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ON SUB-DEFECT OF HADAMARD PRODUCT OF DOUBLY SUBSTOCHASTIC MATRICES

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ABSTRACT. The *sub-defect* of A, defined as $\mathrm{sd}(A) = \lceil n - \mathrm{sum}(A) \rceil$, is the minimum number of rows and columns required to be added to transform a doubly substochastic matrix into a doubly stochastic matrix. Here, n is the size of A and sum(A) is the sum of all entries of matrix A. In this paper, we show that for arbitrary doubly substochastic matrices A and B, the Hadamard product $A \circ B$ is also a doubly substochastic matrix, and $\max\{sd(A), sd(B)\} \leq sd(A \circ B) \leq \max\{n, sd(A) + sd(B)\}$.

1. INTRODUCTION

Definition 1.1. *A matrix $A = (a_{ij})$ of size $n \times n$ is called a* doubly substochastic matrix *if the following conditions are satisfied:*

(1) *All elements of A are non-negative: $a_{ij} \geq 0$ for all $1 \leq i, j \leq n$.*
(2) *The sum of the elements in each row is less than or equal to 1: $\sum_{j=1}^{n} a_{ij} \leq 1$ for all $1 \leq i \leq n$.*
(3) *The sum of the elements in each column is less than or equal to 1: $\sum_{i=1}^{n} a_{ij} \leq 1$ for all $1 \leq j \leq n$.*

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Doubly substochastic matrices are an extension of the well-known doubly stochastic matrices and have found applications in numerous fields, such as probability theory, operations research, and optimization. A basic foundation can be found in [4], by Marshall and Olkin. This study aims to investigate the algebraic properties of doubly substochastic matrices, with a particular emphasis on the Hadamard product operation.

The Hadamard product, denoted by the $\circ$ symbol, is an element-wise multiplication of two matrices of the same size. While the Hadamard product has been extensively studied in the context of doubly stochastic matrices, its behavior when applied to doubly substochastic matrices remains relatively unexplored. The following definition was first introduced by Cao and Koyuncu in [1].

Definition 1.2. *Let A be an $n \times n$ doubly substochastic matrix with elements (a_{ij}) for $1 \leq i, j \leq n$. Denote the sum of all elements of A by sum(A), i.e.,*

$$\textit{sum}(A) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}.$$

The sub-defect *of A is defined as*

$$\mathrm{sd}(A) = \lceil n - \textit{sum}(A) \rceil,$$

where $\lceil x \rceil$ is the ceiling of x, that is, the smallest integer greater than or equal to x.

More details on the sub-defect of doubly substochastic matrices can be found in ([1, 2]). The main contribution of [1] is that the sub-defect of any doubly substochastic matrix can be calculated by taking the ceiling of the difference between its size and the sum of all entries. This number (the sub-defect) gives us the minimum number of rows and columns that is required to complete a doubly substochastic matrix into a stochastic matrix. In [2], some properties of the sub-defect of doubly substochastic matrices were established with respect to usual multiplication. More work on minimal completion and the sub-defect can be found in ([5,6]).

In this paper, we revisit the sub-defect property of a doubly substochastic matrix, which allows for partitioning the set of doubly substochastic matrices into convex subsets. We first show that the Hadamard product of two doubly substochastic matrices belongs to the set of doubly substochastic matrices. We then establish lower and upper bounds for the Hadamard product of two doubly substochastic matrices.

Let Ω_n denote the set of doubly stochastic matrices, ω_n denote the set of doubly substochastic matrices, and $\omega_{n,k}$ denote the set of all doubly substochastic matrices with sub-defect k.

