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A NUMERICAL SOLUTION OF THE FRACTIONAL NAVIER-STOKES EQUATION USING THE CAPUTO-FABRIZIO ABOODH TRANSFORM METHOD WITH THE REDUCED DIFFERENTIAL POLYNOMIALS

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ABSTRACT. A combination of the Aboodh transform method and the reduced differential polynomial technique was employed in this work to solve the Navier-Stokes equations with the Caputo-Fabrizio derivative. Two illustrations are presented to show the efficacy of the used method. The results gotten are showcased with the aid of tables and graphs. It is discovered that the results derived are close to the actual solution of the problems illustrated. This work will thus make it simple to study nonlinear process that arise in various aspect of innovations and researches.

1. INTRODUCTION

Fractional calculus which deals with the concept of fractional derivative was first given by the Greek mathematician Leibniz in 1695. Many researcher have since being motivated as the concept of fractional calculus interprets true nature in a brilliant and methodical way [5-7]. It has also been discovered that calculus of non-integer order derivative are essential in the description of many scientific value problems such as but not limited to rheology and damping laws [10-14].

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Numerous concepts of fractional calculus were given by Kemple and Beyer [8], Momani and Shawagfeh [3], Kilbas and Trujillo [1], Oldham and Spanier [9], Miller and Ross [4], Podlubny [2], Jafari and Seifi [13,14], Caputo [15], Diethelm *et al.*[16] and Kiryakova [20].

In recent time, mathematicians have given huge attention to analytical and approximate solutions of fractional differential equations. Most of the techniques applied are the variational iteration method (VIM) [21], reduced differential transform method (RDTM)[22], homotopy perturbation transform method [23-26] and finite difference method (FDM) [26] to mention but a few.

Stokes and Clade were the first researchers to discover the Navier-Stokes equation (N-SE) in 1822 [27] as an equation of motion of viscous flow. The N-SE regarded as Newton's second law of motion for fluid substances is a combination of the energy equation, continuity and momentum equations. The Navier-Stokes (N-S) model explains many physical processes such as air flow around a wing,water flow in pipes, ocean currents, weather and many more which arises in various aspect of sciences. The N-S equation is also considered a useful tool in the area of meterology and for discovering the relationship between rigid bodies and viscous fluid [28-29]. Various techniques have been used to solve the N-SE by several mathematicians, this include: the modified Laplace decomposition method employed by Kumar *et al.* [29] for the analytical solution of the N-S fractional order equation.

The combined fractional complex transform and He -Laplace transform technique for the solution of N-SE was implemented by Edeki and Akinlabi [30] and the analytical solution of time-fractional Navier-Stokes equation in polar coordinate using homotopy perturbation method by Ganji *et al.* [31]. Singh and Kumar [33] used the fractional reduced differential transformation method (FRDM) to find a time-fractional N-S equation numerical solution.

Several researchers have also used various procedure to obtain the solution for the Navier-Stokes quations. Ragab *et al* [34] used the homotopy analysis method to solve the time-fractional Navier-Stokes equations. The discrete Adomian decomposition method was employed by Birajdar [35] to obtain the numerical solution of time fractional Navier Stokes equations. Momani and Odibat [36] obtained the analytical solution of a time fractional Navier Stokes equation via the Adomian decomposition method. A fractional model of Navier Stokes equations arising in unsteady flow of a viscous fluid was investigated by Kumar et al. [37].

This work present the numerical solution of the fractional Navier-Stokes equations with the aid of the combined Aboodh transform and reduced differential tranform methods(ABRDTM). Basic fundamental notations and definitions on fractional calculus were explained in sections 2 and 3. Section 4 explains the procedure for the ABRDTM scheme for the Caputo-Fabrizio derivative and section 5 analyzes the conclusion drawn from the study.

2. Definitions

Definition 2.1. *The Riemann-Liouville fractional derivative of a function g, is defined as [2-3]:*

$$ {}_0I_\eta^\tau g(\eta) = \frac{1}{\Gamma(\tau)} \int_0^\eta (\eta-\gamma)^{\tau-1} g(\gamma) d\gamma \tag{1} $$

where $\tau > 0, [0,\eta]$ *is the interval,* $\Gamma(.)$ *connotes the gamma function.*

Definition 2.2. *The fractional order of the Caputo derivative* τ *is defined in [4-5] as:*

$$ {}_0^cD_\eta^\tau g(\eta) = \frac{1}{\Gamma(r-\tau)} \int_0^\eta \frac{g^{(r)}(\gamma)}{(\eta-\gamma)^{\tau+1-r}} d\gamma, \tag{2} $$

where $r-1 < \tau \leq r, \quad r \epsilon N.$

Definition 2.3. *The Caputo-Fabrizio fractional derivative of a function g, is given as [7-8]:*

$$ {}_0^{C.F}D_\eta^\tau g(\eta) = \frac{N(\tau)}{\Gamma(1-\tau)} \int_a^\eta e^{\frac{-\tau(\eta-\gamma)}{1-\tau}} g'(\gamma) d\gamma, \tag{3} $$

3. The Aboodh transform method

The Aboodh transform defined for a function of the exponential order in a set R defined [5] by:

$$ R = \left\{ g(\gamma) : S, q_1, q_2 > 0, |g(\gamma)| < Se^{-\nu\gamma} \right\} \tag{4} $$

where S is a constant that is an infinite number and q_1, q_2 may be finite or infinite. The Aboodh transform defined by Aboodh *et al* [5] is denoted by the operator A(.) and defined by the integral;

$$A[g(\gamma)] = H(\nu) = \frac{1}{\nu}\int_0^\infty g(\gamma)e^{-\nu\gamma}d\gamma, \gamma \geq 0, q_1 \leq \nu \leq q_2 \tag{5}$$

Theorem 3.1 (5)**.** *Given that, $H(\nu)$ is the Aboodh transform of $g(\gamma)$ such that*

$$A[g(\gamma)] = H(\nu),$$

then:

(1) $A\left[g'(\gamma)\right] = \nu H(\nu) - \frac{1}{\nu}g(0)$;

(2) $A\left[g''(\gamma)\right] = \nu^2 H(\nu) - \frac{1}{\nu}g'(0) - g(0)$;

(3) $A\left[g^m(\gamma)\right] = \nu^{(m)} H\left(\nu\right) - \sum_{r=0}^{m-1} \frac{g^{(r)}(0)}{\nu^{2-m+r}}$.

Theorem 3.2. *Let $g(\eta)$ be continuous, bounded and integrable then; the Aboodh transform of $g(\eta)$ in Riemann Liouville fractional derivative sense is given as:*

$$A\{{}_0^{R.L}I_\eta^\tau g(\eta)\} = \frac{H(\nu)}{\nu^\tau}. \tag{6}$$

Proof. From the definition of Riemann Liouville integral:

$${}_0^{R.L}I_\eta^\tau g(\eta) = \frac{1}{\Gamma\tau}\int_0^\eta (\eta-\gamma)^{\tau-1}g(\gamma)d\gamma. \tag{7}$$

Applying the definition of convolution, then Aboodh transform of equation (7) is given as:

$$\begin{aligned} A\left[{}_0^{R.L}I_\eta^\tau g(\eta)\right] &= A\left[\frac{1}{\Gamma\tau}\int_0^\eta (\eta-\gamma)^{\tau-1}g(\gamma)d\gamma\right] \\ &= A\left\{\frac{1}{\Gamma\tau}\{\eta^{\tau-1} \times g(\eta)\}\right\}. \end{aligned} \tag{8}$$

The Aboodh transform of equation (8) is further expressed as:

$$A\left[{}_0^{R.L}I_\eta^\tau g(\eta)\right] = \nu\frac{1}{\Gamma\tau}A\{\eta^{\tau-1}\} \times A\{g(\eta)\}. \tag{9}$$

Thus, equation(9) becomes:

$$A\left[{}_0^{R.L}I_\eta^\tau g(\eta)\right] = \frac{1}{\Gamma\tau} \times \frac{\Gamma\tau}{\nu^\tau} \times H(\nu). \tag{10}$$

