ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **13** (2024), no.2, 97–115
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.13.2.1

SOLUTION OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS BY A DEFORMATION BASED SAWI'S TRANSFORM

R.A. ODERINU[1], K.A. SALAUDEEN[2], W.A. TIJANI[3], AND S.O. SANGONIYI[4]

ABSTRACT. In this paper, a new method for solving nonlinear Klein-Gordon and Korteweg-de-Vries (KdV) partial differential equations based on the combination of the Sawi Transform (ST) and the Homotopy Analysis Method (HAM) was proposed. It was shown that Sawi transform can transform nonlinear patial differential equations into an algebraic form which can then be solved using HAM. Three examples were considered to demostrate the effectiveness and efficiency of the proposed method. The results indicated that the proposed method is a promising approach for solving nonlinear partial differential equations.

1. INTRODUCTION

Integral transforms are one of the most easy and effective methods for solving problems arising in mathematical physics, applied mathematics, engineering and sciences, signal processing [8]. They are mathematics operations that connect functions or mathematical objects in one domain to another domain through the use of integration, these transforms simplify the analysis of mathematical models and systems and are of the following advantages such as simplification and compact representation of function, frequency analysis of function by decomposing

[2]*corresponding author*

2020 *Mathematics Subject Classification.* 35F20.
Key words and phrases. Nonlinear Klein-Gordon and Korteweg-de-Vries (KdV), Partial Differential Equation, Sawi Transform, Homotopy Analysis Method.
Submitted: 23.03.2024; *Accepted:* 07.04.2024; *Published:* 16.04.2024.

the function in frequency components. They as well convert differential equations into algebraic equations thereby making them easier to solve [2, 6, 9,13, 14, 27, 28]. They are mostly used mathematical techniques to determine the answers of advance problems of space, science, technology and engineering. Several integral transforms such as Laplace [16, 29], Elzaki [2], Aboodh [27], Sumudu, Mohand [31], Sawi [9, 13, 14, 23] have been used extensively to find analytical solution of linear problems of differential equations be it classical (ordinary differential equation, partial differential equation,integral equations)and fractional order differential equations. These integral transforms were combined with other numerical methods such as homotopy pertubation method [32], homotopy analysis method [16, 18, 29], variational iteration method [7], adomian decomposition method [13,] to find approximate solution of integro-differential equation, ordinary differential equation, partial differential equations of classical type as well as fractional order differential equation [1, 3, 8, 10, 12, 16, 18, 19, 22]. Klein-Gordon equation has important applications in plasma physics together with Zakharov equations describing the interaction of Langmuir wave also known as a plasma oscillation and the ion acoustic wave in plasma, It is a relativistic wave equation that describes the behaviour of scalar particles with spin zero. This equation was proposed by Oskar Klein and Walter Gordon in 1926 as an attempt to incorporate relativistic effects into the Schrödinger equation,it also describes the propagation of particles with mass and satisfies relativistic invariance [7, 8, 17], it as well explains several phenomena in the quantum realm that is the structure of atoms and molecules, Its drawbacks is that it allows for negative probabilities which lead to difficulties in interpreting it as a probability equation [17, 25, 33, 34]. Klein-Gordon equation provides a simple but rich model to describe a self-interacting scalar field.

2. Concept of Homotopy Analysis Methods

The basic idea of HAM is explained in this section by considering the differential equation [18, 19, 21]:

$$N[u(x,t)] = 0, \quad (x,t) \in \Omega, \tag{2.1}$$

where N is the operator both linear and nonlinear, x and t are the independent variables, u is the unknown function in the domain Ω. Next,

$$H(\phi, s) \equiv (1-s)[L(\phi(x;s)) - u_0(x)] - shN(\phi(x;s)). \tag{2.2}$$

The homotopy operator H is defined as in [12], where $s \in [0,1]$ is an embedding parameter and $h \neq 0$ is the convergence control parameter , u_0 is an initial guess of the solution of Eq. (2.1), ϕ is an unknown function and L is the auxiliary linear operator satisfying the feature $L(0) = 0$ when $H(\phi, s)$ is consider to be zero [12, 19]. Therefore,

$$(1-s)[L(\phi(x;t;s)) - u_0(x,t)] = shN(\phi(x;t;s)), \tag{2.3}$$

which is known as the zeroth-order deformation equation. From Eqn. (2.3), it can be observed that if $s = 0$,

$$L(\phi(x,t;0)) - u_0(x,t) = 0, \tag{2.4}$$

which gives $\phi(x;0) = u_0(x,t)$. Conversely, if $s = 1$, Eqn. (2.3) reduces to $N(\phi(x,t;s)) = 0$, and this gives $\phi(x,t;1) = u(x)$. Thus, by replacing s from 0 to 1, the result changes from u_0 to u.

Using Maclaurin Series, the function $\phi(x,t;s)$ with parameter s may be written as [12]

$$\phi(x,t;s) = \phi(x,t;0) + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{\partial^k \phi(x,t;s)}{\partial s^k}\bigg|_{s=0}, \tag{2.5}$$

representing

$$y_k(x,t) = \frac{1}{k!} \frac{\partial^k (x,t;s)}{\partial s^k}\bigg|_{s=0} k = 1,2,3. \tag{2.6}$$

Then Eqn. (2.5) changes to

$$\phi(x,t;s) = u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t) s^k. \tag{2.7}$$

If the series in eqn. (2.6) converged for $s = 1$, then the solution of Eqn. (2.1) is given as [12]

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t). \tag{2.8}$$

To find the function u_k, Eqn. (2.3) is differentiated, k times with respect to s and the result is divided by $k!$ where $s = 0$ [12] through this kth-order deformation equation for $k > 0$ is defined as

$$L(y_k(x,t) - x - ky_{k-1}(x,t)) = hH(x,t)\phi_k(y_{k-1}(x,t)), \tag{2.9}$$

where $H(x,t)$ is the auxiliary function.