2. Bounds for doubly substochastic matrices under the Hadamard product

Lemma 2.1. *Let $A, B \in \omega_n$. Then the set $\mathcal{C} = \{A \circ B \mid A, B \in \omega_n\}$ is convex.*

Proof. Let $A, B \in \omega_n$. To show that the set $\mathcal{C}$ is convex, we need to prove that for any $t \in [0,1]$, and for any matrices $A_1, A_2, B_1, B_2 \in \omega_n$,

$$t(A_1 \circ B_1) + (1-t)(A_2 \circ B_2) \in \omega_n.$$

First, note that since $A_1, A_2, B_1, B_2 \in \omega_n$, all their elements are non-negative. Therefore, all the elements of $t(A_1 \circ B_1)$ and $(1-t)(A_2 \circ B_2)$ are non-negative as well. Thus, their sum also has only non-negative elements.

Now, we first check the row and column sums. For any $1 \leq i, j \leq n$:

$$\sum_{j=1}^{n} [t(A_1 \circ B_1)_{ij} + (1-t)(A_2 \circ B_2)_{ij}] = t\sum_{j=1}^{n}(A_1 \circ B_1)_{ij} + (1-t)\sum_{j=1}^{n}(A_2 \circ B_2)_{ij}.$$

Using the properties of the Hadamard product, we can rewrite the sums as:

$$t\sum_{j=1}^{n}(a_1)_{ij}(b_1)_{ij} + (1-t)\sum_{j=1}^{n}(a_2)_{ij}(b_2)_{ij}.$$

Since $A_1, A_2, B_1, B_2 \in \omega_n$, we know that:

$$\sum_{j=1}^{n}(a_1)_{ij} \leq 1, \quad \sum_{j=1}^{n}(a_2)_{ij} \leq 1, \quad \sum_{j=1}^{n}(b_1)_{ij} \leq 1, \quad \sum_{j=1}^{n}(b_2)_{ij} \leq 1.$$

Therefore, the row sums of the resulting matrix are also less than or equal to 1. A similar argument can be made for the column sums.

Hence, we have shown that for any $t \in [0,1]$ and for any matrices $A_1, A_2, B_1, B_2 \in \omega_n$, the matrix $t(A_1 \circ B_1) + (1-t)(A_2 \circ B_2)$ is also in ω_n. □

Lemma 2.2. *Let $A, B \in \omega_n$. Then the set $\mathcal{H} = \{A \circ B \mid A, B \in \omega_n\}$ is compact.*

Proof. To show that the set $\mathcal{H}$ is compact, we need to demonstrate that it is both closed and bounded.

We will use the Frobenius norm to show that the set $\mathcal{H}$ is bounded. The Frobenius norm of a matrix M is defined as

$$\|M\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}|m_{ij}|^2}.$$

Let $C = A \circ B \in \mathcal{H}$, with $A, B \in \omega_n$. Since $0 \leq a_{ij}, b_{ij} \leq 1$ for all $1 \leq i, j \leq n$, we have $0 \leq c_{ij} = a_{ij}b_{ij} \leq 1$. Then,

$$\|C\|_F = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ij}|^2} \leq \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}1^2} = \sqrt{n^2} = n.$$

Therefore, the set $\mathcal{H}$ is bounded with respect to the Frobenius norm.

Now, let $\{A_k \circ B_k\}_{k=1}$ be a sequence of matrices in $\mathcal{H}$ that converges to some matrix C. We need to show that $C \in \mathcal{H}$. Since $A_k, B_k \in \omega_n$, we know that their row and column sums are less than or equal to 1. As $\{A_k \circ B_k\}_{k=1}$ converges to C, the row and column sums of C must also be less than or equal to 1. Moreover, the elements of C are non-negative, since the elements of $A_k \circ B_k$ are non-negative for all k. Thus, $C \in \omega_n$, and the set $\mathcal{H}$ is closed.

Since $\mathcal{H}$ is both bounded and closed, it is compact. $\square$

Lemma 2.3. *([2]) Let $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ be two sets of real numbers with $0 \leq a_i \leq 1$ and $0 \leq b_i \leq 1$ for $i = 1, 2, \ldots, n$. If $\sum_{i=1}^n a_i = S_1$ and $\sum_{i=1}^n b_i = S_2$, then*

$$\sum_{i=1}^{n} a_i b_i \geq S_1 + S_2 - n.$$

Theorem 2.1. *If $A, B \in \omega_n$, then*

$$\max\{sd(A), sd(B)\} \leq sd(A \circ B) \leq \max\{n, sd(A) + sd(B)\},$$

where $\circ$ is the Hadamard product.