Hence,

$$A\{{}_{0}^{R.L}I_{\eta}^{\tau}g(\eta)\} = \frac{H(\nu)}{\nu^{\tau}}. \tag{11}$$

□

Theorem 3.3. *Let $g(\eta)$ be continuous, bounded and integrable, then the Aboodh transform of $g(\eta)$ in Caputo fractional derivative sense is given as:*

$$A\left\{{}_{0}^{c}D_{\eta}^{\tau}g(\eta)\right\} = \nu^{\tau}H(\nu) - \sum_{r=0}^{m-1}\nu^{\tau-r-2}g^{(r)}(0). \tag{12}$$

Proof. From the definition of Caputo fractional derivative,

$$A\left[{}_{0}^{c}D_{\eta}^{\tau}g\left(\eta\right)\right] = A\left[{}_{0}I^{m-\tau}g^{m}\left(\eta\right)\right]. \tag{13}$$

Let

$$g^{m}(\eta) = k(\eta).$$

Applying the result obtained in equation(11), then

$$A\left[{}_{0}I_{\eta}^{m-\tau}k(\eta)\right] = \frac{K(\nu)}{\nu^{m-\tau}}, \tag{14}$$

where $K(\nu){=}A\{k(\nu)\} = A\{g^{(m)}(\eta)\}$.

Simplifying $A\{g^{(m)}(\eta)\}$ using Theorem 3.1, then

$$A\{g^{m}(\eta)\} = \nu^{m}H(\nu) - \sum_{r=0}^{m-1}\nu^{\tau-r-2}g^{(r)}(0), \tag{15}$$

thus,

$$A\left\{{}_{0}^{c}D_{\eta}^{\tau}g(\eta)\right\} = \frac{K(\nu)}{\nu^{m-\tau}} = \frac{1}{\nu^{m-\tau}}\{\nu^{m}H(\nu) - \sum_{r=0}^{m-1}\nu^{m-r-2}g^{(r)}(0)\}, \tag{16}$$

therefore,

$$A\left\{{}_{0}^{c}D_{\eta}^{\tau}g(\eta)\right\} = \nu^{-(m-\tau)}\left\{\nu^{m}H(\nu) - \sum_{r=0}^{m-1}\nu^{m-r-2}g^{(r)}(0)\right\}, \tag{17}$$

$$= \nu^{\tau}H(\nu) - \sum_{r=0}^{m-1}\nu^{\tau-r-2}g^{(r)}(0), \tag{18}$$

Hence, the Aboodh transform of Caputo derivative of order τ is given as;

$$A\left\{{}_0^c D_\eta^\tau g(\eta)\right\} = \nu^\tau H(\nu) - \sum_{r=0}^{m-1} \nu^{\tau-r-2} g^{(r)}(0) \ . \tag{19}$$

□

Theorem 3.4. *Let $g(\eta)$ be continuous, bounded and integrable then; the Aboodh transform of $g(\eta)$ in Caputo-Fabrizio fractional derivative sense is given as: The Caputo-Fabrizio fractional derivative in a sobolev space given by [5] is defined as:*

$$ {}_a^{C.F} D_\eta^\tau g(\eta) = \frac{N(\tau)}{1-\tau} \int_a^\eta e^{\frac{-\tau(\eta-\gamma)}{1-\tau}} g^{'}(\gamma) d\gamma, \qquad 0 < \tau \leq 1. \tag{20}$$

From the definition of Caputo derivative [4],

$$ {}_a^c D_\eta^\tau g(\eta) \ = \ {}_a I_\eta^{m-\tau} g^{(m)}(\eta) = \frac{1}{\Gamma(m-\tau)} \int_a^\eta (\eta-\gamma)^{m-\tau-1} g^{(m)}(\gamma) d\gamma, \tag{21}$$

$m-1 < \tau \leq m$. *When $m = 1$, $a = 0$, then equation (21) was simplified to obtain:*

$$ {}_a^c D_\eta^\tau g(\eta) = \frac{1}{\Gamma(1-\tau)} \int_0^\eta (\eta-\gamma)^{-\tau} g^{'}(\gamma) d\gamma, \qquad 0 < \tau \leq 1. \tag{22}$$

Let $\tau \epsilon$ [0,1], $g(\eta)\epsilon$ K'(a,b) for a b, then the Caputo-Fabrizio fractional derivative is given as [5]:

$$ {}_a^{C.F} D_\eta^\tau g(\eta) = \frac{N(\tau)}{1-\tau} \int_a^\eta e^{\frac{-\tau(\eta-\gamma)}{1-\tau}} g^{'}(\gamma) d\gamma, \qquad 0 < \tau \leq 1, \tag{23}$$

when, $a = 0$ and $N(\tau) = 1$.

Equation (23) was simplified to obtain:

$$ {}_0^{C.F} D_\eta^\tau g(\eta) = \frac{1}{1-\tau} \int_0^\eta e^{\frac{-\tau(\eta-\gamma)}{1-\tau}} g^{'}(\gamma) d\gamma, \qquad 0 < \tau \leq 1. \tag{24}$$

The Aboodh transform properties is applied on equation (24)to obtain:

$$A\left[{}_0^{C.F} D_\eta^\tau g(\eta)\right] = \frac{1}{1-\tau} \times A\left\{ e^{\frac{-\tau\gamma}{1-\tau}} \times g'(\gamma) \right\}. \tag{25}$$

Equation (25) was further simplified to obtain:

$$A\left[{}_0^{C.F}D_\eta^\tau g(\eta)\right] = \frac{1}{1-\tau} \times \nu \times A\left\{e^{\frac{-\tau\gamma}{1-\tau}}\right\} \times A\{g^{(\tau)}(\gamma)\}$$

$$= \frac{\nu}{\nu^2(1-\tau)+\tau\nu} \times A\{g^{(\tau)}(\gamma)\} \tag{26}$$

since,

$$A[g^\tau(\gamma)] = \nu^\tau H(\nu) - \sum_{r=0}^{m-1} \frac{g^r(0)}{\nu^{2-\tau+r}}. \tag{27}$$

Hence, equation (26) becomes:

$$A\left[{}_a^{C.F}D_\eta^\tau g(\eta)\right] = \frac{\nu}{\nu^2(1-\tau)+\tau\nu} \times \nu^\tau H(\nu) - \sum_{r=0}^{m-1} \frac{g^r(0)}{\nu^{2-\tau+r}}, \tag{28}$$

which is the Aboodh transform of Caputo-Fabrizio derivative of order τ.

4. Procedure of the Aboodh and Reduced Differential Transform Scheme for the Caputo-Fabrizio derivative

Given the general fractional differential equation of the form [31];

$$^{C.F}D_\eta^\tau u(\eta,\gamma) + Ru(\eta,\gamma) + NU(\eta,\gamma) = g(\eta,\gamma) \tag{29}$$

with the given conditions:

$$u^{(m)}(\eta,0) = g(\eta),\ \forall\ \eta\epsilon N,\ m = 1,2,3,\ldots \tag{30}$$

where the Caputo Fabrizio derivative of order τ is given as $^{C.F}D_\eta^\tau u(\eta,\gamma)$, R is the linear differential operator, N the nonlinear term and the source term as $g(\eta,\gamma)$. Applying the properties of the Aboodh transform on equation (28) we get:

$$A\left[^{C.F}D_\eta^\tau u(\eta,\gamma)\right] + A\left[Ru(\eta,\gamma)\right] + A\left[Nu(\eta,\gamma)\right] = A\left[g(\eta,\gamma)\right]. \tag{31}$$

The inverse Aboodh transform is applied on equation (31) with the given condition to give:

$$u(\eta,\gamma) = A^{-1}\left[\frac{\nu^2(1-\tau)+\tau\nu}{\nu^{1+\tau}} A\left[g(\eta,\gamma)\right] + \sum_{r=0}^{m-1} \frac{u^{(r)(0)}}{\nu^{2-\tau+r}}\right] \tag{32}$$

$$A^{-1}\left[\frac{\nu^2(1-\tau)+\tau\nu}{\nu^{1+\tau}}\times A\left[Ru(\eta,\gamma)+Nu(\eta,\gamma)\right]\right]$$

