $u \notin C$. Then, there exits $\bar{u} \in Z(u)$ such that $Z(u) = C[\bar{u}]$.*

Proof. It is sufficient to choose a minimal, C-primitive and a non constant term $\bar{u}$ in $Z(u)$. And by using Proposition 3.1 and Remark 3.2, we get the result. □

Case of characteristic 3

In the sequel, we only consider the case of 3 characteristic field since 2 characteristic is entirely solved.

Proposition 3.2. *If $u = x^2 a + xu_1 + u_0$ is minimal with $a \in C$ such that $a \neq 0$ and the $u_i \in C[xy]$, then $Z(u) = \mathbb{L}[u] \cap A_1$.*

Proof. Let $v \in Z(u)$. Set $w = \dfrac{u}{a}$. We have $[v, u] \iff [v, w]$.

Set $w = x^2 + xl_1 + l_0$ where $l_i = \dfrac{u_i}{a} \in \mathbb{L}[xy]$. Set $d(w) = x^3 - l_1 l_1(+1) l_1(+2)$. Notice that $l_1 l_1(+1) l_1(+2) \in \mathbb{L}$ according to Remark 2.2. It suffices to show that $d(w) \neq 0$ and then, we have $v \in \mathbb{L}[w] = \mathbb{L}[u]$.

To show it, suppose that $d(w) = 0$. Then $l_1 l_1(+1) l_1(+2) = x^3$ and $d_y^o(l_1) = 0$. Hence, $l_1 \in \mathbb{L}$. Consequently, $l_1 l_1(+1) l_1(+2) = (l_1)^3 = x^3$ implies $l_1 = \lambda x$ with $\lambda \in k$ such that $\lambda \neq 0$. This contredits the fact that $l_1 \in \mathbb{L}[xy]$. So, $d(w) \neq 0$ and we obtain our result. □

Corollary 3.2. *Let $u \in A_1$ such that u is minimal. Then, $Z(u) = \mathbb{L}[u] \cap A_1$.*

Proof. $C[xy]$ is a free module over C of rank 3, with $\{1, xy, (xy)^2\}$ as a basis. If $d_3^o(u) = 0$, then, according to Proposition 2.4, $Z(u)$ is still entirely known. Let $d_3^o(u) \in \{1, 2\}$. On one hand, if $u = x^2 a + xu_1 + u_0$ where $a \in C$, then we have the result according to Proposition 3.2. Otherwise, set $u = x^2 u_2 + xu_1 + u_0$ and $u^2 = x^2 h_2 + xh_1 + h_0$ with $u_i, h_i \in C[xy]$. Let $v = x^2 v_2 + xv_1 + v_0 \in Z(u)$. Then, $1, u_2$, h_2 and v_2 are linearly dependent. In other words, there exist $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{L}$ such that $v_2 = \alpha_2 h_2 + \alpha_1 u_2 + \alpha_0$. Therefore, $v = \alpha_2 u^2 + \alpha_1 u + \alpha$ where $\alpha \in \mathbb{L}$ too. Hence, $Z(u) = \mathbb{L}[u] \cap A_1$. □

Remark 3.3. *$Z(u)$ is still a commutative algebra in 2 or 3 characteristic field as in the case of zero characteristic field. See in [3].*

Corollary 3.3. *Let $u, v \in A_1 \setminus C$. If $Z(u) \neq Z(v)$, then $Z(u) \cap Z(v) = C$.*

Proof. Let $d \in Z(u) \cap Z(v)$ such that $d \notin C$. Then $u, v \in Z(d)$. According to Remark 3.3, $[u, v] = 0$. That contradicts the hypothesis. □

Definition 3.1. *Any element $e \in C$ such that $e \neq 0$ is called central irreductible element if for all $a, b \in C$, $e = ab$ implies $a \in k$ or $b \in k$. For examples: x^3, y^3 and all nonzero scalars are central irreductible elements.*

Remark 3.4. *Let $u, v \in A_1$ such that $[u, v] \in C$. For all $w \in Z(u)$, $[v, w] \in Z(u)$. It suffices to show that $[[v, w], u] = 0$ by using Jacobi Identity.*

Corollary 3.4. *If there exist $u, v \in A_1$ such that $[u, v]$ is a central irreductible element, then $Z(u) = C[u]$ and $Z(v) = C[v]$.*

Proof. Set $[u, v] = e$. Let $P = \alpha_2 u^2 + \alpha_1 u + \alpha_0 \in Z(u)$ where $\alpha_i \in \mathbb{L}$ for all $0 \leq i \leq 2$.

If $P = \alpha_1 u + \alpha_0$, then from Remark 3.4, we have $[P, v] \in Z(u)$, that is $\alpha_1 e \in Z(u)$ and then, since e is irreductible and by using Lemma 3.1, we obtain $\alpha_1 \in C$. Therefore, $\alpha_0 \in C$.

Let $P = \alpha_2 u^2 + \alpha_1 u + \alpha_0$. We have $[P, v] = -\alpha_2 eu + \alpha_1 e \in Z(u)$. From precedent result, we have $\alpha_2, \alpha_1 \in C$. And then, $\alpha_0 \in C$. Hence, $Z(u) = C[u]$. Similarly, we show that $Z(v) = C[v]$. □

Remark 3.5. *When u is ad-invertible, i.e there exists $v \in A_1$ such that $[u, v] = 1$, then we have immediately $Z(u) = C[u]$ since it is a particular case of Corollary 3.4. Hence, the result given in zero characteristic field by Jorge A. Guccione, Juan J. Guccione and Christian Valqui [2] is recovered.*

At the end, we discovered that, in 2 or 3 characteristic field, the centralizers are still monomial algebras.

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Laboratory of Mathematics and Applications
University Félix Houphouët Boigny
Abidjan,
Côte d'Ivoire.
Email address: lesagebene@gmail.com

Laboratory of Mathematics and Applications
University Félix Houphouët Boigny
Abidjan,
Côte d'Ivoire.
Email address: makonankouakou@yahoo.fr