ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **13** (2024), no.1, 85–96
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.13.1.6

INVERSE BOUNDARY PROBLEM FOR THE EQUATION OF FORCED VIBRATIONS OF A CANTILEVER BEAM WITH INTEGRAL CONDITIONS

R.M. Taghiyev[1], S.S. Farajova[2], A.T. Ramazonova[3], R. N. Allahverdiyev[4], and M.Y.Shalimov[5]

ABSTRACT. The paper studies the solvability of an inverse boundary value problem with an unknown time-dependent coefficient for the equation of forced vibrations of a cantilever beam with the integral. Bending transverse vibrations of a homogeneous beam under the action of an external force in the absence of rotational motion during bending are described by a fourth-order differential equation. The purpose of the work is to determine the unknown coefficient and solve the problem under consideration. THIS problem under consideration is reduced to an auxiliary equivalent problem. Next, the existence and uniqueness of a solution to the equivalent problem is proved using the contraction mapping principle. As a result, using equivalence, the uniqueness of the existence of the classical solution is proved. Classification of subjects in mathematics 2010: primary education 35R30, 35L10, 35L70; Secondary 35A01, 35A02, 35A09.

1. INTRODUCTION AND PRELIMINARY NOTES

There are many cases where the needs of the practice leads to problems in determining the coefficients or the right-hand side of the differential equations

[2]*corresponding author*
2020 *Mathematics Subject Classification.* Primary 35R30, 35L10, 35L70; Secondary 35A01, 35A02, 35A09.
Key words and phrases. inverse problems, cantilever beam, forced vibrations, nonlocal integral condition; classical solution, existence, uniqueness.
Submitted: 15.02.2024; *Accepted:* 02.03.2024; *Published:* 25.03.2024.

according to some known data of its solutions. Such problems are called inverse value problems of mathematical physics. Inverse value problems arise in various areas of human activity such as seismology, mineral exploration, biology, medicine, quality control of industrial products, etc., that states them in a number of actual problems of modern mathematics. The inverse problems are favorably developing section of up-to-date mathematics. Recently, the inverse problems are widely applied in various fields of science.

Different inverse problems for various types of partial differential equations have been studied in many papers. First of all we note the papers of A. N. Tikhonov [1], M. M. Lavrentyev [2, 3], A. M. Denisov [4], M.I. Ivanchov [5] and their followers.

Recently, a special attention has been paid to the study of a fourth-order linear differential equation describing the bending transverse vibrations of a homogeneous beam under the influence of an external force in the absence of rotational motion during bending [6, 7].

Note that the problems of oscillatory processes of beams, rods and plates play an important role in structural mechanics [8, p. 326].

In this paper, we proved the existence and uniqueness of the solution of the inverse boundary value problem for the equation of forced vibrations of a cantilever beam with integral conditions

2. Problem statement and its reduction to equivalent problem

Let $T > 0$ be some fixed number and denote by $D_T := \{(x,t):\ 0 < x < 1,\ 0 < t < T\}$. Consider the one-dimensional inverse problem of identifying an unknown pair of functions $\{u(x,t)\,,\ a(t)\}$ for the equation of forced vibrations of a cantilever beam [6,7]

$$u_{tt}(x,t) + \alpha u_{xxxx}(x,t) = a(t)u(x,t) + f(x,t) \tag{2.1}$$

with the nonlocal initial conditions

$$\begin{aligned} u(x,0) &= \varphi(x) + \int_0^T p_1(t)u(x,t)dt,\ u_t(x,0) \\ &= \psi(x) + \int_0^T p_2(t)u(x,t)dt\ (0 \le x \le 1). \end{aligned} \tag{2.2}$$

Neumann boundary conditions

(2.3) $$u_x(0,t) = u_x(1,t) = u_{xxx}(0,t) = 0 \ \ (0 \le t \le T),$$

nonlocal integral condition

(2.4) $$\int_0^1 u(x,t)dx = 0(0 \le t \le T)$$

and over determination condition

(2.5) $$u(0,t) = h(t) \ \ (0 \le t \le T)\,,$$

where $\alpha > 0$given number,$f(x,t),\ \varphi(x),\ \psi(x),\ p_i(t)\ (i = 1,2)$, and $h(t)$ are given sufficiently smooth functions of $x \in [0,1]$ and $t \in [0,T]$.