Equation (32) is then written as:

$$u(\eta,\gamma)=G(\eta,\gamma)-A^{-1}\left[\frac{\nu^2(1-\tau)+\tau\nu}{\nu^{1+\tau}}\{A\left[Ru(\eta,\gamma)\right]+A\left[Nu(\eta,\gamma)\right]\}\right], \tag{33}$$

where the expressions $G(\eta,\gamma)$ that rose from the source term after it has been simplified. The approximated solution will be expressed as:

$$u(\eta,\gamma)=\sum_{r=0}^{\infty}u_r(\eta,\gamma). \tag{34}$$

The nonlinear part is reduced as follows:

$$Nu(\eta,\gamma)=\sum_{r=0}^{\infty}A_r, \tag{35}$$

where A_r is expressed as the reduced polynomial which can be gotten from the below formula

$$A_r=U_r(\eta)U_{m-r}(\gamma),\quad r=0,1,\ldots.$$

Substituting equations (34) and (35) into equation (33) gives

$$\begin{aligned}&\sum_{r=0}^{\infty}u_r(\eta,\gamma)\\ &=G(\eta,\gamma)-A^{-1}\left[\frac{\nu^2(1-\tau)+\tau\nu}{\nu^{1+\tau}}\left\{A\left[R\sum_{r=0}^{m}u_r(\eta,\gamma)\right]+A\left[\sum_{r=0}^{m}A_r\right]\right\}\right].\end{aligned} \tag{36}$$

From equation (36), the initial approximation is obtained as

$$u_r(\eta,\gamma)=G(\eta,\gamma),\ \text{when: } r=0. \tag{37}$$

And the recursive relation is defined as

$$u_{r+1}=-A^{-1}\left[\frac{v^2(1-\tau)+\tau\nu}{\nu^{1+\tau}}\{A\left[Ru_r(\eta,\gamma)\right]+A\left[A_r\right]\}\right], \tag{38}$$

where τ =1,2,3 and $r\geq 0$.

The solution $u(\eta,\gamma)$ will then be approximated by the series;

$$u(\eta,\gamma)=\lim_{N\to\infty}\sum_{r=0}^{N}u_r(\eta,\gamma). \tag{39}$$

5. Applications to Fractional Navier-Stokes Equations

5.1. **Illustration I.**

Given the fractional order Navier-Stokes equation:

$$\left.\begin{aligned} D_\eta^\tau \mu &= \theta(\mu_{\phi\phi}+\mu_{\sigma\sigma})-(\mu\mu_\phi+\varphi\mu_\sigma)+\lambda \\ D_\eta^\tau \varphi &= \theta(\varphi_{\phi\phi}+\varphi_{\sigma\sigma})-(\mu\varphi_\phi+\varphi\varphi_\sigma)-\lambda \end{aligned}\right\} \tag{40}$$

subject to the given conditions

$$\begin{aligned} \mu(\phi,\sigma,0) &= -\sin(\phi+\sigma) \\ \varphi(\phi,\sigma,0) &= \sin(\phi+\sigma) \end{aligned}. \tag{41}$$

Applying the differential properties of the Aboodh transform of Caputo-Fabrizio on equation (40):

$$\begin{aligned} A\left[{}^{C.F}D_\eta^\tau \mu\right] &= A\left[\theta(\mu_{\phi\phi}+\mu_{\sigma\sigma})-(\mu\mu_\phi+\varphi\mu_\sigma)+\lambda\right] \\ \frac{\nu^{1+\tau}}{\nu^2(1-\tau)+\tau\nu}A\left[\mu(\phi,\sigma)\right] &- \sum_{r=0}^{m-1}\frac{\mu^{(r)}(0)}{\nu^2-\tau+r} \\ =&A\left[\theta(\mu_{\phi\phi}+\mu_{\sigma\sigma})-(\mu\mu_\phi+\varphi\mu_\sigma)+\lambda\right], \end{aligned} \tag{42}$$

$$\begin{aligned} A\left[{}^{C.F}D_\eta^\tau \varphi\right] &= A\left[\theta(\varphi_{\phi\phi}+\varphi_{\sigma\sigma})-(\mu\varphi_\phi+\varphi\varphi_\sigma)-\lambda\right] \\ \frac{\nu^{1+\tau}}{\nu^2(1-\tau)+\tau\nu}A\left[\varphi(\phi,\sigma)\right] &- \sum_{r=0}^{m-1}\frac{\varphi^{(r)}(0)}{\nu^2-\tau+r} \\ =&A\left[\theta(\varphi_{\phi\phi}+\varphi_{\sigma\sigma})-(\mu\varphi_\phi+\varphi\mu_\sigma)+\lambda\right]. \end{aligned} \tag{43}$$

The inverse Aboodh transform of equations (42) and (43) alongside the given conditions is expressed as

$$\begin{aligned} &\mu(\phi,\sigma) = A^{-1}\left[G(\phi,\sigma,0)\right] \\ &+A^{-1}\left\{\frac{\nu^2(1-\tau)+\tau\nu}{\nu^{1+\tau}}\left[A\left[\theta(\mu_{\phi\phi}+\mu_{\sigma\sigma})-(\mu\mu_\phi+\varphi\mu_\sigma)+\lambda\right]\right]\right\} \\ &\sum_{r=0}^{\infty}\mu_r(\phi,\sigma,\psi) = -\sin(\phi+\sigma)+\left(\frac{\psi^\tau}{\Gamma(\tau+1)}\right)\lambda \\ &+A^{-1}\left[\frac{\nu^2(1-\tau)+\tau\nu}{\nu^{1+\tau}}A\left[\theta(\mu_{\phi\phi}+\mu_{\sigma\sigma})-\left[N(\mu)_{\phi\sigma}\right]\right]\right] \end{aligned} \tag{44}$$

(45)
$$\varphi(\phi,\sigma) = A^{-1}\left[G(\phi,\sigma,0)\right]$$
$$+A^{-1}\left\{\frac{\nu^2(1-\tau)+\tau\nu}{\nu^{1+\tau}}\left[A\left[\theta(\varphi_{\phi\phi}+\varphi_{\sigma\sigma})-(\mu\varphi_{\phi}+\varphi\varphi_{\sigma})-\lambda\right]\right]\right\}$$
$$\sum_{r=0}^{\infty}\varphi_r(\phi,\sigma,\psi) = \sin(\phi+\sigma)-\left(\frac{\psi^{\tau}}{\Gamma(\tau+1)}\right)\lambda$$
$$+A^{-1}\left[\frac{\nu^2(1-\tau)+\tau\nu}{\nu^{1+\tau}}A\left[\theta(\varphi_{\phi\phi}+\varphi_{\sigma\sigma})-\left[N(\varphi)_{\phi\sigma}\right]\right]\right]$$

Thus, the first iterate is given as:

(46)
$$\left.\begin{aligned}\mu_0 &= -\sin(\phi+\sigma)+\left(\frac{\psi^{\tau}}{\Gamma(\tau+1)}\right)\lambda\\ \varphi_0 &= \sin(\phi+\sigma)-\left(\frac{\psi^{\tau}}{\Gamma(\tau+1)}\right)\lambda\end{aligned}\right\},$$

where $N(\mu)$ and $N(\varphi)$ are the reduced polynomials defined as:

(47)
$$N(\mu) = \mu\mu_{\phi} = \sum_{r=0}^{m} A_r$$

$$A_r = \mu_m(\mu_{m-r})_{\phi}, \quad A_0 = \mu_0\mu_{0,\phi}\{r=0\}, \quad A_1 = \mu_0\mu_{1,\phi}+\mu_1\mu_{0,\phi}\ \{r=1\}$$

