ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **13** (2024), no.2, 117–143
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.13.2.2

EXPONENTIAL ATTRACTORS FOR THE VISCOUS CAHN-HILLIARD SYSTEM

GAEL ALBAN OSSEBI, JEAN DE DIEU MANGOUBI[1], MAIMOS MAHAMAT,
DELMAS VITAL MABONZO, AND TATHY CHRISTIAN ARMAND ANICET

ABSTRACT. In this article, we explore a viscous Cahn-Hilliard system, which has applications in biology. By imposing appropriate boundary and initial conditions, we examine the asymptotic behavior of its solutions. First, we show that the problem of initial and limit values generates by a continuous semigroup on an appropriate phase space, which has a global attractor denoted $\mathcal{A}$. Subsequently, we establish the existence of an exponential attractor $\mathcal{M}$. Therefore, the global attractor $\mathcal{A}$ has a finite fractal dimension.

1. INTRODUCTION

In [3], [5] (also see [6]), G. Calginalp introduced the following phase-field system:

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta\theta, \tag{1.1}$$

$$\frac{\partial \theta}{\partial t} - \Delta\theta = -\frac{\partial u}{\partial t}. \tag{1.2}$$

The system consists of the order parameter u, the temperature θ, and the derivative of a double-well potential F, denoted as f. The typical choice for F is $F(s) =$

[1]*corresponding author*

2020 *Mathematics Subject Classification.* **MSC:** 35K55, 35L15.
Key words and phrases. Rivers silting process, numerical simulation, digital data, direct code, indirect code.
Submitted: 21.03.2024; *Accepted:* 10.04.2024; *Published:* 02.05.2024.

$\frac{1}{4}(s^2-1)^2$, which results in the usual cubic nonlinear term $f(s) = s^3 - s$. It is important to note that all physical parameters in this system have been set equal to one. This model is commonly used to describe phase transition phenomena, such as melting-solidification processes, in certain types of materials, and have been extensively studied from a mathematical standpoint. For more details, we refer the reader to [9] , [4] and [8]. To derive our system, we introduce the following total Ginzburg-Landau free energy

$$\Psi_{GL} = \int_\Omega (\frac{1}{2}|\nabla u|^2 + F(u) - u\theta - \frac{1}{2}\theta^2)dx, \tag{1.3}$$

where Ω represents the domain occupied by the material. We then define the enthalpy H as

$$H = u + \theta. \tag{1.4}$$

For the evolution of the order parameter, we assume the relaxation dynamics with the relaxation parameter set to one, resulting in:

$$\frac{\partial u}{\partial t} = \Delta \frac{D\Psi_{GL}}{Du}, \tag{1.5}$$

where $\frac{D}{Du}$ represents the variational derivative with respect to u, leading to (1.1). Next, we derive the energy equation as follows:

$$\frac{\partial H}{\partial t} = -\mathrm{div} q, \tag{1.6}$$

where q denotes the heat flux. Finally, we make the assumption of the usual Fourier law for heat conduction, which can be expressed as

$$q = -\nabla\theta, \tag{1.7}$$

We recover (1.2) from the previous derivations. In equation (1.3), the term $|\nabla u|^2$ represents short-ranged interactions. It is worth noting that such a term is obtained by truncating higher-order terms (as discussed in [7]), and it can also be viewed as a first-order approximation of a nonlocal term that accounts for long-ranged interactions (as discussed in [14]). Consequently, our focus shifts to the system where the Cahn-Hilliard equation is substituted by the Cahn-Hilliard equation accompanied by a mass source. Specifically, instead of equation (1.1), we contemplate the following

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta f(u) = -\Delta\theta. \tag{1.8}$$

Therefore, we are concerned with

$$\frac{\partial}{\partial t}(u + \epsilon(-\Delta)u) + \Delta^2 u - \Delta f(u) = -\Delta\theta, \text{ in } \Omega \times \mathbb{R}_+^* \tag{1.9}$$

$$\frac{\partial \theta}{\partial t} - \Delta\theta = -\frac{\partial u}{\partial t}, \text{ in } \Omega \times \mathbb{R}_+^*, \tag{1.10}$$

$$u = \Delta u = \theta = 0 \text{ on } \Gamma \times \mathbb{R}_+^*, \tag{1.11}$$

$$u(0,x) = u_0(x);\ \theta(0,x) = \theta_0(x),\ \forall x \in \Omega. \tag{1.12}$$

We assume that Ω is a bounded and regular domain in $\mathbb{R}^n$, where $n = 1, 2, 3$. Additionally, Moreover, we consider the subsequent polynomial of order $2p - 1$:

$$f(s) = \sum_{i=1}^{2p-1} a_i s^i, \text{ where } a_{2p-1} > 0 \text{ and } p \geq 2. \tag{1.13}$$

It should be noted that the motivation for this article comes from the recent work of Batoul, A. Miranville and others who have worked on this (see [4]- [20]), where the authors established the existence and uniqueness solutions as well as the existence of an exponential attractor of a cahn -Hilliard equation. Building on this basis, our contribution consists of proving the existence of exponential attractors for a viscous Cahn-Hilliard system which in turn guarantees the existence of finite-dimensional global attractors.