Definition 2.1. *The pair* $\{u(x,t)\,,\, a(t)\}$ *is said to be a classical solution to the problem* (2.1)-(2.5)*, if the functions* $u(x,t) \in \tilde{C}^{4,2}(\bar{D}_T)$ *and* $a(t) \in C[0,T]$ *satisfies an (Eq 1) in the region* D_T, *the condition* (2.2) *on* $[0,1]$, *and the statements* (2.3)-(2.5) *on the interval* $[0,T]$, *where*

$$\tilde{C}^{(4,2)}(\bar{D}_T) = \left\{u(x,t):\ u(x,t) \in C^2(\bar{D}_T), u_{xxxx}(x,t) \in C(\bar{D}_T)\right\}.$$

In order to investigate the problem (2.1) - (2.5), first we consider the following auxiliary problem

(2.6) $$y''(t) = a(t)y(t),\ \ t \in [0,T],$$

(2.7) $$y(0) = \int_0^T p_1(t)y(t)dt,\ \ y'(0) = \int_0^T p_2(t)y(t)dt,$$

where $p_1(t),\ p_2(t),\ a(t) \in C[0,T]$ are given functions, and $y = y(t)$ is desired function. Moreover, by the solution of the problem (2.6), (2.7), we mean a function $y(t)$ belonging to $C^2[0,T]$ and satisfying the conditions (2.6), (2.7) in the usual sense.

Lemma 2.1. *Assume that* $p_1(t), p_2(t) \in C[0,T],\ a(t) \in C[0,T], \|a(t)\|_{C[0,T]} \le R = const$, *and the condition*

$$\left(T\,\|p_2(t)\|_{C[0,T]} + \|p_1(t)\|_{C[0,T]} + \frac{T}{2}R\right)T < 1$$

hold. Then the problem (2.6), (2.7) *has a unique trivial solution.*

Now along with the inverse boundary-value problem (2.1) - (2.5), we consider the following auxiliary inverse boundary-value problem: It is required to determine a pair $\{u(x,t), a(t)\}$ of functions $u(x,t) \in \tilde{C}^{4,2}(\bar{D}_T)$ and $a(t) \in C[0,T]$, from relations (2.1)-(2.3), and

$$u_{xxx}(1,t) = 0 \,(0 \le t \le T), \tag{2.8}$$

$$a\,(t)\,h\,(t) + f\,(0,t) = h''\,(t) + \alpha u_{xxxx}\,(0,t) \qquad (\,0 \le t \le T)\,. \tag{2.9}$$

Using Lemma 2.1, similarly to [8]. we prove the following

Theorem 2.1. *Suppose that* $f(x,t) \in C(\bar{D}_T)$, $\varphi(x), \psi(x) \in C[0,1]$, $p_i(t) \in C[0,T]$ $(i = 1,2)$*,* $h(t) \in C^2[0,T]$*,* $h(t) \neq 0$*,* $\int_0^1 f(x,t)dx = 0 \quad (0 \le t \le T)$ *and the compatibility conditions*

$$\int_0^1 \varphi(x)dx = 0, \quad \int_0^1 \psi(x)dx = 0\,, \tag{2.10}$$

$$\varphi(0) + \int_0^T p_1(t)h(t)dt = h(0), \;\; \psi(0) + \int_0^T p_2(t)h(t)dt = h'(0)\ , \tag{2.11}$$

holds. Then the following assertions are valid:

(1) *each classical solution* $\{u(x,t), a(t)\}$ *of the problem* (2.1)-(2.5) *is a solution of problem* (2.1)-(2.3)*,* (2.10)*,*(2.11)*, as well;*
(2) *each solution* $\{u(x,t), a(t)\}$ *of the problem* (2.1)-(2.3)*,* (2.10)*,* (2.11)*, if*

$$\left(T\,\|p_2(t)\|_{C[0,T]} + \|p_1(t)\|_{C[0,T]} + \frac{T}{2}\,\|a(t)\|_{C[0,T]}\right) T < 1 \tag{2.12}$$

is a classical solution of problem (2.1)-(2.5).