3. Definition of Sawi Transform

Sawi transform of the function $f(t)$ for all $t \geq 0$ is defined as:

$$S(f(t)) = \frac{1}{w^2}\int_0^\infty f(t)e^{-(\frac{t}{w})}dt = f(w), \quad w > 0, \tag{3.1}$$

where S stands for Sawi transform operator , w stands for complex frequency domain parameter and $t = \mathbb{R} \geq 0$ [14]

$$X_n = \begin{cases} 0, & n \leq 1 \\ 1, & n > 1 \end{cases}$$

and

$$\phi k(y_{k-1}(x,t)) = \frac{1}{(k-1)!}\left(\frac{\partial^{k-1}}{\partial p^{k-1}} N\left(\sum_{j=1}^{\infty} y_j(x)p^j\right)\right).$$

Table 1. Sawi transform of some functions [31]

S/N	$F(t)$	$S(F(t)) = f(w)$
1	1	$\frac{1}{w}$
2	t	1
3	t^2	$2!w$
4	$t^k, k \in K$	kw^{k-1}
5	$t^k, k > -1$	$\Gamma(k+1)w^{k-1}$
6	e^{xt}	$\frac{1}{w(1-xw)}$
7	$\sin xt$	$\frac{x}{1+x^2w^2}$
8	$\cos xt$	$\frac{1}{w(1+x^2w^2)}$
9	$\sinh xt$	$\frac{x}{1-x^2w^2}$
10	$\cosh xt$	$\frac{1}{w(1-x^2w^2)}$

3.1. **Deformation Based Sawi's Transform (SHAM).** Consider the nonlinear differential equation

$$N(y(x,t)) = f(x,t), \tag{3.2}$$

where N is the general nonlinear operator including both linear and nonlinear terms in which the linear term is divided into $G + L$ where L is the highest order linear operator and G is the remaining of the linear operator,$F(y)$ is the nonlinear operator and $g(x,t)$ is the source term. Then the equation can be written as

$$L(y) + R(y) + F(y) = g(x,t). \tag{3.3}$$

Applying Sawi transform on both sides of eqn.(3.3) gives

$$S[L(y)] + S[R(y)] + S[F(y)] = S[g(x,t)], \tag{3.4}$$

where

$$S[L(y)] = \frac{Y(w)}{w^n} - \sum_{j=0}^{n-1} \frac{y^j(0)}{w^{n-j+1}}. \tag{3.5}$$

Substituting Eqn.(3.5) into Eqn.(3.4), then

$$\bar{y}(x,w) - \sum_{j=0}^{n-1} \frac{y^j(0)}{w^{1-j}} + w^n S[L(y)] + w^n S[R(y)] - w^n S[g(x,t)] = 0. \tag{3.6}$$

By applying n-th order deformation equation to find $\bar{y}_0, \bar{y}_1, \ldots$, we have

$$S[\bar{y}_n(x,w)] - \chi_n \bar{y}_{n-1}(x,w) = hD_{n-1}(N(\phi(x,w,q))),$$

and so,

$$\bar{y}_n(x,w) - \chi_n \bar{y}_{n-1}(x,w) \tag{3.7}$$

$$= hD_{n-1}\Bigg(\bar{\phi}(x,w;q) - \sum_{j=0}^{n-1} \frac{y^j(0)}{w^{1-j}} \tag{3.8}$$

$$+ w^n(S(L(y)) + S[R(y)]) - w^n S[\bar{g}(x,t)]\Bigg),$$

where

$$y_n(x,w) = S^{-1}\Bigg(\chi_n \bar{y}_{n-1}(x,w)) + h\Bigg(\bar{y}_{n-1}(x,w) - (1-\bar{\chi}_{n-1})\Bigg(\sum_{j=0}^{n-1}\frac{y^j(0)}{w^{1-j}} - w^n S[\bar{g}(x,t)]\Bigg) + D_{n-1}\Bigg(w^n(S(L(y))) + S(R(y))\Bigg)\Bigg)\Bigg), \tag{3.9}$$

where y_n is the solution of the equation, h is the convergent control parameter, w is the complex number, t is the real number greater than or equal to zero

$$\bar{\chi}_{n-1} = \begin{cases} 0, & n-1<1 \\ 1, & n-1\geq 1 \end{cases},$$

where y_0 is evaluated from the simplified expression that emerges from the given initial condition and the source term.The successful iteration is obtained by taking the Sawi inverse of equation (15).

3.2. **Application 1.** Consider the nonlinear Klein-Gordon equation [17]:

$$\frac{d^2y}{dt^2}(x,t) - \frac{d^2y}{dx^2}(x,t) + y^2(x,t) = x^2t^2, \tag{3.10}$$

with the initial conditions

$$y(x,0) = 0, \quad y_t(x,0) = x. \tag{3.11}$$

The exact solution of eqn. (3.9) is given by

$$y(x,t) = xt. \tag{3.12}$$

Apply Sawi transform on both sides of Eq. (3.9) subject to the initial conditions in eqn. (3.10). then,

$$\bar{y_n}(x,w) - x + w^2S\Big(y^2 - \frac{d^2y}{dx^2})\Big) - 2w^3x^2 = 0. \tag{3.13}$$

Applying n-th order deformation equation on Eq. (3.12) by subjecting $h=-1$ and using homotopy derivative properties, we have

$$\bar{y_n}(x,w) = \chi_n \bar{y}_{n-1}(x,w) - \Big(\bar{y}_{n-1}(x,w) + (1-\bar{\chi}_{n-1})\Big(-x-2w^3x^2\Big)\Big) \tag{3.14}$$

$$+ D_{n-1}\Big(w^2 S\Big(y^2 - \frac{d^2y}{dx^2}\Big)\Big), \tag{3.15}$$

where

$$D_{n-1}\Big(w^2 S\Big(\phi^2(x,w,q)\Big)\Big) = w^2 S\Big(\sum_{i=0}^{n-1} y_i y_{n-1-i}\Big)$$

and

$$S\Big(y(x,t)\Big) = x + 2w^3x^2 + w^2 S\Big(\frac{d^2 y_{n-1}}{dx^2} - \sum_{i=0}^{n-1} y_i y_{n-1-i}\Big). \tag{3.16}$$