The organization of this article is as follows: In Section 2, we introduce the notations, function spaces, and main assumptions. In Section 3, we establish a priori estimates, prove the existence and uniqueness of the solution, and demonstrate that the problem generates a continuous semigroup on a suitable phase space. Moreover, we establish the presence of a global attractor and an exponential attractor within the dynamical system.

1.1. **Notations and Functions spaces.** We denote by $(.,.)$ the inner product in $L^2(\Omega)$ associated with the classical norm $\| \,.\, \|$ and $\| \,.\, \|_{-1} = \|(-\Delta)^{-\frac{1}{2}}.\|$, where $-\Delta$ is the Laplace operator associated with homogeneous Dirichlet conditions. Generally, $\| \,.\, \|_X$ refers to the norm in the Banach space X.

We also introduce the following phase space:

$$\Psi = ((H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega),$$

which is a complete metric space with respect to the metric associated with the norm

$$\|(u,\theta)\|_{\Psi} := \|u\|^2_{H^2(\Omega)} + \|\theta\|^2_{H^1(\Omega)}.$$

1.2. **Main assumptions.** We establish the following assumptions:

$$f \in C^2(\mathbb{R}),\ f(0) = 0, \tag{1.14}$$

$$f(s)s \geq c_1 F(s) - c_2,\ s \in \mathbb{R},\ c_1, c_2 \geq 0, \tag{1.15}$$

$$-c_3 \leq \frac{a_{2p-1}}{2p} s^{2p-2} - c_4 \leq f^{'}(s) \leq 3pa_{2p-1}s^{2p-2} + c_4,\ s \in \mathbb{R},\ c_3, c_4 \geq 0, \tag{1.16}$$

$$\frac{a_{2p-1}}{4p} s^{2p} - c_5 \leq F(s) \leq \frac{a_{2p-1}}{4p} s^{2p} + c_5,\ s \in \mathbb{R},\ c_5 \geq 0, \tag{1.17}$$

where $F(s) = \int_0^s f(\tau)d\tau$. Furthermore, we have

Throughout the article, c, c', and c'' represent the constants that can vary from line to line or even within the same line. Similarly, the symbol Q represents monotone increasing functions that may vary from line to line or even within the same line.

2. GLOBAL ATTRACTOR AND EXPONENTIAL ATTRACTORS.

In this section, our first step is to derive a priori estimates for the solution of (1.9)-(1.12) using formal arguments. These estimates will play a crucial role in establishing the existence and uniqueness of the solution, as well as the existence of a continuous dissipative semigroup. Additionally, we obtain certain regularity results that are essential for further analysis. Subsequently, we demonstrate the existence of both global and exponential attractors.

3. A PRIORI ESTIMATES.

Let us begin with the following theorem

Theorem 3.1. *Under the above assumptions hold. If $(u(t),\theta(t))$ is a global solution to (1.9)-(1.12) originating from $(u_0,\theta_0)\in\Psi$. Then, the following dissipative estimate is valid:*

$$\begin{aligned}&\|u(t)\|^2_{H^2(\Omega)}+\|\theta(t)\|^2_{H^1(\Omega)}\\ &\le e^{-ct}Q(\|u_0\|_{H^2(\Omega)},\|\theta_0\|_{H^1(\Omega)})+c',\ c>0,\ t\ge 0.\end{aligned}\tag{3.1}$$

Proof. Multiplying (1.9) by $(-\Delta)^{-1}u$ and integrate over Ω, we obtain, using (1.15)

$$\begin{aligned}&\frac{d}{dt}(\|u\|^2_{-1}+\epsilon\|u\|^2)+2\|\nabla u\|^2+2c_1\int_\Omega F(u)dx=2(\theta,u)+2c_2|\Omega|\\ &\le 2(cc_p(-\Delta)^{\frac12}\theta,\frac{1}{cc_p}(-\Delta)^{-\frac12}u)+2c_2|\Omega|\end{aligned}$$

impliies

$$\frac{d}{dt}(\|u\|^2_{-1}+\epsilon\|u\|^2)+c\Big(\|u\|^2_{H^1(\Omega)}+\int_\Omega F(u)dx`\Big)\le cc_p\|\nabla\theta\|^2+c',\ \forall c>0.\tag{3.2}$$