(48)
$$\varphi\mu_{\sigma} = \sum_{r=0}^{m} B_r$$

$$B_0 = \varphi_0\mu_{0,\sigma}\ \{r=0\}, \quad B_1 = \varphi_0\mu_{1,\sigma}+\varphi_1\mu_{0,\sigma}\ \{r=1\}$$

(49)
$$N(\varphi) = \mu\varphi_{\phi} = \sum_{r=0}^{m} C_r, \varphi\varphi_{\sigma} = \sum_{r=0}^{m} D_r$$

The recursive relation is given as:

(50)
$$\mu_{r+1}(\phi,\sigma,\psi)$$
$$=A^{-1}\left\{\frac{\nu^2(1-\tau)+\tau\nu}{\nu^{1+\tau}}\left[A\left[\theta(\mu_{\phi\phi}+\mu_{\sigma\sigma})-\left(\sum_{r=0}^{m}A_r+\sum_{r=0}^{m}B_r\right)\right]\right]\right\}$$
$$\varphi_{r+1}(\phi,\sigma,\psi)$$
$$=A^{-1}\left\{\frac{\nu^2(1-\tau)+\tau\nu}{\nu^{1+\tau}}\left[A\left[\theta(\varphi_{\phi\phi}+\varphi_{\sigma\sigma})-\left(\sum_{r=0}^{m}C_r+\sum_{r=0}^{m}D_r\right)\right]\right]\right\}$$

when $r = 0$:

$$\begin{aligned} \mu_1(\phi, \sigma, \psi) &= \sin(\phi + \sigma)\frac{2\theta\psi^\tau}{\Gamma(\tau+1)} \\ \varphi_1(\phi, \sigma, \psi) &= -\sin(\phi + \sigma)\frac{2\theta\psi^\tau}{\Gamma(\tau+1)}, \end{aligned} \tag{51}$$

when $r = 1$:

$$\begin{aligned} \mu_2(\phi, \sigma, \psi) &= -\sin(\phi + \sigma)\frac{(2\theta)^2\psi^{2\tau}}{\Gamma(2\tau+1)} \\ \varphi_2(\phi, \sigma, \psi) &= \sin(\phi + \sigma)\frac{(2\theta)^2\psi^{2\tau}}{\Gamma(2\tau+1)} \end{aligned} \tag{52}$$

when $r = 2$:

$$\begin{aligned} \mu_3(\phi, \sigma, \psi) &= -\sin(\phi + \sigma)\frac{(2\theta)^3\psi^{3\tau}}{\Gamma(3\tau+1)} \\ \varphi_3(\phi, \sigma, \psi) &= \sin(\phi + \sigma)\frac{(2\theta)^3\psi^{3\tau}}{\Gamma(3\tau+1)}. \end{aligned} \tag{53}$$

The approximated solution is obtained as:

$$\begin{aligned} \mu(\phi, \sigma, \psi) &= \mu_0(\phi, \sigma, \psi) + \mu_1(\phi, \sigma, \psi) + \mu_2(\phi, \sigma, \psi) + \mu_3(\phi, \sigma, \psi) + \dots \\ &= -\sin(\phi + \sigma) + \left(\frac{\psi^r}{\Gamma(\tau+1)}\right)\lambda + \sin(\phi + \sigma)\frac{2\theta\psi^r}{\Gamma(\tau+1)} \\ &\quad - \sin(\phi + \sigma)\frac{(2\theta)^2\psi^{2\tau}}{\Gamma(2\tau+1)} + \sin(\phi + \sigma)\frac{(2\theta)^3\psi^{3\tau}}{\Gamma(3\tau+1)} \end{aligned} \tag{54}$$

$$\begin{aligned} \psi(\phi, \sigma, \psi) &= \varphi_0(\phi, \sigma, \psi) + \varphi_1(\phi, \sigma, \psi) + \varphi_2(\phi, \sigma, \psi) + \varphi_3(\phi, \sigma, \psi) + \dots \\ &= \sin(\phi + \sigma) - \left(\frac{\psi^r}{\Gamma(\tau+1)}\right)\lambda - \sin(\phi + \sigma)\frac{2\theta\psi^r}{\Gamma(\tau+1)} \\ &\quad - \sin(\phi + \sigma)\frac{(2\theta)^2\psi^{2\tau}}{\Gamma(2\tau+1)} - \sin(\phi + \sigma)\frac{(2\theta)^3\psi^{3\tau}}{\Gamma(3\tau+1)} \end{aligned} \tag{55}$$

Equations (54) and (55) are the solution of equation (40) which converges to the exact solution, (when τ=1 and $\lambda = 0$):

$$\mu(\phi, \sigma, \psi) = -e^{-2\theta\psi}\sin(\phi + \sigma), \tag{56}$$

$$\varphi(\phi, \sigma, \psi) = e^{-2\theta\psi}\sin(\phi + \sigma). \tag{57}$$

TABLE 1. Comparisons between the numerical and analytical solutions for equation (37), $\mu(\phi,\sigma,\psi)$ at $\sigma=\psi=\theta=10^{-3}$.

ϕ	$ANALYTICAL$	$ABRDTM$	$FRTM[16]$	$\|E-ABRDTM\|$
0.1	-0.1097563473	-0.1097552362	-0.1097552362	4.20161×10^{-7}
0.2	-0.2084182120	-0.2084181103	-0.2084181103	4.61745×10^{-8}
0.3	-0.3049976308	-0.3049967275	-0.3049967275	7.67804×10^{-7}
0.4	-0.3985296141	-0.3985285031	-0.3985285031	3.46567×10^{-9}
0.5	-0.4880796212	-0.4880795120	-0.4880795120	5.03011×10^{-8}
0.6	-0.5727528981	-0.5727527870	-0.5727527870	5.12184×10^{-7}
0.7	-0.6517034173	-0.6517023063	-0.6517023063	3.62834×10^{-6}
0.8	-0.7241423315	-0.7241422304	-0.7241422304	2.48523×10^{-7}
0.9	-0.7893458547	-0.7893457436	-0.7893457436	3.31621×10^{-9}
1.0	-0.8466624952	-0.8466623841	-0.8466623841	3.86431×10^{-8}

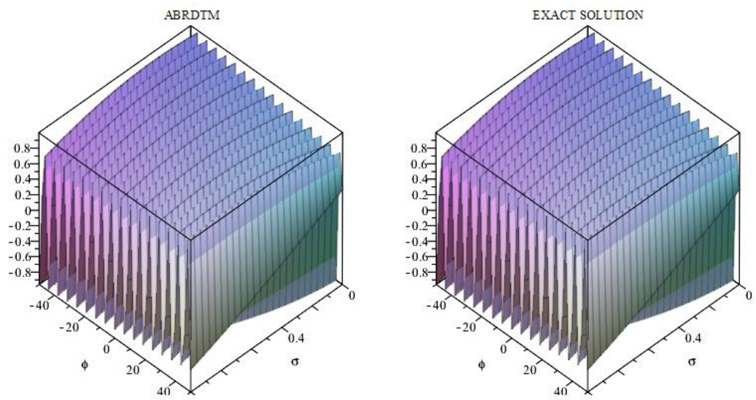

FIGURE 1. Graph of $\mu(\phi,\sigma,\psi)$ for equation (40) at $\tau=1$

TABLE 2. Comparisons between the numerical and analytical solutions for equation (40), $\varphi(\phi,\sigma,\psi)$ at $\sigma=\psi=\theta=10^{-3}$.