Thus,

$$y_0(x,t) = S^{-1}\Big(x + 2w^3x^2\Big) = xt + \frac{x^2t^4}{12}. \tag{3.17}$$

For $n \geq 1$, recursive relation is given as

$$y_n(x,t) = S^{-1}\Big(w^2 S\Big(\frac{d^2 y_{n-1}}{dx^2} - \sum_{i=0}^{n-1} y_i y_{n-1-i}\Big)\Big). \tag{3.18}$$

The following iterations were obtained from eqn. (24) as follows,

$$y_0 = xt + \frac{x^2t^4}{12}, \qquad y_1 = \frac{t^6}{180} - \frac{x^4t^4}{12} - \frac{x^3t^7}{252} - \frac{x^4t^{10}}{12960}$$

$$y_2 = \frac{t^6}{180} - \frac{x^2t^{12}}{71280} - \frac{xt^9}{5670} + \frac{x^3t^7}{252} = \frac{11x^4t^{10}}{45360} + \frac{37x^5x^{13}}{7076160} + \frac{x^6t^{16}}{18662400}$$

$$y(x,t) = \sum_{i=0}^{n-1} y_n(x,t) = y_0 + y_1 + y_2 + \ldots$$

$$y(x,t) = xt + \frac{x^4t^{10}}{6048} - \frac{x^2t^{12}}{71280} - \frac{xt^9}{5670} + \frac{37x^5t^{13}}{7076160} + \frac{x^6t^{16}}{18662400}$$

3.3. **Application 2.** Consider the nonlinear partial differential equations [10]:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial v^2}{\partial x} - v + v^2 = 0, \tag{3.19}$$

with initial conditions $v(x,0) = e^{-x}$.

Applying Sawi transform derivative property using given initial conditions and applying nth order deformation as well as homotopy derivative property, then the recursive relation is given as

$$v_n(x,w) = -S^{-1}\left(\frac{1}{w}S\left(\frac{1}{2}\frac{\partial}{\partial v}\sum_{i=0}^{n-1} v_i v_{n-1-i} - v_{n-1} + \sum_{i=0}^{n-1} v_i v_{n-1-i}\right)\right). \tag{3.20}$$

By varying the values of n from 1 to 5 in eqn. (3.18) then the following iterates were obtained

$$v_0 = e^{-x}, \qquad v_1 = te^{-x}, \qquad v_2 = \frac{t^2}{2}e^{-x}$$
$$v_3 = \frac{t^3}{6}e^{-x}, \qquad v_4 = \frac{t^4}{24}e^{-x}, \qquad v_5 = \frac{t^5}{120}e^{-x}.$$

The solution of Eqn. (3.17) is given as

$$y(x,t) = \sum_{i=0}^{n-1} y_n(x,t) = y_0 + y_1 + y_2 + \dots$$

$$y(x,t) = e^{-x}\left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120}\right) \tag{3.21}$$

The closed form solution of (3.19) is given as:

$$y(x,t) = e^{t-x}. \tag{3.22}$$

3.4. **Application 3.** Consider the nonlinear KDV equation [3]:

$$\frac{\partial y}{\partial t} - 6y\frac{\partial y}{\partial x} + \frac{\partial^3 y}{\partial x^3} = 0, \tag{3.23}$$

with the initial conditions

$$y(x,0) = \frac{x}{6}. \tag{3.24}$$

The exact solution of Eqn. (3.21) is given by

$$y(x,t) = \frac{x}{6(1-t)}. \tag{3.25}$$

Applying Sawi transform on both sides of Eqn. (3.21) subject to the initial condition (3.22) gives,

$$\bar{y_n}(x,w) - \frac{x}{6w} - w\Big[S\Big(-\frac{d^3y}{dx^3} + 6y\frac{dy}{dx}\Big)\Big] = 0. \tag{3.26}$$

Simplifying Eqn. (3.24) using n-th-order deformation equation, leads to

$$\begin{aligned}\bar{y}_n(x,w) - \chi_n\bar{y}_{n-1}(x,w) &= hD_{n-1}(N(\bar{\phi}(x,w,q)))\\ &= hD_{n-1}\Big((\bar{\phi}(x,w,q)) - \frac{x}{6w} + wS\Big(\frac{\partial^3 y}{\partial x^3} - 6y\frac{\partial y}{\partial x}\Big)\Big)\end{aligned} \tag{3.27}$$

Applying Homotopy derivatives property on Eqn. (3.25) with $h=-1$ gives

$$\begin{aligned}\bar{y}_n(x,w) &= -(1-\chi_n)\bar{y}_{n-1}(x,w) + (1-\bar{\chi}_{n-1})\Big(\frac{x}{6w}\Big)\\ &+ \Big(wS\Big(6\sum_{i=0}^{n-1} y_i\frac{\partial y_{n-1-i}}{\partial x} - \frac{\partial^3 y_{n-1}}{\partial x^3}\Big)\Big).\end{aligned} \tag{3.28}$$

The initial approximation y_0 is obtaned from Eqn.(35) having taken inverse Sawi transform thus:

$$y_0(x,t) = S^{-1}\Big(\frac{x}{6w}\Big) = \frac{x}{6}. \tag{3.29}$$

The recursive relation for the solution of Eqn.(30) is given as

$$y_n(x,t) = S^{-1}\Big(wS\Big(6\sum_{i=0}^{n-1} y_i\frac{\partial y_{n-1-i}}{\partial x}\Big) - \frac{\partial^3 y_{n-1}}{\partial x^3}\Big). \tag{3.30}$$