Now, multiplying (1.10) by $-\Delta\theta$ and integrate over Ω, we get

$$\frac{d}{dt}\|\nabla\theta\|^2+2\|\Delta\theta\|^2=2\left(\frac{\partial u}{\partial t},\Delta\theta\right).\tag{3.3}$$

Another multiplication of (1.9) by $\frac{\partial u}{\partial t}$ and integrate over Ω leads to

$$\frac{d}{dt}\|\Delta u\|^2+2\epsilon\left\|\nabla\frac{\partial u}{\partial t}\right\|^2+2\left\|\frac{\partial u}{\partial t}\right\|^2=2\left(\Delta f(u),\frac{\partial u}{\partial t}\right)+2\left(-\Delta\theta,\frac{\partial u}{\partial t}\right).\tag{3.4}$$

Let's observe that

$$\left(\Delta f(u),\frac{\partial u}{\partial t}\right)\ \le\ c\|f(u)\|^2_{H^2(\Omega)}+\frac12\left\|\frac{\partial u}{\partial t}\right\|^2.$$

Then

$$\frac{d}{dt}\|\Delta u\|^2+2\epsilon\|\nabla\frac{\partial u}{\partial t}\|^2+\|\frac{\partial u}{\partial t}\|^2\le c\|f(u)\|^2_{H^2(\Omega)}+2\left(-\Delta\theta,\frac{\partial u}{\partial t}\right).\tag{3.5}$$

Taking into account (1.14) and the continuous injection $H^2(\Omega)\subset C(\overline{\Omega})$, we have

$$\|f(u)\|^2_{H^2(\Omega)}\le Q(\|u\|_{H^2}),\tag{3.6}$$

which implies

$$\frac{d}{dt}\|\Delta u\|^2 + 2\epsilon \left\|\nabla \frac{\partial u}{\partial t}\right\|^2 + \|\frac{\partial u}{\partial t}\|^2 \leq Q(\|u\|_{H^2}) + 2\left(-\Delta\theta, \frac{\partial u}{\partial t}\right). \tag{3.7}$$

Finally, summing (3.3) and (3.7), we are led to

$$\frac{d}{dt}(\|\Delta u\|^2 + \|\nabla\theta\|^2) + 2\epsilon \left\|\nabla \frac{\partial u}{\partial t}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 + 2\|\Delta\theta\|^2 \leq Q(\|\Delta u\|^2 + \|\nabla\theta\|^2). \tag{3.8}$$

Setting

$$y = \|\Delta u\|^2 + \|\nabla\theta\|^2,$$

we deduce a differential inequality of the form

$$y' \leq Q(y). \tag{3.9}$$

Let z be the solution of the ordinary differential equation

$$z^{'} = Q(z),\ z(0) = y(0) = \|\Delta u_0\|^2 + \|\nabla\theta_0\|^2.$$

It follows from the comparison principle that there exists a time

$$T_0 = T_0(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}) > 0,$$

belonging, for example, to the interval $(0, \frac{1}{2})$ such that

$$y(t) \leq z(t),\ \forall t \in [0, T_0]. \tag{3.10}$$

Therefore

$$\|u(t)\|^2_{H^2(\Omega)} + \|\theta(t)\|^2_{H^1(\Omega)} \leq Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}),\ \forall t \leq T_0. \tag{3.11}$$

Next, we multiply (1.9) by $(-\Delta)^{-1}\frac{\partial u}{\partial t}$ and integrate over Ω, we obtain

$$\frac{d}{dt}\left(\|\nabla u\|^2 + 2\int_\Omega F(u)dx\right) + 2\epsilon\left\|\frac{\partial u}{\partial t}\right\|^2 + 2\left\|\frac{\partial u}{\partial t}\right\|^2_{-1} = 2\left(\theta, \frac{\partial u}{\partial t}\right). \tag{3.12}$$

Similarly, let's multiply (1.10) by θ and integrate over Ω, we find

$$\frac{d}{dt}\|\theta\|^2 + 2\|\nabla\theta\|^2 = -2\left(\frac{\partial u}{\partial t}, \theta\right). \tag{3.13}$$