ϕ	$ANALYTICAL$	$ABRDTM$	$FRTM[16]$	$\|E-ABRDTM\|$
0.1	0.1097563473	0.1097552362	0.1097552362	4.20161×10^{-7}
0.2	0.2084182120	0.2084181103	0.2084181103	4.61745×10^{-8}
0.3	0.3049976308	0.3049967275	0.3049967275	7.67804×10^{-7}
0.4	0.3985296141	0.3985285031	0.3985285031	3.46567×10^{-9}
0.5	0.4880796212	0.4880795120	0.4880795120	5.03011×10^{-8}
0.6	0.5727528981	0.5727527870	0.5727527870	5.12184×10^{-7}
0.7	0.6517034173	0.6517023063	0.6517023063	3.62834×10^{-6}
0.8	0.7241423315	0.7241422304	0.7241422304	2.48523×10^{-7}
0.9	0.7893458547	0.7893457436	0.7893457436	3.31621×10^{-9}
1.0	0.8466624952	0.8466623841	0.7893457436	3.86431×10^{-8}

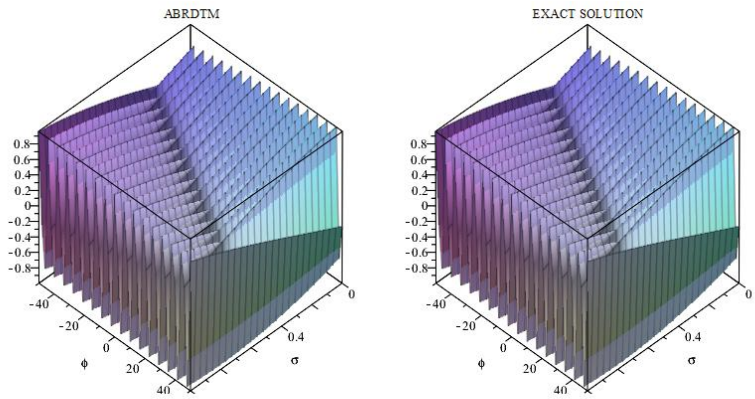

FIGURE 2. Graph of $\varphi(\phi, \sigma, \psi)$ for equation (40) at τ =1

TABLE 3. Comparisons between the numerical and analytical solutions for equation (40), $\varphi(\phi, \sigma, \psi)$ at $\sigma = \psi = \theta = 10^{-3}$, $a = \tau = 0.25$, $b = \tau = 0.75$.

ϕ	ANALYTICAL	ABRDTM (a)	ABRDTM(b)	$\lvert$E-ABRDTM$\rvert$	$\lvert$E-ABRDTM$\rvert$
0.1	0.1097563473	0.08989609518	0.08477066486	1.9860×10^{-2}	2.4985×10^{-2}
0.2	0.2084182120	0.18855896930	0.18343353900	1.9859×10^{-2}	2.4984×10^{-2}
0.3	0.3049976308	0.28513758650	0.28001215620	1.9860×10^{-2}	2.4985×10^{-2}
0.4	0.3985296141	0.37866936210	0.37354393180	1.9860×10^{-2}	2.4984×10^{-2}
0.5	0.4880796212	0.46822037100	0.46309494070	1.9859×10^{-2}	2.4984×10^{-2}
0.6	0.5727528981	0.55289364600	0.54776821570	1.9859×10^{-2}	2.4985×10^{-2}
0.7	0.6517034173	0.63184316530	0.62671773500	1.9860×10^{-2}	2.4984×10^{-2}
0.8	0.7241423315	0.70428308940	0.69915765910	1.9859×10^{-2}	2.4984×10^{-2}
0.9	0.7893458547	0.76948660260	0.76436117230	1.9859×10^{-2}	2.4984×10^{-2}
1.0	0.8466624952	0.82680324310	0.82167781280	1.9859×10^{-2}	2.4984×10^{-2}

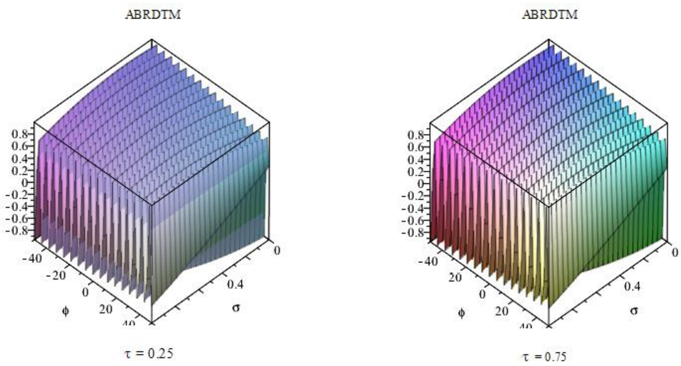

FIGURE 3. Graph of $\mu(\phi, \sigma, \psi)$ for equation (40) at τ =0.25 and 0.75

5.2. **Illustration II.**

Given the fractional order Navier-Stokes equation:

$$\left.\begin{aligned} D_\eta^\tau \mu &= \theta(\mu_{\phi\phi} + \mu_{\sigma\sigma} + \mu_{\varrho\varrho}) - (\mu\mu_\phi + \varphi\mu_\sigma + \rho\mu_\varrho) + \lambda_1 \\ D_\eta^\tau \varphi &= \theta(\varphi_{\phi\phi} + \varphi_{\sigma\sigma} + \varphi_{\varrho\varrho}) - (\mu\varphi_\phi + \varphi\varphi_\sigma + \rho\varphi_\varrho) + \lambda_2 \\ D_\eta^\tau \rho &= \theta(\rho_{\phi\phi} + \rho_{\sigma\sigma} + \rho_{\varrho\varrho}) - (\mu\rho_\phi + \varphi\rho_\sigma + \rho\rho_\varrho) + \lambda_3 \end{aligned}\right\} \tag{58}$$

subject to the initial condition

$$\left.\begin{aligned} \mu(\phi,\sigma,\varrho,0) &= -0.5\phi + \sigma + \varrho \\ \varphi(\phi,\sigma,\varrho,0) &= \phi - 0.5\sigma + \varrho \\ \rho(\phi,\sigma,\varrho,0) &= \phi + \sigma - 0.5\varrho \end{aligned}\right\} \tag{59}$$

Applying the differential properties of the Aboodh transform of Caputo-Fabrizio on equation (58):

$$\begin{aligned} &A\left[{}^{C.F}D_\eta^\tau \mu\right] = A\left[\theta(\mu_{\phi\phi} + \mu_{\sigma\sigma} + \mu_{\varrho\varrho}) - (\mu\mu_\phi + \varphi\mu_\sigma + \rho\mu_\varrho) + \lambda_1\right] \\ &\frac{\nu^{1+\tau}}{\nu^2(1-\tau)+\tau\nu} A\left[\mu(\phi,\sigma,\varrho)\right] - \sum_{r=0}^{m-1} \frac{\mu^{(r)}(0)}{\nu^2-\tau+r} \\ =& A\left[\theta(\mu_{\phi\phi} + \mu_{\sigma\sigma} + \mu_{\varrho\varrho}) - (\mu\mu_\phi + \varphi\mu_\sigma + \rho\mu_\varrho) + \lambda_1\right] \end{aligned} \tag{60}$$

$$\begin{aligned} &A\left[{}^{C.F}D_\eta^\tau \varphi\right] = A\left[\theta(\varphi_{\phi\phi} + \varphi_{\sigma\sigma} + \varphi_{\varrho\varrho}) - (\mu\varphi_\phi + \varphi\varphi_\sigma + \rho\varphi_\varrho) + \lambda_2\right] \\ &\frac{\nu^{1+\tau}}{\nu^2(1-\tau)+\tau\nu} A\left[\varphi(\phi,\sigma,\varrho)\right] - \sum_{r=0}^{m-1} \frac{\varphi^{(r)}(0)}{\nu^2-\tau+r} \\ =& A\left[\theta(\varphi_{\phi\phi} + \varphi_{\sigma\sigma} + \varphi_{\varrho\varrho}) - (\mu\varphi_\phi + \varphi\varphi_\sigma + \rho\varphi_\varrho) + \lambda_2\right] \end{aligned} \tag{61}$$

$$\begin{aligned} &A\left[{}^{C.F}D_\eta^\tau \rho\right] = A\left[\theta(\rho_{\phi\phi} + \rho_{\sigma\sigma} + \rho_{\varrho\varrho}) - (\mu\rho_\phi + \varphi\rho_\sigma + \rho\rho_\varrho) + \lambda_3\right] \\ &\frac{\nu^{1+\tau}}{\nu^2(1-\tau)+\tau\nu} A\left[\rho(\phi,\sigma,\varrho)\right] - \sum_{r=0}^{m-1} \frac{\varphi^{(r)}(0)}{\nu^2-\tau+r} \\ =& A\left[\theta(\rho_{\phi\phi} + \rho_{\sigma\sigma} + \rho_{\varrho\varrho}) - (\mu\rho_\phi + \varphi\rho_\sigma + \rho\rho_\varrho) + \lambda_3\right]. \end{aligned} \tag{62}$$