For $n \geq 1$ in Eqn. (3.28), the following iterations were obtained thus:

$$y_0(x,t) = \frac{x}{6}, \qquad y_1(x,t) = \frac{xt}{6}, \qquad y_2(x,t) = \frac{xt}{6}$$
$$y_3(x,t) = \frac{xt^3}{6}, \qquad y_4(x,t) = \frac{xt^4}{6}, \qquad y_5(x,t) = \frac{xt^5}{6}$$

The solution of Eqn. (3.21) is obtained as

$$\begin{aligned}y(x,t) &= \sum_{i=0}^{n} y_n(x,t)\\ y(x,t) &= \frac{x}{6}\Bigg(1 + t + t^2 + t^3 + t^4 + t^5\Bigg).\end{aligned} \tag{3.31}$$

The closed form solution of Eqn. (3.29) is given as

$$y(x,t) = \frac{x}{6(1-t)}. \tag{3.32}$$

TABLE 2. Comparison of Solution of SHAM, PITM with Exact and absolute error for x=1 at different values of t

t	SHAM	PITM	Exact	AESHAM	AEPITM [17]
0.1	0.1000000000	0.1000000000	0.1000000000	0.0000000000	0.0000000000
0.2	0.1999999998	0.1999999998	0.2000000000	0.0000000002	0.0000000002
0.3	0.2999999915	0.2999999910	0.3000000000	0.0000000085	0.0000000090
0.4	0.3999998900	0.3999998820	0.4000000000	0.0000001100	0.0000001180
0.5	0.4999992114	0.4999991372	0.5000000000	0.0000007886	0.0000008628
0.6	0.5999960883	0.5999956314	0.6000000000	0.0000039117	0.0000043686
0.7	0.6999849554	0.5999828312	0.7000000000	0.0000150446	0.0000171688
0.8	0.7999519818	0.7999439424	0.8000000000	0.0000480182	0.0000560576
0.9	0.8998671262	0.8998411081	0.9000000000	0.0001328738	0.0001588919
1.0	0.9996715886	0.9995971250	1.0000000000	0.0003284114	0.000402875
MAE				5.29167×10^{-5}	6.403519×10^{-4}

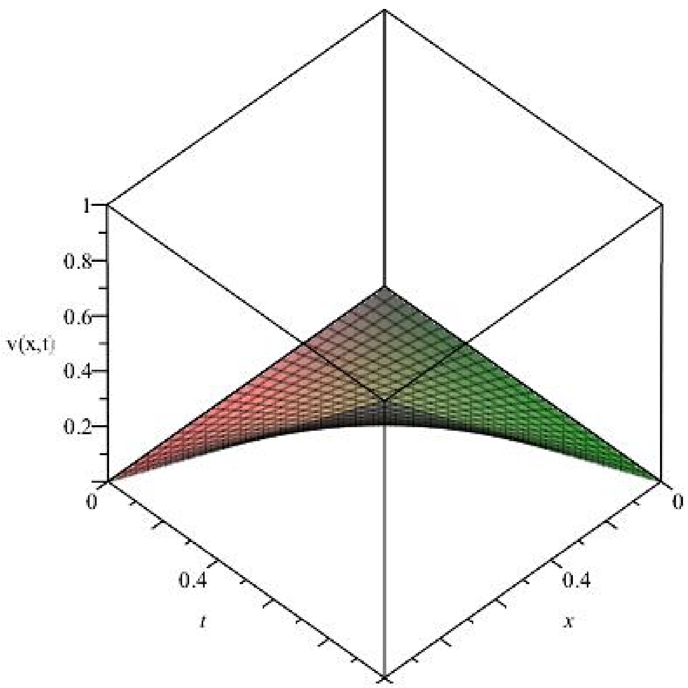

FIGURE 1. SHAM of $y(t)$

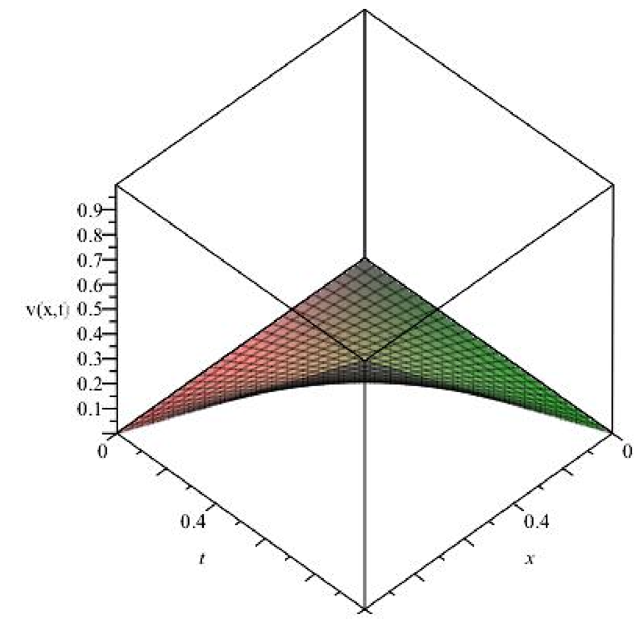

FIGURE 2. HPM of y(t)