Now, adding (3.12) and (3.13), we conclude that

$$\frac{d}{dt}\left(\|\nabla u\|^2+2\int_\Omega F(u)dx+\|\theta\|^2\right)+2\epsilon\left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+2\|\nabla\theta\|^2\le c'. \tag{3.14}$$

which implies

$$\frac{d\Gamma_1(t)}{dt}+c\left(\epsilon\left\|\frac{\partial u(t)}{\partial t}\right\|^2+\left\|\frac{\partial u(t)}{\partial t}\right\|_{-1}^2+\|\nabla\theta(t)\|^2\right)\le c^{'}\ c>0, \tag{3.15}$$

where, $\Gamma_1(t)$ is define as

$$\Gamma_1(t)=\|\nabla u(t)\|^2+2\int_\Omega F(u(t))dx+\|\theta(t)\|^2.$$

Summing (3.2) and ϵ_3(3.15), we have leads to

$$\frac{d\Gamma_2(t)}{dt}+c\left(\Gamma_2(t)+\left\|\frac{\partial u(t)}{\partial t}\right\|_{-1}^2+\|\nabla\theta(t)\|^2\right)\le c^{'},\ c>0, \tag{3.16}$$

where

$$\Gamma_2(t)=\epsilon_3\Gamma_1(t)+\epsilon\|u(t)\|^2+\|u(t)\|_{-1}^2,$$

satisfies, owing to (1.17)

$$\Gamma_2(t)\ge c\left(\|u(t)\|_{H^1(\Omega)}^2+\|\theta(t)\|^2\right)-c^{'},\ c,c^{'}>0 \tag{3.17}$$

and

$$\Gamma_2(t)\le c^{'}\left(\|u(t)\|_{H^1(\Omega)}^2+\|\theta(t)\|^2\right)+c^{''},\ c^{'},c^{''}>0. \tag{3.18}$$

The Gronwall's lemma applied (see, for example [12, 13, 21]) to (3.16) leads, thanks to (3.17)

$$\|u(t)\|_{H^1(\Omega)}^2+\|\theta(t)\|^2 \le c^{'}e^{-ct}\left(\|u_0\|_{H^1(\Omega)}^2+\|\theta_0\|^2\right)+c^{''},\ c,c^{'}>0,\ t\ge 0. \tag{3.19}$$

Eventually, from (3.16) we deduce that

$$\int_t^{t+1}\left(\epsilon\left\|\frac{\partial u(\tau)}{\partial t}\right\|^2+\left\|\frac{\partial u(\tau)}{\partial t}\right\|_{-1}^2+\|\nabla\theta(\tau)\|^2\right)d\tau$$
$$\leq c^{'}e^{-ct}\left(\|u_0\|^2_{H^1(\Omega)}+\|\theta_0\|^2\right)+c^{''},\ c,c^{'}>0,\ t\geq 0. \tag{3.20}$$

Next, we multiply (1.9) by u and integrate over Ω. This leads us to conclude that

$$\frac{d}{dt}(\|u\|^2+\epsilon\|\nabla u\|^2)+\frac{3}{2}\|\Delta u\|^2+2(f(u),-\Delta u)\leq c\|\theta\|^2. \tag{3.21}$$

Note that, thanks to (1.16) and an appropriate interpolation inequality, we have

$$\begin{aligned}(f(u),-\Delta u)=(f^{'}(u)\nabla u,\nabla u) &\geq -c_3\|\nabla u\|^2\\ &\geq -c_3\|u\|^2-\frac{1}{2}\|\Delta u\|^2.\end{aligned} \tag{3.22}$$

Thus, it follows from the above estimates that

$$\frac{d}{dt}(\|u\|^2+\epsilon\|\nabla u\|^2)+\|u\|^2_{H^2(\Omega)}\leq c(\|u\|^2_{H^1(\Omega)}+\|\theta\|^2)+c^{'}. \tag{3.23}$$

Integrating (3.23) over $(0,t)$, we find, with the help of (3.19)

$$\int_0^t\|u(\tau)\|^2_{H^2(\Omega)}d\tau\leq c^{'}\left(\|u_0\|^2_{H^1(\Omega)}+\|\theta_0\|^2\right)+c^{''},\ \forall t\geq 0. \tag{3.24}$$

Now, differentiating (1.9) with respect to time, we find the following differential equation

$$\frac{\partial}{\partial t}\frac{\partial u}{\partial t}+\epsilon(-\Delta)\frac{\partial}{\partial t}\frac{\partial u}{\partial t}+\Delta^2\frac{\partial u}{\partial t}-\Delta(f^{'}(u)\frac{\partial u}{\partial t})=-\Delta\frac{\partial\theta}{\partial t}. \tag{3.25}$$