The inverse Aboodh transform of equations (60-62) alongside the given conditions is expressed as:

$$\begin{aligned} &\mu(\phi,\sigma,\varrho) = A^{-1}\left[G(\phi,\sigma,\varrho,0)\right] \\ &+A^{-1}\left\{\frac{\nu^2(1-\tau)+\tau\nu}{\nu^{1+\tau}}\left[A\left[\theta(\mu_{\phi\phi}+\mu_{\sigma\sigma}+\mu_{\varrho\varrho})-(\mu\mu_\phi+\varphi\mu_\sigma+\rho\mu_\varrho)+\lambda_1\right]\right]\right\} \end{aligned}$$

$$\begin{aligned} &\sum_{r=0}^{\infty}\mu_r(\phi,\sigma,\varrho,\psi) = (-0.5\varphi+\sigma+\rho)+\left(\frac{\psi^\tau}{\Gamma(\tau+1)}\right)\lambda_1 \\ &+A^{-1}\left[\frac{\nu^2(1-\tau)+\tau\nu}{\nu^{1+\tau}}A\left[\theta(\mu_{\phi\phi}+\mu_{\sigma\sigma}+\mu_{\varrho\varrho})-[N(\mu)_{\phi\sigma}]\right]\right] \end{aligned} \tag{63}$$

$$\begin{aligned} &\varphi(\phi,\sigma,\varrho) = A^{-1}\left[G(\phi,\sigma,\varrho,0)\right] \\ &+A^{-1}\left\{\frac{\nu^2(1-\tau)+\tau\nu}{\nu^{1+\tau}}\left[A\left[\theta(\varphi_{\phi\phi}+\varphi_{\sigma\sigma}+\varphi_{\varrho\varrho})\right.\right.\right. \\ &-\left.\left.\left.(\mu\varphi_\phi+\varphi\varphi_\sigma+\rho\varphi_\varrho)+\lambda_2\right]\right]\right\} \end{aligned} \tag{64}$$

$$\begin{aligned} &\sum_{r=0}^{\infty}\varphi_r(\phi,\sigma,\varrho,\psi) = (\phi-0.5\sigma+\rho)+\left(\frac{\psi^\tau}{\Gamma(\tau+1)}\right)\lambda_2 \\ &+A^{-1}\left[\frac{\nu^2(1-\tau)+\tau\nu}{\nu^{1+\tau}}A\left[\theta(\varphi_{\phi\phi}+\varphi_{\sigma\sigma}+\varphi_{\varrho\varrho})-[N(\varphi)_{\phi\sigma}]\right]\right] \end{aligned} \tag{65}$$

$$\begin{aligned} &\rho(\phi,\sigma,\varrho) = A^{-1}\left[G(\phi,\sigma,\varrho,0)\right] \\ &+A^{-1}\left\{\frac{\nu^2(1-\tau)+\tau\nu}{\nu^{1+\tau}}\left[A\left[\theta(\rho_{\phi\phi}+\rho_{\sigma\sigma}+\rho_{\varrho\varrho})-(\mu\rho_\phi+\varphi\rho_\sigma+\rho\rho_\varrho)+\lambda_2\right]\right]\right\} \end{aligned}$$

$$\begin{aligned} &\sum_{r=0}^{\infty}\rho_r(\phi,\sigma,\varrho,\psi) = (\phi-0.5\sigma+\rho)+\left(\frac{\psi^\tau}{\Gamma(\tau+1)}\right)\lambda_2 \\ &+A^{-1}\left[\frac{\nu^2(1-\tau)+\tau\nu}{\nu^{1+\tau}}A\left[\theta(\rho_{\phi\phi}+\rho_{\sigma\sigma}+\rho_{\varrho\varrho})-[N(\rho)_{\phi\sigma}]\right]\right] \end{aligned} \tag{66}$$

Thus, the first iterate is given as:

$$\left.\begin{aligned} \mu_0 &= -0.5\phi+\sigma+\varrho \\ \varphi_0 &= \phi-0.5\sigma+\varrho \\ \rho_0 &= \phi+\sigma-0.5\varrho \end{aligned}\right\} \tag{67}$$

where $N(\mu)$, $N(\varphi)$ and $N(\rho)$are the reduced polynomials defined as:

$$N(\mu) = \mu\mu_\phi = \sum_{r=0}^{m} A_r$$

$$A_r = \mu_m(\mu_{m-r})_\phi, \quad A_0 = \mu_0\mu_{0,\phi}\{r=0\}, \quad A_1 = \mu_0\mu_{1,\phi} + \mu_1\mu_{0,\phi}\ \{r=1\}$$

(68)
$$\varphi\mu_\sigma = \sum_{r=0}^{m} B_r\ \rho\mu_\varrho = \sum_{r=0}^{m} C_r$$

$$B_0 = \varphi_0\mu_{0,\sigma}\ \{r=1\}, \quad B_1 = \varphi_0\mu_{1,\sigma} + \varphi_1\mu_{0,\sigma}\ \{r=1\}$$

$$N(\varphi) = \mu\varphi_\phi = \sum_{r=0}^{m} D_r,\ \varphi\varphi_\sigma = \sum_{r=0}^{m} E_r\rho\varphi_\varrho = \sum_{r=0}^{m} F_r$$

(69)
$$N(\rho) = \mu\rho_\phi = \sum_{r=0}^{m} G_r,\ \varphi\rho_\sigma = \sum_{r=0}^{m} H_r\rho\rho_\varrho = \sum_{r=0}^{m} I_r$$

The recursive relation is given as:

$$\mu_{r+1}(\phi,\sigma,\varrho,\psi) = A^{-1}\left\{\frac{\nu^2(1-\tau)+\tau\nu}{\nu^{1+\tau}}\left[A\left[\theta(\mu_{\phi\phi}+\mu_{\sigma\sigma}+\mu_{\varrho\varrho}) - \left(\sum_{r=0}^{m} A_r + \sum_{r=0}^{m} B_r + \sum_{r=0}^{m} C_r\right)\right]\right]\right\}$$

(70)
$$\varphi_{r+1}(\phi,\sigma,\varrho,\psi) = A^{-1}\left\{\frac{\nu^2(1-\tau)+\tau\nu}{\nu^{1+\tau}}\left[A\left[\theta(\varphi_{\phi\phi}+\varphi_{\sigma\sigma}+\varphi_{\varrho\varrho}) - \left(\sum_{r=0}^{m} D_r + \sum_{r=0}^{m} E_r + \sum_{r=0}^{m} F_r\right)\right]\right]\right\}$$

(71)
$$\rho_{r+1}(\phi,\sigma,\varrho,\psi) = A^{-1}\left\{\frac{\nu^2(1-\tau)+\tau\nu}{\nu^{1+\tau}}\left[A\left[\theta(\rho_{\phi\phi}+\rho_{\sigma\sigma}+\rho_{\varrho\varrho}) - \left(\sum_{r=0}^{m} G_r + \sum_{r=0}^{m} H_r + \sum_{r=0}^{m} I_r\right)\right]\right]\right\}$$

when $r=0$:

(72)
$$\mu_1(\phi,\sigma,\varrho,\psi) = \frac{-2.25\phi\psi^\tau}{\Gamma(\tau+1)}$$

$$\varphi_1(\phi,\sigma,\varrho,\psi) = \frac{-2.25\sigma\psi^\tau}{\Gamma(\tau+1)}$$

$$\rho_1(\phi,\sigma,\varrho,\psi) = \frac{-2.25\rho\psi^\tau}{\Gamma(\tau+1)}$$

when $r = 1$:

$$\mu_2(\phi,\sigma,\varrho,\psi) = \frac{2(2.25)\phi\psi^{\tau}}{\Gamma(2\tau+1)}(-0.5\phi+\sigma+\varrho)$$