Proof. For the lower bound, we need to show that:

$$\max\{sd(A), sd(B)\} \leq sd(A \circ B).$$

Given $A, B \in \omega_n$, their Hadamard product $A \circ B$ is also in ω_n, and each element $(A \circ B)_{ij} = a_{ij} \cdot b_{ij}$ is less than or equal to both a_{ij} and b_{ij}.

Let $m = \max\{\text{sum}(A), \text{sum}(B)\}$. Then we have:

$$\text{sum}(A \circ B) \leq m.$$

Since m is the sum of the matrix with the smaller sum, $n - \text{sum}(A \circ B)$ will be greater than or equal to $n - m$. Therefore:

$$sd(A \circ B) = \lceil n - \text{sum}(A \circ B) \rceil \geq \lceil n - m \rceil = \min\{sd(A), sd(B)\}$$

For the upper bound, we need to show that:

$$sd(A \circ B) \leq \max\{n, sd(A) + sd(B)\}$$

Since $sd(A) = \lceil n - \text{sum}(A) \rceil$,

$$n - \text{sum}(A) \leq sd(A) \leq n - \text{sum}(A) + 1,$$

which implies

$$n - sd(A) \leq \text{sum}(A) \leq n - sd(A) + 1. \tag{2.1}$$

Similarly,

$$n - sd(B) \leq \text{sum}(B) \leq n - sd(B) + 1. \tag{2.2}$$

By (2.1), (2.2) and Lemma 2.3,

$$\begin{aligned} \text{sum}(A \circ B) &\geq \text{sum}(A) + \text{sum}(B) - n \geq (n - sd(A)) + (n - sd(B)) - n \\ &= n - sd(A) - sd(B). \end{aligned}$$

Hence $sd(A \circ B) = \lceil n - \text{sum}(\text{A} \circ \text{B}) \rceil \leq n - (n - sd(A) - sd(B)) = sd(A) + sd(B)$. $\square$

Example 1. *Consider two 2×2 doubly substochastic matrices A and B:*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Their Hadamard product $A \circ B$ is:

$$A \circ B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The sums of matrices A, B, and $A \circ B$ are:

$$\textit{sum}(A) = \textit{sum}(B) = 2, \quad \textit{sum}(A \circ B) = 0.$$

The sub-defects are calculated as:

$$sd(A) = \lceil 2 - 2 \rceil = 0, \quad sd(B) = \lceil 2 - 2 \rceil = 0, \quad sd(A \circ B) = \lceil 2 - 0 \rceil = 2.$$

Therefore, we have:

$$sd(A) = 0, \quad sd(B) = 0, \quad sd(A \circ B) = 2.$$

Example 2. *Consider two 3×3 doubly substochastic matrices A and B:*

$$A = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Their Hadamard product $A \circ B$ is:

$$A \circ B = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0.25 & 0.25 & 0 \\ 0.25 & 0.25 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The sums of matrices A, B, and $A \circ B$ are:

$$\textit{sum}(A) = \textit{sum}(B) = 2, \quad \textit{sum}(A \circ B) = 1.$$

The sub-defects are calculated as follows for a 3×3 matrix:

$$sd(A) = \lceil 3 - 2 \rceil = 1, \quad sd(B) = \lceil 3 - 2 \rceil = 1, \quad sd(A \circ B) = \lceil 3 - 1 \rceil = 2.$$

Therefore, we have:

$$sd(A) = 1, \quad sd(B) = 1, \quad sd(A \circ B) = 2.$$

Corollary 2.1. *If $A, B \in \omega_n$, then*

$$sd(A \circ B) \geq sd(AB).$$

Corollary 2.2. *If $A \in \omega_{n,k}$ and $B \in \Omega_n$, then $A \circ B \in \omega_{n,k}$.*

Proof. Let $A \in \omega_{n,k}$ be an $n \times n$ doubly substochastic matrix with sub-defect k, and let $B \in \Omega_n$ be an $n \times n$ doubly stochastic matrix. We want to show that the Hadamard product $A \circ B$ is an $n \times n$ doubly substochastic matrix in $\omega_{n,k}$.