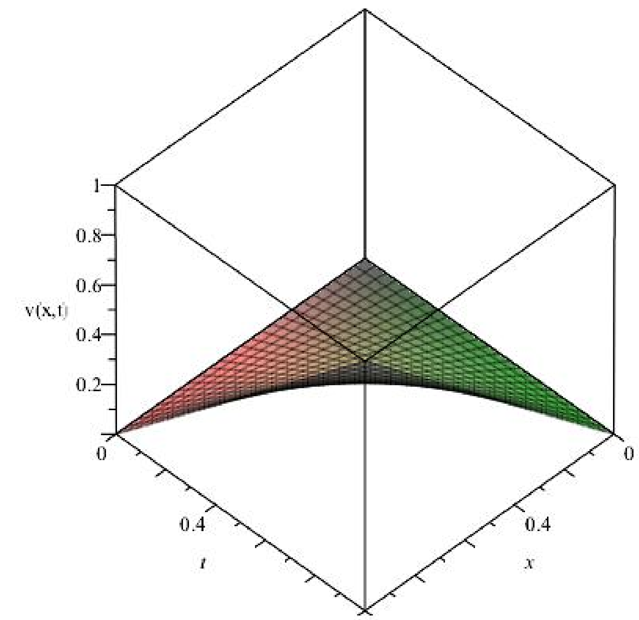

FIGURE 3. Exact of y(t)

Solution of Exact and SHAM against t in application 1

TABLE 3. Comparison of SHAM, HPM with Exact and absolute errors for different values of t at $x = 1$ for aplication 2

t	SHAM	HPM	Exact	AESHAM	AEHPM [10]
0.1	0.4065696592	0.4065696592	0.4065696597	0.00000000005	0.00000000005
0.2	0.4493289304	0.4493289304	0.4493289641	0.0000003370	0.0000003370
0.3	0.4965849149	0.4965849149	0.4965853038	0.00000003887	0.00000003887
0.4	0.5488094175	0.5488094175	0.5488116361	0.00000221860	0.00000221860
0.5	0.6065220683	0.6065220683	0.6065306597	0.00000859140	0.00000859140
0.6	0.6702940001	0.6702940001	0.6703200460	0.00002604590	0.00002604590
0.7	0.7407515275	0.7407515275	0.7408182207	0.00006669320	0.00006669320
0.8	0.8185798263	0.8185798263	0.8187307531	0.00015092680	0.00015092680
0.9	0.9045266111	0.9045266111	0.9048374180	0.00031080690	0.00031080690
1.0	0.9994058152	0.9994058152	1.0000000000	0.00059418480	0.00059418480
MAE				$1.15954022 \times 10^{-4}$	$1.15954022 \times 10^{-4}$

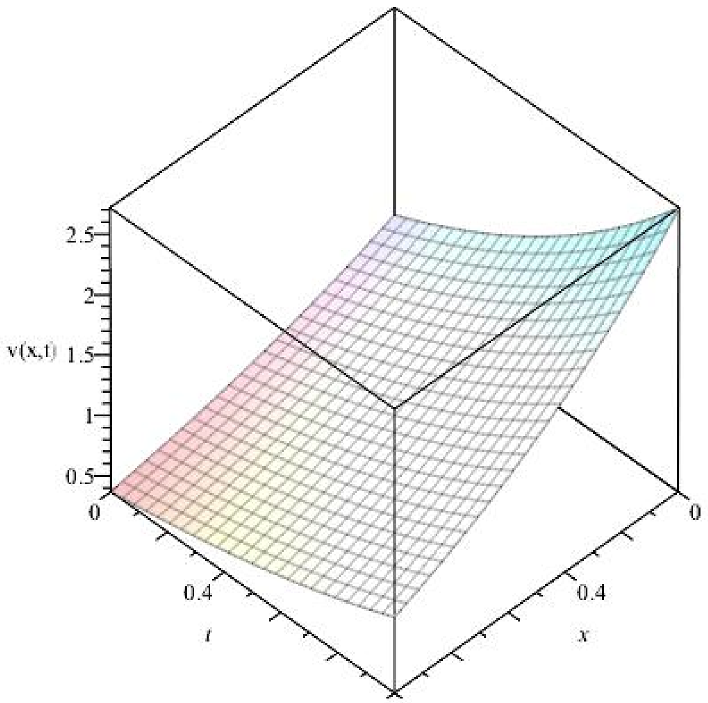

FIGURE 4. SHAM of v(t)

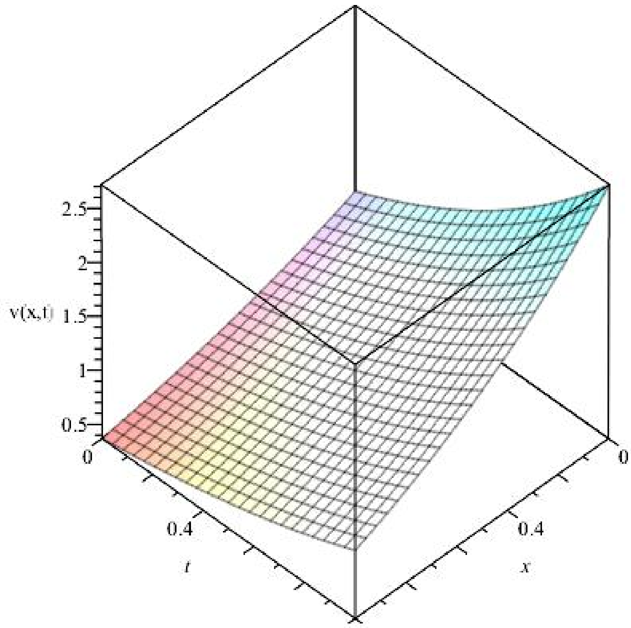

FIGURE 5. Exact of v(t)