(73)
$$\varphi_2(\phi,\sigma,\varrho,\psi) = \frac{2(2.25)\phi\psi^{\tau}}{\Gamma(2\tau+1)}(\phi-0.5\sigma+\varrho)$$

$$\rho_2(\phi,\sigma,\varrho,\psi) = \frac{2(2.25)\phi\psi^{\tau}}{\Gamma(2\tau+1)}(\phi+\sigma-0.5\varrho)$$

when $r = 2$:

$$\mu_3(\phi,\sigma,\varrho,\psi) = -\frac{(2.25)^2\phi(4(\Gamma(\tau+1))^2+\Gamma(2\tau+1))\psi^{3\tau}}{\Gamma(2\tau+1)(\Gamma(\tau+1))^2}$$

(74)
$$\varphi_3(\phi,\sigma,\varrho,\psi) = -\frac{(2.25)^2\sigma(4(\Gamma(\tau+1))^2+\Gamma(2\tau+1))\psi^{3\tau}}{\Gamma(2\tau+1)(\Gamma(\tau+1))^2}$$

$$\rho_3(\phi,\sigma,\varrho,\psi) = -\frac{(2.25)^2\rho(4(\Gamma(\tau+1))^2+\Gamma(2\tau+1))\psi^{3\tau}}{\Gamma(2\tau+1)(\Gamma(\tau+1))^2}$$

The approximated solution is obtained as:

(75)
$$\begin{aligned}
&\mu(\phi,\sigma,\varrho,\psi)\\
&= \mu_0(\phi,\sigma,\varrho,\psi)+\mu_1(\phi,\sigma,\varrho,\psi)+\mu_2(\phi,\sigma,\varrho,\psi)+\mu_3(\phi,\sigma,\varrho,\psi)+\dots\\
&= -0.5\phi+\sigma+\varrho-\frac{2.25\phi\psi^{\tau}}{\Gamma(\tau+1)}+\frac{2(2.25)\phi\psi^{2\tau}}{\Gamma(2\tau+1)}\\
&\times(-0.5\phi+\sigma+\varrho)-\frac{(2.25)^2\phi\psi^{3\tau}}{\Gamma(3\tau+1)}\left(4+\frac{\Gamma(2\tau+1)}{(\Gamma(\tau+1))^2}\right)+\dots
\end{aligned}$$

(76)
$$\begin{aligned}
&\varphi(\phi,\sigma,\varrho,\psi)\\
&= \varphi_0(\phi,\sigma,\varrho,\psi)+\varphi_1(\phi,\sigma,\varrho,\psi)+\varphi_2(\phi,\sigma,\varrho,\psi)+\varphi_3(\phi,\sigma,\varrho,\psi)+\dots\\
&= \phi-0.5\sigma+\varrho-\frac{2.25\sigma\psi^{\tau}}{\Gamma(\tau+1)}+\frac{2(2.25)\sigma\psi^{2\tau}}{\Gamma(2\tau+1)}\\
&\times(\phi-0.5\sigma+\varrho)-\frac{(2.25)^2\sigma\psi^{3\tau}}{\Gamma(3\tau+1)}\left(4+\frac{\Gamma(2\tau+1)}{(\Gamma(\tau+1))^2}\right)+\dots
\end{aligned}$$

$$\begin{aligned}
&\rho(\phi,\sigma,\varrho,\psi)\\
&= \rho_0(\phi,\sigma,\varrho,\psi)+\rho_1(\phi,\sigma,\varrho,\psi)+\rho_2(\phi,\sigma,\varrho,\psi)+\rho_3(\phi,\sigma,\varrho,\psi)+\dots
\end{aligned}$$

$$
\begin{aligned}
&= \phi + \sigma - 0.5\varrho - \frac{2.25\varrho\psi^{\tau}}{\Gamma(\tau+1)} + \frac{2(2.25)\varrho\psi^{2\tau}}{\Gamma(2\tau+1)} \\
&\times (\phi + \sigma - 0.5\varrho) - \frac{(2.25)^2\varrho\psi^{3\tau}}{\Gamma(3\tau+1)}\left(4 + \frac{\Gamma(2\tau+1)}{(\Gamma(\tau+1))^2}\right) + \dots
\end{aligned} \tag{77}
$$

Equations (74-76) is the solution of equation (55) which converges to the exact solution,(when τ=1):

$$
\begin{aligned}
\mu(\phi,\sigma,\varrho,\psi) &= \frac{-0.5\phi + \sigma + \varrho - 2.25\phi\psi}{1 - 2.25\psi^2} \\
\psi(\phi,\sigma,\varrho,\psi) &= \frac{\phi - 0.5\sigma + \varrho - 2.25\sigma\psi}{1 - 2.25\psi^2} \\
\rho(\phi,\sigma,\varrho,\psi) &= \frac{\phi + \sigma - 0.5\varrho - 2.25\varrho\psi}{1 - 2.25\psi^2}.
\end{aligned} \tag{78}
$$

TABLE 4. Comparisons between the numerical and analytical solutions for equation (58) $\mu(\phi,\sigma,\varrho,\psi)$ at $\sigma = \varrho = \psi = 10^{-3}$.

ϕ	*Analytical*	*ABRDTM*	*FRTM*	$\lvert E - ABRDTM\rvert$
0.1	0.03225073125	0.03225062024	0.03225062024	3.09891×10^{-7}
	0.08450221250	0.08450210149	0.08450210149	4.17898×10^{-8}
	0.13675444380	0.13675434270	0.13675434270	3.01948×10^{-9}
	0.18900742500	0.18900731400	0.18900731400	2.06452×10^{-6}
0.5	0.24126115620	0.24126114510	0.24126114510	3.21061×10^{-7}
	0.29351563750	0.29351452640	0.29351452640	3.07221×10^{-9}
	0.34577086880	0.34577075770	0.34577075770	2.09879×10^{-7}
	0.39802685000	0.39802684967	0.39802684967	4.08559×10^{-6}
	0.45028358120	0.45028347019	0.45028347019	5.76776×10^{-7}
1.0	0.50254106250	0.50254095140	0.50254095140	5.76776×10^{-8}

6. DISCUSSION OF RESULTS

Aboodh transform of convolution of two functions was shown to exist in Theorem 3.2. In addition, the formula for Aboodh transform of Riemann Liouville derivative and Caputo derivative were also shown to exist in Theorem 3.3 and 3.4 respectively which were then used in obtaining solutions of two Navier-Stokes equations of the Caputo-Fabrizio type.

TABLE 5. Comparisons between the numerical and analytical solutions for equation (58) $\mu(\phi,\sigma,\varrho,\psi)$ at $\sigma = \varrho = \psi = 10^{-3}$ $a = \tau = 0.25$, $b = \tau = 0.75$

ϕ	$Analytical$	$ABRDTM(a)$	$ABRDTM(b)$	$\|E - ABRDTM\|$
0.1	0.03225073125	0.03007115150	0.30711550980	2.17957×10^{-3}
	0.08450221250	0.08014230370	0.08142317700	4.35990×10^{-3}
	0.13675444380	0.13021345670	0.13213487790	6.54098×10^{-3}
	0.18900742500	0.18028461050	0.18284665390	8.72281×10^{-3}
0.5	0.24126115620	0.23035576500	0.23355850490	1.09053×10^{-2}
	0.29351563750	0.28042692020	0.28427043090	1.30887×10^{-2}
	0.34577086880	0.33049807620	0.33498243190	1.52727×10^{-2}
	0.39802685000	0.38056923300	0.38569450780	1.74576×10^{-2}
	0.45028358120	0.43064039040	0.43640665880	1.96431×10^{-2}
1.0	0.50254106250	0.48071154870	0.48711888480	2.18295×10^{-2}