3. Existence and uniqueness of the classical solution

We seek the first component $u(x,t)$ of classical solution $\{u(x,t)\,,\, a(t)\}$ of the problem (2.1)-(2.3), (2.10), (2.11) in the form

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x \quad (\lambda_k = k\pi)\,, \tag{3.1}$$

where

$$u_k(t) = l_k \int_0^1 u(x,t)\cos\lambda_k x dx \ (k = 0,1,\ldots), l_k = \begin{cases} 1, & k = 0, \\ 2, & k = 1,2,\ldots. \end{cases}$$

Then applying the formal scheme of the Fourier method, from (2.1) and (2.2) we have

$$u_k''(t) + \alpha\lambda_k^4 u_k(t) = F_k(t;u,a) \ \ (0 \le t \le T;\ k = 0,1,\ldots), \tag{3.2}$$

$$u_k(0) = \varphi_k + \int_0^T p_1(t)u_k(t)dt, \quad u_k'(0) = \psi_k + \int_0^T p_2(t)u_k(t)dt \quad (k = 0,1,\ldots), \tag{3.3}$$

where

$$F_k(t;u,a) = f_k(t) + a(t)u_k(t), f_k(t) = l_k \int_0^1 f(x,t)\cos\lambda_k x dx \ (k = 0,1,\ldots)\,,,$$

$$\varphi_k = l_k \int_0^1 \varphi(x)\cos\lambda_k x dx, \psi_k = l_k \int_0^1 \psi(x)\cos\ \lambda_k x dx \ \ (k = 0,1,\ldots).$$

Solving the problem (3.2), (3.3) gives

$$\begin{aligned} u_0(t) = \varphi_0 + \int_0^T p_1(t)u_0(t)dt + t\ \left(\psi_0 + \int_0^T p_2(t)u_0(t)dt\right) \\ + \int_0^t (t-\tau)F_0(\tau;u,a)d\tau \ (0 \le t \le T)\,, \end{aligned} \tag{3.4}$$

(3.5)

$$\begin{aligned} u_k(t) = \left(\varphi_k + \int_0^T p_1(t)u_k(t)dt\right)\cos\beta_k t + \frac{1}{\beta_k}\left(\psi_k + \int_0^T p_2(t)u_k(t)dt\right)\sin\beta_k t \\ + \frac{1}{\beta_k}\int_0^t F_k(\tau;u,a)\sin\beta_k\,(t-\tau)\,d\tau(\,k = 1,2,\ldots;\ 0 \le t \le T), \end{aligned}$$

where

$$\beta_k = \lambda_k^2\sqrt{\alpha}.$$

To determine the first component of the classical solution to the problem (2.1)-(2.3), (2.10), (2.11) we substitute the expressions $u_k(t) \ \ (k = 0,\,1,\ldots)$ into (3.1) and, we obtain

$$\begin{aligned} u(x,t) &= \varphi_0 + \int_0^T p_1(t)u_0(t)dt + t\left(\psi_0 + \int_0^T p_2(t)u_0(t)dt\right) \\ &+ \int_0^t (t-\tau)F_0(\tau;u,a)d\tau + \sum_{k=1}^{\infty}\left\{\left(\varphi_k + \int_0^T p_1(t)u_k(t)dt\right)\cos\beta_k t\right. \\ &+ \frac{1}{\beta_k}\left(\psi_k + \int_0^T p_2(t)u_k(t)dt\right)\sin\beta_k t \\ &+ \left.\frac{1}{\beta_k}\int_0^t F_k(\tau;u,a)\sin\beta_k(t-\tau)\,d\tau\right\}\cos\lambda_k x\,. \end{aligned} \tag{3.6}$$

It follows from (2.11) and (3.1) that

$$a(t) = [h(t)]^{-1}\left\{h''(t) - f(0,t) + \sum_{k=1}^{\infty}\alpha\lambda_k^4 u_k(t)\right\}. \tag{3.7}$$

Then from (3.7), taking into account (3.5), we find:

$$\begin{aligned} a(t) = [h(t)]^{-1}&\left\{h''(t) - f(0,t) + \sum_{k=1}^{\infty}\alpha\lambda_k^4\left[\left(\varphi_k + \int_0^T p_1(t)u_k(t)dt\right)\cos\beta_k t\right.\right. \\ &+ \frac{1}{\beta_k}\left(\psi_k + \int_0^T p_2(t)u_k(t)dt\right)\sin\beta_k t \\ &\left.\left.+\frac{1}{\beta_k}\int_0^t F_k(\tau;u,a)\sin\beta_k(t-\tau)\,d\tau\right]\right\}. \end{aligned} \tag{3.8}$$

Thus, the solution of problem (2.1) - (2.3), (2.10), (2.11) is reduced to the solution of system (3.6), (3.8) with respect to unknown functions $u(x,t)$ and $a(t)$.