First, we show that the Hadamard product $A \circ B$ is a doubly substochastic matrix. Recall that a matrix is doubly substochastic if the sum of its elements in each row and each column is less than or equal to 1. Since A is a doubly substochastic matrix and B is a doubly stochastic matrix, we have:

$$\sum_{i=1}^{n} a_{ij} \leq 1, \quad \sum_{j=1}^{n} a_{ij} \leq 1, \quad \sum_{i=1}^{n} b_{ij} = 1, \quad \text{and} \quad \sum_{j=1}^{n} b_{ij} = 1.$$

Now, consider the Hadamard product $A \circ B = (c_{ij})_{i,j=1}^{n}$, where $c_{ij} = a_{ij}b_{ij}$. We need to show that $\sum_{i=1}^{n} c_{ij} \leq 1$ and $\sum_{j=1}^{n} c_{ij} \leq 1$.

For the row sums of $A \circ B$:

$$\sum_{i=1}^{n} c_{ij} = \sum_{i=1}^{n} a_{ij}b_{ij} \leq \sum_{i=1}^{n} a_{ij} \leq 1,$$

and for the column sums of $A \circ B$:

$$\sum_{j=1}^{n} c_{ij} = \sum_{j=1}^{n} a_{ij}b_{ij} \leq \sum_{j=1}^{n} a_{ij} \leq 1.$$

Thus, $A \circ B$ is a doubly substochastic matrix.

Next, we need to show that the sub-defect of $A \circ B$ is equal to k. Recall that the sub-defect is defined as $\mathrm{sd}(C) = \lceil n - \mathrm{sum}(C) \rceil$, where $\mathrm{sum}(C)$ is the sum of all elements in the matrix C. Since B is a doubly stochastic matrix, the sum of its elements is equal to n. Thus,

$$\mathrm{sum}(A \circ B) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}b_{ij} \leq \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = \mathrm{sum}(A).$$

Now, we have:

$$\mathrm{sd}(A \circ B) = \lceil n - \mathrm{sum}(A \circ B) \rceil \geq \lceil n - \mathrm{sum}(A) \rceil = \mathrm{sd}(A) = k.$$

Since $A \circ B$ is a doubly substochastic matrix with a sub-defect equal to k, we have $A \circ B \in \omega_{n,k}$. □

Corollary 2.3. *If $A, B \in \Omega_n$, then $A \circ B \in \Omega_n$.*

Proof. Let $A, B \in \Omega_n$. We know that doubly stochastic matrices have row and column sums equal to 1. Since A and B are doubly stochastic matrices, for each row and column, we have:

$$\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} b_{ij} = 1.$$

Now, first consider the row sums of the Hadamard product $A \circ B$:

$$\sum_{j=1}^{n}(a_{ij}b_{ij}) = \sum_{j=1}^{n} a_{ij} \sum_{j=1}^{n} b_{ij} = \left(\sum_{j=1}^{n} a_{ij}\right)\left(\sum_{j=1}^{n} b_{ij}\right) = 1 \cdot 1 = 1.$$

Similarly, we can show that the column sums of the Hadamard product $A \circ B$ are also equal to 1:

$$\begin{aligned}\sum_{i=1}^{n}(a_{ij}b_{ij}) &= \sum_{i=1}^{n} a_{ij} \sum_{i=1}^{n} b_{ij} \\ &= \left(\sum_{i=1}^{n} a_{ij}\right)\left(\sum_{i=1}^{n} b_{ij}\right) \\ &= 1 \cdot 1 \\ &= 1.\end{aligned}$$

Since the Hadamard product $A \circ B$ has row and column sums equal to 1, it is also a doubly stochastic matrix. Thus, $A \circ B \in \Omega_n$. □

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