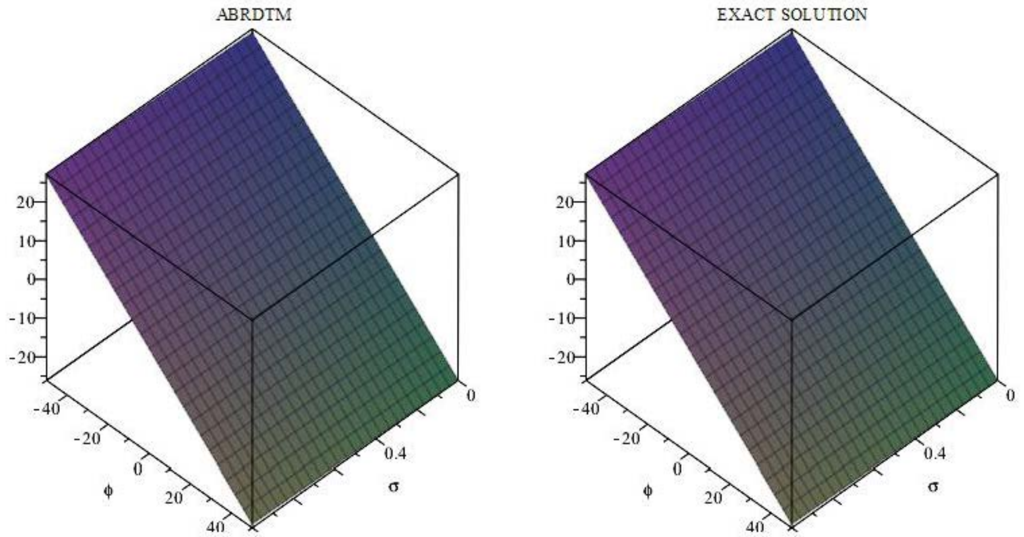

FIGURE 4. Graph of $\mu(\phi,\sigma,\varrho,\varphi)$ for equation (55) at $\tau = 1$

Tables 1, 2 3, and 4, 5, show the results of equations (40) and (58), respectively, which compared the numerical results obtained in this work with the exact solution at $\tau = 1$. Different values of τ at 0.25 and 0.75 were computed and compared to verify their effect on the solution of the problems considered. Tables 3 and 5 displays the values and errors obtained when compared with the values obtained at $\tau = 1$ for both problems solved. These results agree with the exact solutions

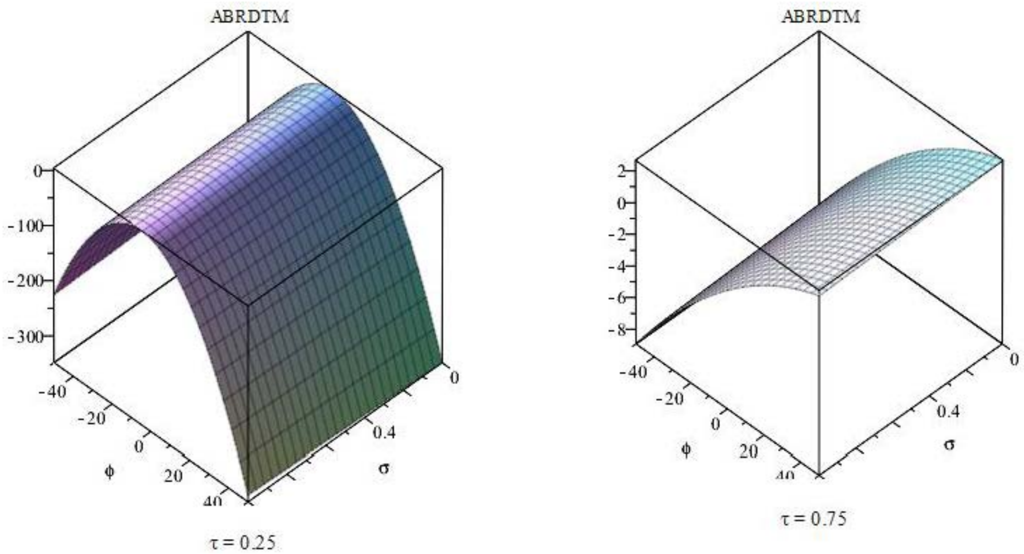

FIGURE 5. Graph of $\mu(\phi, \sigma, \varrho, \varphi)$ for equation (58) at $\tau = 0.25$ and 0.75

TABLE 6. Comparisons between the numerical and analytical solutions for equation (58) for $\varphi(\phi, \sigma, \varrho, \psi)$ at $\sigma = \varrho = \psi = 10^{-3}$.

ϕ	$Analytical$	$ABRDTM$	$FRTM$	$E - ABRDTM$
0.1	0.1047747300	0.1047636279	0.1047636279	3.09891×10^{-9}
	0.2047744488	0.2047743487	0.2047743487	4.17898×10^{-8}
	0.3047741674	0.3047741674	0.3047741674	3.01948×10^{-5}
	0.4047738862	0.4047738862	0.4047738862	4.06452×10^{-3}
0.5	0.5047736050	0.5047625940	0.5047625940	5.21061×10^{-7}
	0.6047733238	0.6047622127	0.6047622127	4.07221×10^{-6}
	0.7047730424	0.7047629393	0.7047629393	3.09879×10^{-8}
	0.8047727612	0.8047616601	0.8047616601	4.08559×10^{-7}
	0.9047724800	0.9047613985	0.9047613985	5.76776×10^{-9}
1.0	1.0047721990	1.0047610989	1.0047610989	5.76776×10^{-7}

as the errors calculated are very negligible. The choice of $\tau = 1$ is the only point where exact solutions exists for the two problems.

Figures 1,2,3,4,5 and 6 also depicts the pictorial properties of the problems considered at different values of fractional order τ. The shapes of the graphs shows the effect of the various values obtained for each problem with different values of τ considered.

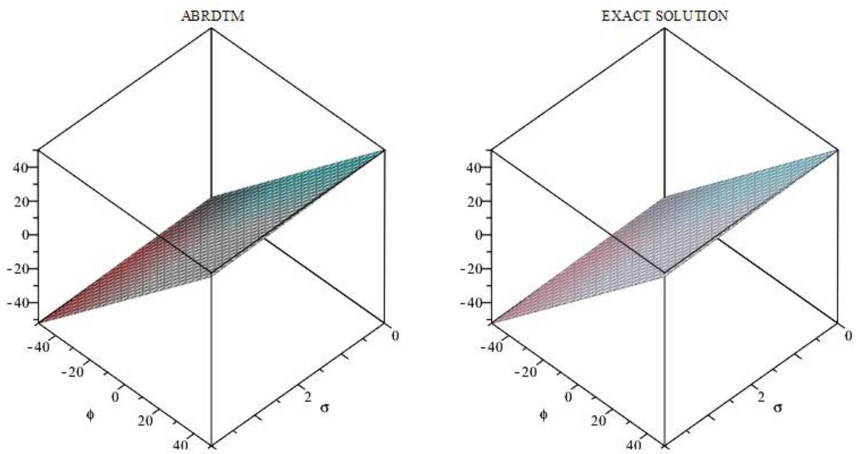

FIGURE 6. Graph of $\varphi(\phi, \sigma, \varrho, \psi)$ for equation (58) at $\tau = 1$

7. CONCLUSION

In this work, we have investigated the solutions of the N–S equations of fractional order with the aid of the Aboodh and reduced differential transform methods(ABRDTM) of the Caputo-Fabrizio type. The proposed method is a combination of Aboodh transform method [32] and reduced differential transform method [33, 34]. The combined method has been used for two nonlinear partial differential Navier -Stokes equations and provide the actual solutions in the form of convergent series. The solutions are calculated for both fractional and integer orders of the problems. The results gotten are explained and verified using graphs and tables. It is analyzed that the present technique provides the solutions of fractional-order problems in a very simple and straightforward procedure and thus suitable to compute the solutions of other nonlinear problems in various branches of applied sciences.

COMPETING INTERESTS

The authors declare that there is no competing interests.

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