Taking into account $\frac{\partial\theta}{\partial t}=-\frac{\partial u}{\partial t}+\Delta\theta$, we are led to

$$\frac{\partial}{\partial t}\frac{\partial u}{\partial t}+\epsilon(-\Delta)\frac{\partial}{\partial t}\frac{\partial u}{\partial t}+\Delta^2\frac{\partial u}{\partial t}-\Delta(f^{'}(u)\frac{\partial u}{\partial t})=\Delta\frac{\partial u}{\partial t}-\Delta^2\theta. \tag{3.26}$$

Multiplying (3.26) by $t(-\Delta)^{-1}\frac{\partial u}{\partial t}$ and integrate over Ω yields, for $t\leq T_0$

$$t\frac{d}{dt}\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\epsilon\left\|\frac{\partial u}{\partial t}\right\|^2\right)+2t\left\|\nabla\frac{\partial u}{\partial t}\right\|^2+2t\left(f^{'}(u)\frac{\partial u}{\partial t},\frac{\partial u}{\partial t}\right)$$
$$=-2t\left(\frac{\partial u}{\partial t},\frac{\partial u}{\partial t}\right)+2t\left(\Delta\theta,\frac{\partial u}{\partial t}\right), \tag{3.27}$$

which, gives

$$\frac{d}{dt}\left(t\left\|\frac{\partial u}{\partial t}\right\|_{-1}^{2}+\epsilon t\left\|\frac{\partial u}{\partial t}\right\|^{2}\right)+2t\left\|\nabla\frac{\partial u}{\partial t}\right\|^{2}+2t\left(f^{'}(u)\frac{\partial u}{\partial t},\frac{\partial u}{\partial t}\right)$$

$$\leq 2t\left\|\frac{\partial u}{\partial t}\right\|^{2}+2t\left|\left(\Delta\theta,\frac{\partial u}{\partial t}\right)\right|+\left\|\frac{\partial u}{\partial t}\right\|_{-1}^{2}+\epsilon\left\|\frac{\partial u}{\partial t}\right\|^{2}. \tag{3.28}$$

Note that $H^2(\Omega)\subset L^\infty(\Omega)$ by continuous embedding (see, for example [17]), we get due to (3.11) Furthermore, due to (1.16), we have

$$\begin{aligned}\left(f^{'}(u)\frac{\partial u}{\partial t},\frac{\partial u}{\partial t}\right) &\leq \int_\Omega |f^{'}(u)|\Big|\frac{\partial u}{\partial t}\Big|\Big|\frac{\partial u}{\partial t}\Big|dx\\ &\leq \int_\Omega (3pa_{2p-1}|u|^{2p-2}+c_4)\Big|\frac{\partial u}{\partial t}\Big|\Big|\frac{\partial u}{\partial t}\Big|dx\\ &\leq (3pa_{2p-1}\|u\|_{L^\infty(\Omega)}^{2p-2}+c_4)\Big\|\frac{\partial u}{\partial t}\Big\|\Big\|\frac{\partial u}{\partial t}\Big\|\\ &\leq Q(\|u_0\|_{H^2},\|\theta_0\|_{H^1(\Omega)})\Big\|\frac{\partial u}{\partial t}\Big\|\Big\|\frac{\partial u}{\partial t}\Big\|. \end{aligned} \tag{3.29}$$

Besides

$$\left|\left(\Delta\theta,\frac{\partial u}{\partial t}\right)\right| \leq c\|\nabla\theta\|^2+\frac{1}{2}\left\|\nabla\frac{\partial u}{\partial t}\right\|^{2}. \tag{3.30}$$

Combining the above estimates into (3.28), we arrive at the following differential inequality

$$\begin{aligned}&\frac{d}{dt}\left(t\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^{2}+\epsilon\left\|\frac{\partial u}{\partial t}\right\|^{2}\right)\right)+t\left\|\nabla\frac{\partial u}{\partial t}\right\|^{2}\\ &\leq Q(\|u_0\|_{H^2(\Omega)},\|\theta_0\|_{H^1(\Omega)})\left(t\left\|\frac{\partial u}{\partial t}\right\|_{-1}^{2}+\epsilon t\left\|\frac{\partial u}{\partial t}\right\|^{2}\right)\\ &+ct\|\nabla\theta\|^2+\left\|\frac{\partial u}{\partial t}\right\|_{-1}^{2}+\epsilon\left\|\frac{\partial u}{\partial t}\right\|^{2}, \end{aligned} \tag{3.31}$$

where we used the interpolation inequality

$$\|w\|^2\leq c\|w\|_{-1}\|\nabla w\|,\ \forall\ w\in H_0^1(\Omega). \tag{3.32}$$