Lemma 3.1. *If* $\{u(x,t), a(t)\}$ *is any solution to problem* (2.1) - (2.3)*,* (2.10)*,* (2.11)*, then the functions*

$$u_k(t) = l_k\int_0^1 u(x,t)\cos\lambda_k x dx \ \ (\ k = 0,1,2\ldots),$$

satisfies the system in $C[0,T]$*.*

It follows from Lemma 3.1 that

Corollary 3.1. *Let system (16), (18) have a unique solution. Then problem* (2.1) - (2.3)*,* (2.10)*,* (2.11) *cannot have more than one solution, i.e. if the problem* (2.1) - (2.3)*,* (2.10)*,* (2.11) *has a solution, then it is unique.*

With the purpose to study the problem (2.1) - (2.3), (2.10), (2.11), we consider the following functional spaces.

Denote by $B_{2,T}^5$ [10] a set of all functions of the form

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) \cos \lambda_k x, \quad \lambda_k = k\pi,$$

considered in the region D_T, where each of the function $u_k(t) \ \ (k = 0, \, 1, \, 2, \ldots)$ is continuous over an interval $[0\,,\,T]$ and satisfies the following condition:

$$J(u) \equiv \|\, u_0(t) \,\|_{C[0\,,\,T]} + \left\{ \sum_{k=1}^{\infty} \left(\lambda_k^5 \, \|\, u_k(t) \,\|_{C[0\,,\,T]} \right)^2 \right\}^{\frac{1}{2}} < +\infty \, .$$

The norm in this set is defined by

$$\|\, u(x,t) \,\|_{B_{2,T}^5} = J(u) \, .$$

It is known that $B_{2,T}^5$ is Banach space . Obviously, $E_T^5 = B_{2,T}^5 \times C[0,T]$ with the norm $\|z(x,t)\|_{E_T^5} = \|u(x,t)\|_{B_{2,T}^5} + \|a(t)\|_{C[0,T]}$ is also Banach space.

Now consider the operator

$$\Phi(u,a) = \{\Phi_1(u,a)\,,\, \Phi_2(u,a)\,\}\,,$$

in the space E_T^3, where

$$\Phi_1(u,a) = \tilde{u}(x,t) \equiv \sum_{k=0}^{\infty} \tilde{u}_k(t) \cos \lambda_k x \,, \Phi_2(u,a) = \tilde{a}(t)$$

and the functions $\tilde{u}_0(t)$,$\tilde{u}_k(t)$, $\ k = \ 1, \, 2, \ldots$, and $\tilde{a}(t)$ are equal to the right-hand sides of (3.4), (3.5), and (3.8), respectively.

With the help of simple transformations, we find:

$$\begin{aligned} \|\tilde{u}_0(t)\|_{C[0,T]} &\leq |\varphi_0| + T\,|\psi_0| + T(\|p_1(t)\|_{C[0,T]} + T\,\|p_2(t)\|_{C[0,T]})\,\|u_0(t)\|_{C[0,T]} \\ &+ T\sqrt{T}\left(\int_0^T |f_0(\tau)|^2\,d\tau\right)^{\frac{1}{2}} + T^2\,\|a(t)\|_{C[0,T]}\,\|u_0(t)\|_{C[0,T]} \ , \end{aligned} \tag{3.9}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k^5\,\|\tilde{u}_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \leq \sqrt{6}\left(\sum_{k=1}^{\infty}(\lambda_k^5\,|\varphi_k|)^2\right)^{\frac{1}{2}} + \sqrt{\frac{6}{\alpha}}\left(\sum_{k=1}^{\infty}(\lambda_k^3\,|\psi_k|)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{6}\left(\|p_1(t)\|_{C[0,T]}+\frac{1}{\sqrt{\alpha}}\|p_2(t)\|_{C[0,T]}\right)T\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\sqrt{\frac{6}{\alpha}T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3|f_k(\tau)|)^2d\tau\right)^{\frac{1}{2}} \tag{3.10}$$