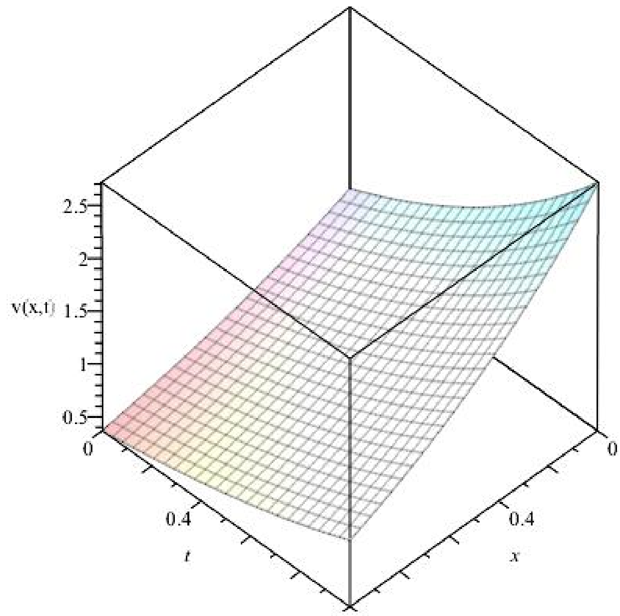

FIGURE 6. HPM of $v(t)$

Solution of Exact and SHAM of application 2

TABLE 4. Comparison of Solutions of SHAM with Exact and absolute errors for different values t of aplication 3

t	x	SHAM	Exact	AESHAM	AEHPM [3]
0.2	0.01	0.002083200000	0.002083333333	$8.320000000 \times 10{-5}$	$8.320000000 \times 10{-5}$
0.4	0.01	0.002766400000	0.002777777778	$4.330666670 \times 10{-4}$	$4.330666670 \times 10{-4}$
0.6	0.01	0.003972266667	0.004166666667	$1.30560000 \times 10{-3}$	$1.30560000 \times 10{-3}$
0.8	0.01	0.006148800000	0.008333333333	$3.148800000 \times 10{-3}$	$3.148800000 \times 10{-3}$
0.10	0.01	0.01000000000	0.003333333333	$6.666666667 \times 10{-3}$	$6.666666667 \times 10{-3}$
0.2	0.05	0.01041600000	0.01000000000	$4.160000000 \times 10{-4}$	$4.160000000 \times 10{-4}$
0.4	0.05	0.01383200000	0.01166666667	$6.252800000 \times 10{-3}$	$2.165300000 \times 10{-3}$
0.6	0.05	0.01986133333	0.013333333333	$6.252800000 \times 10{-3}$	$6.252800000 \times 10{-3}$
0.8	0.05	0.0374400000	0.01500000000	$1.574400000 \times 10{-2}$	$1.574400000 \times 10{-2}$
1.0	0.05	0.05000000000	0.016666666667	$1.33333333 \times 10{-2}$	$1.33333333 \times 10{-2}$