Now, we deduce from (3.20), (3.31) and the Gronwall's lemma that

$$t\left\|\frac{\partial u(t)}{\partial t}\right\|_{-1}^2+\epsilon t\left\|\frac{\partial u(t)}{\partial t}\right\|^2\leq Q(\|u_0\|_{H^2},\|\theta_0\|_{H^1(\Omega)}),$$

which implies

$$\left\|\frac{\partial u(t)}{\partial t}\right\|_{-1}^2\leq\frac{1}{t}Q(\|u_0\|_{H^2(\Omega)},\|\theta_0\|_{H^1(\Omega)}),\ \forall t\in(0,T_0]. \tag{3.33}$$

and

$$\epsilon\left\|\frac{\partial u(t)}{\partial t}\right\|^2\leq\frac{1}{t}Q(\|u_0\|_{H^2(\Omega)},\|\theta_0\|_{H^1(\Omega)}),\ \forall t\in(0,T_0]. \tag{3.34}$$

Multiplying (3.26) by $(-\Delta)^{-1}\dfrac{\partial u}{\partial t}$ and integrate over Ω, we conclude that, for $t\geq T_0$

$$\begin{aligned}\frac{d}{dt}\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\epsilon\left\|\frac{\partial u}{\partial t}\right\|^2\right)+2\left\|\nabla\frac{\partial u}{\partial t}\right\|^2+2\left(f^{'}(u)\frac{\partial u}{\partial t},\frac{\partial u}{\partial t}\right)\\=-2\left(\frac{\partial u}{\partial t},\frac{\partial u}{\partial t}\right)+2\left(\Delta\theta,\frac{\partial u}{\partial t}\right).\end{aligned} \tag{3.35}$$

Taking into account (1.16) again, we find

$$\frac{d}{dt}\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\epsilon\left\|\frac{\partial u}{\partial t}\right\|^2\right)+2\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\leq 2c_3\left\|\frac{\partial u}{\partial t}\right\|^2+2\left\|\frac{\partial u}{\partial t}\right\|^2+2\left(\Delta\theta,\frac{\partial u}{\partial t}\right).$$

Then,

$$\begin{aligned}&\frac{d}{dt}\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\epsilon\left\|\frac{\partial u}{\partial t}\right\|^2\right)+\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\\&\leq Q(\|u\|_{H^2(\Omega)})\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\epsilon\left\|\frac{\partial u}{\partial t}\right\|^2\right)+c\left\|\frac{\partial u}{\partial t}\right\|^2+\|\nabla\theta\|^2,\end{aligned} \tag{3.36}$$

which leads, using the interpolation inequality (3.32) to

$$\begin{aligned}&\frac{d}{dt}\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\epsilon\left\|\frac{\partial u}{\partial t}\right\|^2\right)+\frac{1}{2}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\\&\leq Q(\|u\|_{H^2(\Omega)})\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\epsilon\left\|\frac{\partial u}{\partial t}\right\|^2\right)+\|\nabla\theta\|^2.\end{aligned} \tag{3.37}$$

Applying the Gronwall's lemma, we obtain owing to (3.20) and (3.24)

$$\left\|\frac{\partial u(t)}{\partial t}\right\|_{-1}^{2}+\epsilon\left\|\frac{\partial u(t)}{\partial t}\right\|^{2} \leq e^{ct}Q(\|u_0\|_{H^2(\Omega)},\|\theta_0\|_{H^1(\Omega)}) \left(\left\|\frac{\partial u(T_0)}{\partial t}\right\|_{-1}^{2}+\epsilon\left\|\frac{\partial u(T_0)}{\partial t}\right\|^{2}\right),\ c\geq 0,\ t\geq T_0. \tag{3.38}$$

Finally, it follows from (3.33) and (3.38) that

$$\left\|\frac{\partial u(t)}{\partial t}\right\|_{-1}^{2} \leq e^{ct}Q(\|u_0\|_{H^2(\Omega)},\|\theta_0\|_{H^1(\Omega)}),\ c\geq 0,\ t\geq T_0, \tag{3.39}$$

and

$$\epsilon\left\|\frac{\partial u(t)}{\partial t}\right\|^{2} \leq e^{ct}Q(\|u_0\|_{H^2(\Omega)},\|\theta_0\|_{H^1(\Omega)}),\ c\geq 0,\ t\geq T_0. \tag{3.40}$$