$$+\sqrt{\frac{6}{\alpha}}T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}},$$

$$\|\tilde{a}(t)\|_{C[0,T]}\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t)-f(0,t)\right\|_{C[0,T]}+\right.$$

$$+\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\left[\alpha\left(\sum_{k=1}^{\infty}(\lambda_k^5|\varphi_k|)^2\right)^{\frac{1}{2}}+\sqrt{\alpha}\left(\sum_{k=1}^{\infty}(\lambda_k^3|\psi_k|)^2\right)^{\frac{1}{2}}\right.$$

$$+T\left(\alpha\|p_1(t)\|_{C[0,T]}+\sqrt{\alpha}\|p_2(t)\|_{C[0,T]}\right)\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \tag{3.11}$$

$$+\sqrt{\alpha T}\left(\int_0^T\sum_{k=1}^{\infty}(\lambda_k^3|f_k(\tau)|)^2d\tau\right)^{\frac{1}{2}}$$

$$\left.\left.+\ \sqrt{\alpha}T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k^5\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right\}.$$

Suppose that the data for problem (2.1)-(2.3), (2.10), (2.11) satisfy the assumptions:

1. $\varphi(x)\in C^4[0,1],\ \varphi^{(2.5)}(x)\in L_2(0,1)$ and $\varphi'(0)=\varphi'(2.1)=\varphi'''(0)=\varphi'''(2.1)=0$.
2. $\psi(x)\in C^2[0,1],\ \psi^{(2.3)}(x)\in L_2(0,1)$ and $\psi'(0)=\psi'(1)=0$.
3. $f(x,t),\ f_x(x,t),\ f_{xx}(x,t)\in C(\bar{D}_T),\ f_{xxx}(x,t)\in L_2(D_T)$,
   $f_x(0,t)=f_x(1,t)=0\ (0\leq t\leq T)$.

4. $\alpha > 0, p_i(t) \in C[0,T]\ \ (i=1,2), h(t) \in C^2[0,T],\ \ h(t) \neq 0,\ \ (0 \leq t \leq T)$. Then from (3.9)-(3.11) we get:

$$\begin{aligned} \|\tilde{u}(x,t)\|_{B^5_{2,T}} \leq A_1(T) + B_1(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}} \\ + C_1(T)\,\|u(x,t)\|_{B^5_{2,T}}\ , \end{aligned} \tag{3.12}$$

$$\begin{aligned} \|\tilde{a}(t)\|_{C[0,T]} \leq A_2(T) + B_2(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}} \\ + C_2(T)\,\|u(x,t)\|_{B^5_{2,T}}\ , \end{aligned} \tag{3.13}$$

where

$$A_1(T) = \|\varphi(x)\|_{L_2(0,1)} + T\,\|\psi(x)\|_{L_2(0,1)} + T\sqrt{T}\,\|f(x,t)\|_{L_2(D_T)} + \sqrt{6}\,\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)}$$
$$+ \sqrt{\frac{6}{\alpha}}\,\left\|\psi^{(3)}(x)\right\|_{L_2(0,1)} + \sqrt{\frac{6}{\alpha}T}\,\|f_{xxx}(x,t)\|_{L_2(D_T)},$$
$$B_1(T) = \left(T + \sqrt{\frac{6}{\alpha}}\right)T, C_1(T) = T(1+\sqrt{6})\,\|p_1(t)\|_{C[0,T]}$$
$$+ T\left(T + \sqrt{\frac{6}{\alpha}}\right)\|p_2(t)\|_{C[0,T]}\ ,$$
$$A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|h''(t) - f(0,t)\right\|_{C[0,T]} + \left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\right.$$
$$\left.\left[\alpha\,\left\|\varphi^{(5)}(x)\right\|_{L_2(0,1)} + \sqrt{\alpha}\,\left\|\psi^{(3)}(x)\right\|_{L_2(0,1)} + \sqrt{\alpha T}\,\|f_{xxx}(x,t)\|_{L_2(D_T)}\right]\right\},$$
$$B_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}\sqrt{\alpha}T\,,$$
$$C_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-2}\right)^{\frac{1}{2}}T\left(\alpha\,\|p_1(t)\|_{C[0,T]} + \sqrt{\alpha}\,\|p_2(t)\|_{C[0,T]}\right).$$