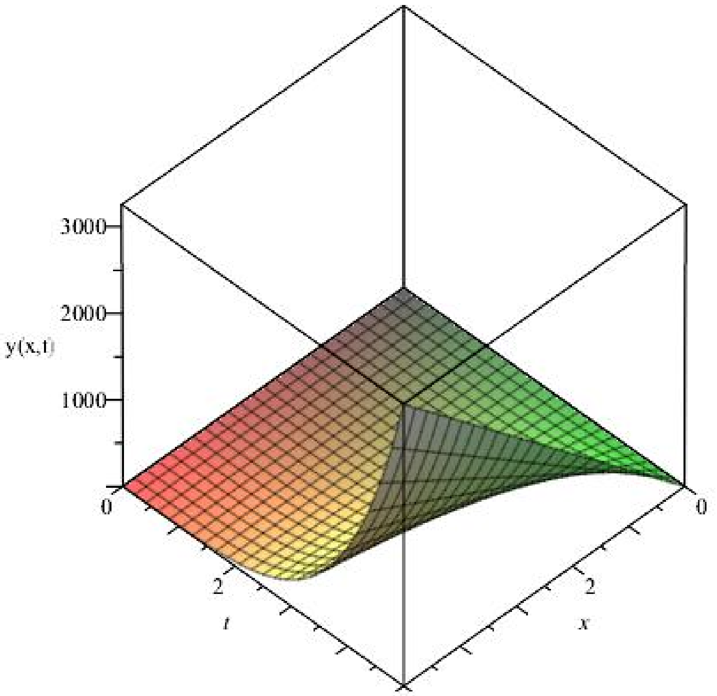

FIGURE 7. SHAM of $y(t)$

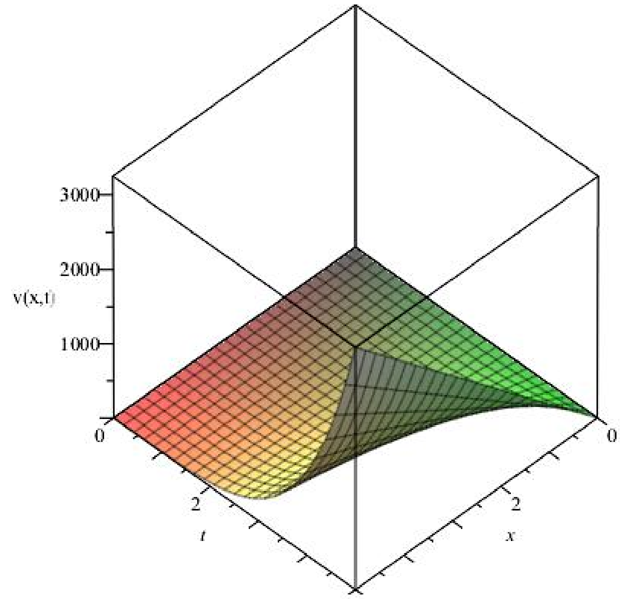

FIGURE 8. HPLTM of $y(t)$

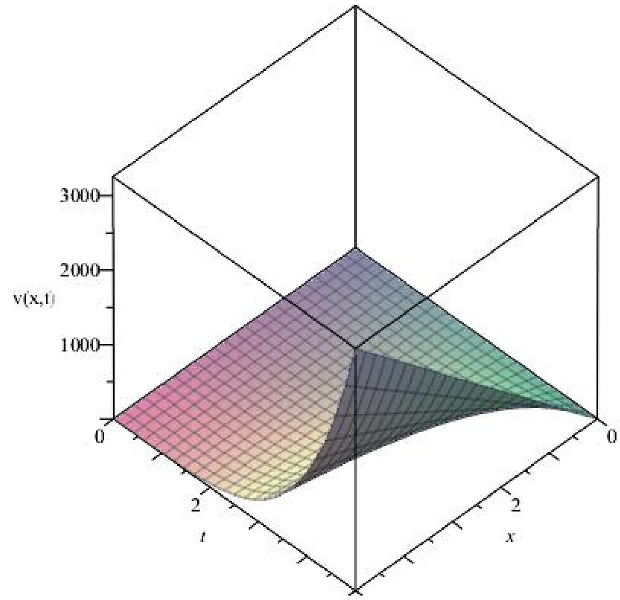

FIGURE 9. Exact of $y(t)$

Solution of Exact and SHAM of application 3

4. Results and Discussion

Table 1 shows the results of the Eqn.(17) with its coresponding absolute error. The maximum and minimum error calculated is 1.3287×10^{-3} and 2.0000×10^{-9}. These errors were compared with PITM in [18] which were $1.5889 \times 10^{-3}$and 2.0000×10^{-9}.Also, the MAE value for SHAM is 5.2845726×10^{-5} while that of [18] was 6.403519×10^{-4} and this shows that SHAM performed slightly better than the result of the referenced literature. The 3D plot of SHAM, PITM and Exact for Eqn.(18) were presented and the shape of the figure are the same for the three methods which demonstrated the behaviour of the classical equation of motion for a free massive scalar field Klein Gordon equation.Table 2 shows the results of the non linear gas dynamic partial differential together with absolute error which was obtained by subtracting each result obtained at each values of t from the exact solution ,it was observed that the Mean Absolute Error obtained from SHAM was the same with the solution of [10] which shows that SHAM compared favourably with the referenced solution. the graph agrees with each other as it describes the behaviour of gases in motion under the influence of various forces.

The solution of Eqn.(25) was shown in Table 3 together with its absolute error from which it was observed that maximum and minimum error of SHAM is 1.5093×10^{-3} and 5×10^{-9}.Which is the same result with that of [15]. The MAE calculate for SHAM and that of HPM were the same which is 1.1595×10^{-4}. figure 4, 5 and 6 are the graphical representation of the method considered with its Exact, this graph show the behaviour of airflow over an aeroplane wing to flow of water in a pipe which in essence means the effect of pressure, temperature as well as density. The graphs were in excellent agreement with one another.The solution of Eqn.(29) is presented in Table 3 together with its exact solution and its absolute error. The maximum and minimum error for SHAM is 2.0686×10^{-3} and 1.852×10^{-5} respectively.These errors were obtained by finding the absolute difference between the calculated values and the Exact values at each points, the same result were obtained with that of [3]. MAE was as well the same and was calculated as 1.9361×10^{-3}. Figure 7, 8 and 9 were the graphical representation of Eqn.(29) for SHAM, HPM and Exact. The figures shows the shape of waves in shallow water and it was discovered that there is an agreement between the solution of the methods. The solution of the non linear KdV equation was presented in Table 3 with the Mean Absolute Error (MAE) calculted, the obtained MAE was the same with that of Ahmet (2009) and this demonstrated the efficiency and applicability of SHAM, the graphs also agrees with that of the reference solution.

5. Conclusion

This work has achieved successful applications of the homotopy analysis Sawi transform method to solve nonlinear Klein-Gordon and Korteweg-deVries (KDV) equations. Through the consideration of three examples, the results obtained demonstrate the efficiency and effectiveness of the proposed method. Furthermore, the method's versatility is highlighted, as it can be applied to solve various types of higher-order nonlinear differential equations, whether they are partial or ordinary differential equations.

References

[1] B.S. Abbas: *The application of homotopy analysis method to solve a generalized hirota-satsuma coupled kdv equation*, Physics letters, section A, **362**(6) (2007), 478-483.

[2] S. Aggarwal, M. Rashmi, A. Kumar: *A comparative study of Mohand and El-zaki transforms*, Global Journal of Engineering, Science and Researches, **6**(2) (2019), 203-213.

[3] Y. Ahmet: *on the solution of the nonlinear Korteweg-de-Vries equation by the homotopy perturbation method*, Commun. numer. meth. Engng, **2009**(25) (2009), 1127-1136.

[4] S. Ali, R. Essam, M.D.A. El-Zahar, D.C. Jae: *An efficient approach for solution of Fractional-order Helmholtz Equations*, Advance in Difference equation, **2021**(14) (2021), 1-15.

[5] G.B. Arfken, H.J. Weber, E.E. Harris: *Mathematical methods for physicists (7th Ed.), Academic press, 2012.*

[6] T.G. Atanaska, P. Albena*: Application of the double Fuzzy Sawi transform for solving a telegraph equation, Symmetry,* **15) (2023), art. np. 854.**

[7] B. Batiha B: *Variation Iterative method for solving the nonlinear Klein-Gordon equation*, **Australian Journal of Basic and Applied sciences, 3(4) (2009), 3876-3890.**

[8] M.S.H. Chowdhury,I. Hashim: *Application of homotopy perturbation method to klein-Gordon and sine-Gordon equations*, **Chaos Solitons Fractals, 39(4) (2009), 1928-1935.**

[9] P.P. Dinkar: *Application of Sawi transform of Error Function for evaluating improper integral,Journal of research and development, A multidisciplinary international level referred and peer reviewed journal,* **11***(20) (2021), 41-45.*

[10] A. Eman, F. Asad*: Analytical and numerical solution for fractional gas dynamic equations using Residual Power Series Method, 2018, 1-7.*

[11] G.B. FOLLAND*: Fourier Analysis and its applications (2nd Ed.), American Mathematical Society, 2013.*

[12] E. HETMANIOK, D. SLOTA, T. TRAWINSKI, R. WITULA*: Usage of the homotopy analysis method for solving the nonlinear and linear integral equations of the second kind, Numerical Algorithms,* **67***(1) (2014), 163-185.*