Now, let's rewrite, for a fixed $t\geq T_0$, (1.9) in the form

$$-\Delta u+f(u)=h_u(t),\ u=0 \text{ on } \Gamma, \tag{3.41}$$

where

$$h_u(t)=-(-\Delta)^{-1}\frac{\partial u}{\partial t}+\epsilon\frac{\partial u}{\partial t}+\theta. \tag{3.42}$$

Multiplying (3.42) by $h_u(t)$ and integrate over Ω, we find

$$\|h_u(t)\|^2\leq c\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^{2}+\left\|\frac{\partial u}{\partial t}\right\|^{2}+\|\theta\|^2\right). \tag{3.43}$$

Exploiting (3.20),(3.39) and (3.43), we infer that

$$\|h_u(t)\|^2 \leq e^{ct}Q(\|u_0\|_{H^2(\Omega)},\|\theta_0\|_{H^1(\Omega)}),\ c>0,\ t\geq T_0. \tag{3.44}$$

Multiplying now (3.41) by u and integrate over Ω, we obtain

$$\|\nabla u\|^2+(f(u),u)=(h_u(t),u), \tag{3.45}$$

which gives, thanks to (1.15) and (1.17)

$$\|u\|_{H^1(\Omega)}^2\leq c\|h_u(t)\|^2+c',\ c_1>0. \tag{3.46}$$

Now, multiplying (3.41) by $-\Delta u$ and integrate over Ω, we obtain

$$\|\Delta u\|^2+2(f'(u)\nabla u,\nabla u)\leq c\|h_u(t)\|^2,$$

which implies, using again (1.15) and (1.17)

$$\|\Delta u\|^2 \leq c\|h_u(t)\|^2 + c^{'}\|u\|^2_{H^1(\Omega)} + c^{''}. \tag{3.47}$$

Summing (3.46) and ϵ_4(3.47), where $\epsilon_4 > 0$ is sufficiently small, leads to

$$\|u\|^2_{H^1(\Omega)} + c\|u\|^2_{H^2(\Omega)} \leq c^{'}\|h_u(t)\|^2 + c^{''},\ c > 0. \tag{3.48}$$

We deduce from (3.44) and (3.48) that

$$\|u(t)\|^2_{H^2(\Omega)} \leq e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}) + c^{''},\ c \geq 0,\ t \geq T_0. \tag{3.49}$$

Returning to the estimate (3.3), we have

$$\frac{d}{dt}\|\nabla\theta\|^2 + \|\Delta\theta\|^2 \leq c\left\|\frac{\partial u}{\partial t}\right\|^2. \tag{3.50}$$

Integrating (3.50) over (T_0, t) leads us to

$$\|\nabla\theta(t)\|^2 + \int_{T_0}^t \|\Delta\theta(s)\|^2 ds \leq c\int_{T_0}^t \left\|\frac{\partial u(s)}{\partial t}\right\|^2 ds + \|\nabla\theta(T_0)\|^2, \tag{3.51}$$

Note that, thanks to (3.20), (3.37) and (3.40)

$$\int_{T_0}^t \left\|\frac{\partial u(s)}{\partial t}\right\|^2 ds \leq e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}),\ c > 0,\ t \geq T_0. \tag{3.52}$$

Then, we deduce from (3.50) and (3.52) that

$$\|\nabla\theta(t)\|^2 + \int_{T_0}^t \|\Delta\theta(s)\|^2 ds \leq e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}),\ c > 0,\ t \geq T_0,$$

which leads in particular to

$$\|\nabla\theta(t)\|^2 \leq e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}),\ c > 0,\ t \geq T_0. \tag{3.53}$$

Combining (3.49) and (3.53), we are led to

$$\begin{aligned} &\|u(t)\|^2_{H^2(\Omega)} + \|\theta(t)\|^2_{H^1(\Omega)} \\ &\leq e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}) + c^{''},\ c \geq 0,\ t \geq T_0. \end{aligned} \tag{3.54}$$

Finally, we deduce from (3.11) and (3.54), that

$$\begin{aligned} &\|u(t)\|^2_{H^2(\Omega)} + \|\theta(t)\|^2_{H^1(\Omega)} \\ &\leq e^{ct}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}) + c^{''},\ c \geq 0,\ t \geq 0. \end{aligned} \tag{3.55}$$

In reality, it is possible to obtain a dissipative estimate here, instead of (3.55), which grows exponentially as time approaches $+\infty$. To do this, we deduce from (3.20) that

$$\int_0^1 \|\nabla\theta(t)\|^2 dt \le Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}) + c^{''}. \tag{3.56}$$