From inequalities (3.12), (3.13) we conclude:

$$\begin{aligned} &\|\tilde{u}(x,t)\|_{B^5_{2,T}} + \|\tilde{a}(t)\|_{C[0,T]} \\ &\leq A(T) + B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B^5_{2,T}} + C(T)\,\|u(x,t)\|_{B^5_{2,T}} \end{aligned} \tag{3.14}$$

where

$$A(T) = A_1(T) + A_2(T)\,,\ B(T) = B_1(T) + B_2(T)\,,\ C(T) = C_1(T) + C_2(T)\,.$$

Thus, we can prove the following theorem

Theorem 3.1. *Assume that statements 1)-4) and the condition*

$$(B(T)(A(T)+2)+C(T))(A(T)+2)<1\ , \tag{3.15}$$

holds, then problem (2.1)-(2.3)*,* (2.10)*,* (2.11) *has a unique solution in the ball* $K=K_R(\|z\|_{E_T^5}\le R\le A(T)+2)$ *of the space* E_T^5*.*

Proof. In the space E_T^5, consider the operator equation

$$z=\Phi z\,, \tag{3.16}$$

where $z=\{u,\,a\}$, and the components $\Phi_i(u,a)\ \ (i=1,2)$, of operator $\Phi(u,a)$ defined by the right sides of (3.6) and (3.8).

Consider the operator $\Phi(u,a)$in the ball $K=K_R$ out of E_T^5 .Similarly to (3.14), we obtain that for any of the estimates are valid:

$$\begin{aligned}
&\|\Phi z\|_{E_T^5}\\
&\le A(T)+B(T)\,\|a(t)\|_{C[0,T]}\,\|u(x,t)\|_{B_{2,T}^5}+C(T)\,\|u(x,t)\|_{B_{2,T}^5}\\
&\le A(T)+B(T)R^2+C(T)R\\
&=A(T)+(B(T)(A(T)+2)+C(T))(A(T)+2)\,,
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
\|\Phi z_1-\Phi z_2\|_{E_T^5}\le B(T)R(&\|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^5}+\|a_1(t)-a_2(t)\|_{C[0,T]}\\
&+C(T)\,\|u_1(x,t)-u_2(x,t)\|_{B_{2,T}^5}\ ,
\end{aligned} \tag{3.18}$$

Then it follows from (3.15), (3.17), and (3.18) that the operator Φ acts in the ball $K=K_R$, and satisfy the conditions of the contraction mapping principle. Therefore the operator Φ has a unique fixed point $\{z\}=\{u\,,\,a\}$ in the ball $K=K_R$, which is a solution of equation (3.16); i.e. the pair $\{u,a\}$ is the unique solution of the systems (3.6) and (3.8) in $K=K_R$.

Hence the function $u(x,t)$ as an element of space $B_{2,T}^5$ is continuous and has continuous derivatives $u_x(x,t),u_{xx}(x,t),u_{xxx}(x,t)$ and $u_{xxxx}(x,t)$ in D_T.

Similarly [9], one can prove that $u_t(x,t)$, $u_{tt}(x,t)$ are continuous in D_T .

It is easy to verify that Eq. (2.1) and conditions (2.2), (2.3), (2.10), (2.11) satisfy in the usual sense. So, $\{u(x,t),a(t)\}$ is a solution of (2.1)-(2.3), (2.10), (2.11), and by Lemma 2 it is unique in the ball $K=K_R$. The proof is complete. □

Finally, from Theorem 2.1 and Theorem 3.1, implies the unique solvability of the original problem (2.1) - (2.5).