[13] M. HIGAZY, A. SUDHANSHU, N.A. TAHER*: Sawi Decomposition method for volterral integral Equation with application, Hindawi Journal of Mathematics, (2020), Art. id. 6687134, 1-13.*

[14] M. HIGAZY, A. SUDHANSHU*: Sawi transformation for system of ordinary differential equations with Application, Ain Shams Engineering Journal,* **2021***(12) (2021), 3173-3182.*

[15] J. HOSSEIN, M. ZABIHI, M. SAIDY*: Application of homotopy perturbation method for solving gas dynamics equations, Applied Mathematics Sciences,* **2***(48) (2008), 2393-2396.*

[16] V.K. INGLE, J.G. PROAKIS*: Digiyal Signal processing using MATLAB Gengage Learning, 2014.*

[17] V. JAVAD*: The combined Laplace Homotopy analysis method for partial differential equations, Journal of Mathematics and computer Science,* **2016***(16) (2016), 88-102.*

[18] M. KHALID, M. SULTANA, F. ZAID, A. UROOSA*: Solving linear and non-linear Klein-Gordon Equations by New perturbation iteration Transform method, TWMSd, App. Eng.Math.* **6***(1) (2016), 115-125.*

[19] S.J. LIAO*: The proposed homotopy analysis technique for the solution of nonlinear problems, Ph.D thesis, Shanghai Jiao Tong University, 1992.*

[20] S.J. LIAO*: On the homotopy analysis method for nonlinear problems, Applied Mathematics and Computation,* **147***(2) (2004a), 499-513.*

[21] S.J. LIAO*: Beyond pertubation: Introduction to the Homotopy Analysis method, Modern Mechanics and Mathematics, Chapman & Hall/CRC, Boca Raton, Fla, USA, Vol. 2 of CRC series, 2004b.*

[22] S.J. LIAO*: Compaison between the homotopy Analysis method and homotopy perturbation method, Applied Mathematics and Computation,* **169***(2) (2005), 1186-1194.*

[23] S.J. LIAO*: Notes on the homotopy analysis method: some definitions and theorems, Commun. Nonlinear Sci. Numer. Simul., [***2009(14) (2009), 983-997.**

[24] M.A.M. MOHAND: *The New Integral Transform "Sawi transform"*, **Advances in Theoretical and Applied Mathematics, 14(1) (2019), 81-87.**

[25] F.K. MOHANNED, A. ALI: *Solving Fredholm Integrol Differential equation of fractional order using Sawi homotopy perturbation Method*, **3rd International Conference on Mathematics and Applied Science (ICMAS 2022), Journal of Physics: Conference Series, 2322.**

[26] ODIBAT, S. AND MONAMI: *A reliable treatment of homotopy perturbation method for Klein-Gordon equations*, **Physics letters A, 365) (2007), 351-357.**

[27] A.W. OGUNSOLA, R.A. ODERINU, T. MUSILIMU, J.A. OWOLABI: *Application of Laplace Decomposition Method to Boundary value Equation in a semi-infinite Domain*, **International Journal of Difference equation, 17(1) (2022), 75-86.**

[28] M. RASHED, O.K. PASHAE: *Integral transforms and their applications*, **CRC press, 2017.**

[29] R.A> SAMEEHAH, A.H.M. AHMAD, S.H. MBROKA: *Triple Shehu transformand its properties with applications*, **African journal of Mathematics and Computer Science Research, 14(1) (2021), 4-12.**

[30] M. SHEHU, Z. WEIDONG: *New homotopy Analysis transform method for solving multidimensional fractional diffusion equations*, **Arab Journal of Basic and Applied Sciences, 27(1) (2020), 27-44.**

[31] A. SUDHANSHU, D.S. SWARG, V. AAKANSHA: *Sawi Transform of Bessel's functions with Application for Evaluating Definite integrals*, **International journal of Latest Technology in Engineering, Management and Applied Science(IJLTEMAS), ix(vii) (2020a), 12-18.**

[32] A. SUDHANSHU, V. AAKANSHA, D.S. SWARG: *Mohand transform for handling Convolution Type Volteral integro-differential equation of first kind*, **International Journal of Latest Technology in Engineering, Management & Applied Science, 9(7) (2020b), 13-19.**

[33] G. SUMIT, S. JAGDEV, D. KUMAR: *Application of Homotopy perturbation transform method for solving Time-dependent fractional differential equations*, **International Journal of Nonlinear Science, 16(1 (2013), 37-49.**

[34] M.A. USMAN, M.T. SHITTU, O.O. SOLANKE, S.A. ONITILO: *Analytical solution of the Relativistic Klein-Gordon wave equation*, **LAUTECH Journal of Engineering and technology, 13(2** *(2019), 19-29.*

[35] Q.H. YAHYA, A.O. ADEGBINDIN, M.D. NUHA, A.A. SOMAIA, G.O. SUMAYAH, A.A. ZAINAB*: Approximate Analytical Solution of Klein-Gordon equations by the modified Adomian Decomposition Method, Advances in Mathematics: Scientific Journal,* **9***(11* (2020), 9079-9087.

[1,3]LADOKE AKINTOLA UNIVERSTY OF TECHNOLOGY, DEPARTMENT OF PURE AND APPLIED MATHEMATICS, OGBOMOSO, NIGERIA.

Email address: [1]`raoderinu@lautech.edu.ng`.

[2,4]EMMANUEL ALAYANDE UNIVERSITY OF EDUCATION, DEPARTMENT OF MATHEMATICS AND COMPUTING SCIENCE EDUCATION, OYO, OYO STATE, NIGERIA.

Email address: [2]`salaudeenka@eauedoyo.edu.ng`

Email address: [3]`adiguntijani111@gmail.com`

Email address: [4]`sangoniyisunday@eauedoyo.edu.ng`