Theorem 3.2. *Suppose that all assumptions of Theorem 3.1, and the conditions*

$$\int_0^1 f(x,t)dx = 0\,, \quad (0 \le t \le T), \ \int_0^1 \varphi(x)dx = 0, \ \ \int_0^1 \psi(x)dx = 0\,,$$

$$\varphi(0) + \int_0^T p_1(t)h(t)dt = h(0), \ \ \psi(0) + \int_0^T p_2(t)h(t)dt = h'(0)\,.$$

holds. Then problem (2.1) - (2.5) *has a unique classical solution in the ball* $K = K_R(\|z\|_{E_T^5} \le A(T) + 2)$ *of the space* E_T^5.

4. Conclusion

The existence and uniqueness of the solution of one inverse boundary value problem for the equation of forced vibrations of a cantilever beam with integral conditions. First, the original problem is reduced to an equivalent problem, for which the theorem of existence and uniqueness of the solution is proved. Using these facts, the existence and uniqueness of the classical solution of one inverse boundary value problem for the equation of forced vibrations of a cantilever beam with integral conditions is proved.

References

[1] A.I. Tikhonov, *On stability of inverse problems,* Dokl. AN SSSR, **39**(5) (1943), 195-198. (in Russian)

[2] M.M. Lavrent'ev, *On an inverse problem for a wave equation,* Dokl. AN SSSR, **157**(3) (1964), 520-521. (in Russian)

[3] M.M. Lavrent'ev, V.G. Romanov, S.T. Shishatsky, *Ill-Posed Problems of Mathematical Physics and Analysis,* M. Nauka, 1980. (in Russian)

[4] A.M. Denisov, *Introduction to Theory of Inverse Problems,* M: MSU, 1994, (in Russian).

[5] M.I. Ivanchov, *Inverse Problems for Equation of Parabolic Type,* Ukraine, 2003.

[6] Tikhonov A. N., Samarskii A. A., *Uravneniia matematicheskoi fiziki [Equations of Mathematical Physics].* Moscow, Nauka, 1966, 724 pp. (In Russian)

[7] Rayleigh L., *Teoriia zvuka [The Theory of Sound].* Moscow, Gostehizdat, 1955, 503 pp. (In Russian)

[8] Krylov A. N., *Vibratsiia sudov [The Ship Vibration].* Leningrad, Moscow, 1936, 442 pp. (In Russian)

[9] YA.T.MEHRALIEV AND F.KH. ALIZADE, *Inverse boundary value problem for a Boussinesq type equation of fourth order with nonlocal time integral conditions of the second kind,* Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki, **26**(4) (2016), 503-514. (in Russian).

[10] YA.T. MEHRALIEV, *On an inverse boundary value problem for a second order elliptic equation with integral condition,* Visnyk of the Lviv Univ. Series Mech. Math. 2012. Issue 77, 145–156.

DEPARTMENT OF LINEAR ALGEBRA AND MATHEMATICAL ANALYSIS, MATHEMATICS
WESTERN CASPIAN UNIVERSITY
BAKU, 27, ISTIGLALIYYAT STR.,
AZERBAIJAN.
Email address: `rauf.tagiyev1953@gmail.com`

DEPARTMENT OF LINEAR ALGEBRA AND MATHEMATICAL ANALYSIS, MATHEMATICS
UNIVERSITY OF WESTERN CASPIAN UNIVERSITY
BAKU, 27, ISTIGLALIYYAT STR.,
AZERBAIJAN.
Email address: `sona.ferecova@wcu.edu.az`

DEPARTMENT OF MATHEMATICS, NONLINEAR OPTIMIZATION ...
UNIVERSITY DUISBURG-ESSEN,
ESSEN, GERMANY, THEA-LEYMAN STR.9; D-45127
COUNTRY.
Email address: `aysel.ramazanova@uni-due.de`

DEPARTMENT OF LINEAR ALGEBRA AND MATHEMATICAL ANALYSIS, MATHEMATICS
UNIVERSITY OF WESTERN CASPIAN UNIVERSITY
BAKU, 27, ISTIGLALIYYAT STR.,
AZERBAIJAN.
Email address: `ramizallahverdiyev198@gmail.com`

DEPARTMENT OF GENERAL AND APPLIED MATHEMATICS
AZERBAIJAN STATE OIL AND INDUSTRY
BAKU,STREET AZADLIQ, 20
AZERBAIJAN.
Email address: `matlab.salimov@asoiu.edu.az`