Furthermore, multiplying (1.9) by u and integrate over Ω, we obtain

$$\frac{d}{dt}(\|u\|^2 + \epsilon\|\nabla u\|^2) + 2\|\Delta u\|^2 + 2(f^{'}(u)\nabla u, \nabla u) = 2(-\Delta\theta, u), \tag{3.57}$$

which implies, due to (1.17)

$$\frac{d}{dt}(\|u\|^2 + \epsilon\|\nabla u\|^2) + \|\Delta u\|^2 \le c(\|\nabla u\|^2 + \|\nabla\theta\|^2) + c^{'}. \tag{3.58}$$

Taking into account (3.19), we find

$$\frac{d}{dt}(\|u\|^2 + \epsilon\|\nabla u\|^2) + \|\Delta u\|^2 \le Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}). \tag{3.59}$$

Integrating (3.59) over $(0, 1)$, we conclude that

$$\int_0^1 \|u(t)\|^2_{H^2(\Omega)} dt \le Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}) + c^{''}. \tag{3.60}$$

Therefore, the estimates (3.56) and (3.60) allow us to state that there exists $T \in (0, 1)$ such that

$$\|u(T)\|^2_{H^2(\Omega)} + \|\theta(T)\|^2_{H^1(\Omega)} \le Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}) + c^{''}. \tag{3.61}$$

Repeating the previous estimates, and starting at $t = T$ instead of $t = 0$, we observe that (3.61) is satisfied for $T = 1$, meaning that we have the following smoothing property

$$\|u(1)\|^2_{H^2(\Omega)} + \|\theta(1)\|^2_{H^1(\Omega)} \le Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}) + c^{''}. \tag{3.62}$$

Particularly, with this smoothing property, we see that it is not difficult to prove, thanks to (3.19), (3.55), and (3.62), the following estimate

$$\|u(t)\|^2_{H^2(\Omega)} + \|\theta(t)\|^2_{H^1(\Omega)} \le e^{-ct}Q(\|u_0\|_{H^2(\Omega)}, \|\theta_0\|_{H^1(\Omega)}) + c^{'},\ c > 0,\ t \ge 0. \tag{3.63}$$

The theorem is proved. □

4. Existence and uniqueness of the solution

Theorem 3.1 plays a crucial role in establishing the existence of a solution to (1.9-(1.12). The proof of Theorem 3.1 follows from the aforementioned a priori estimates and, for instance, a standard Galerkin scheme, which we will outline below.

Proof. We note that since the operator $A = -\Delta$, associated with the Dirichlet boundary conditions of the domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, is bounded, linear, self-adjoint, and strictly positive. It has a compact inverse from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$. We can then consider a family $\{\nu_j, j \geq 1\}$ of eigenvectors of A associated with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, forming an orthonormal basis in $L^2(\Omega)$ and orthogonal to $H_0^1(\Omega)$. The family of $\{\nu_j, j \geq 1\}$ may be assumed to be normalized in the norm of $L^2(\Omega)$, i.e.

$$\delta_{ij} = (\nu_i, \nu_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j. \end{cases}$$

We denote by V_m the space

$$V_m = \text{span}\{\nu_1, \cdots, \nu_m\},$$

and by P_m the orthogonal projection of $L^2(\Omega)$ onto V_m

$$P_m\beta = \sum_{j=1}^{m} (\beta, \nu_j)\nu_j, \text{ for all } m \in \mathbb{N}.$$

We look for functions of the form

$$u_m(t) = \sum_{j=1}^{m} u_{jm}(t)\nu_j \text{ and } \theta_m(t) = \sum_{j=1}^{m} \theta_{jm}(t)\nu_j,$$

solving the approximate problem below:

$$(-\Delta)^{-1}\frac{du_m}{dt} + \epsilon\frac{du_m}{dt} - \Delta u_m + f(u_m) = \theta_m, \tag{4.1}$$

$$\frac{d\theta_m}{dt} - \Delta\theta_m = -\frac{du_m}{dt}, \tag{4.2}$$

together with suitable initial conditions, namely,

$$u_m(0) = P_m u_0; \; \theta_m(0) = P_m\theta_0. \tag{4.3}$$

This is equivalent to the following, for $i = 1, \cdots, m$

$$((-\Delta)^{-1}\frac{du_m}{dt}, \nu_i) + (\epsilon\frac{du_m}{dt}, \nu_i) + (\nabla u_m, \nabla \nu_i) + (f(u_m), \nu_i) = (\theta_m, \nu_i), \tag{4.4}$$

$$(\frac{d\theta_m}{dt}, \nu_i) + (\nabla \theta_m, \nabla \nu_i) = -(\frac{du_m}{dt}, \nu_i), \tag{4.5}$$

$$u_m(0) = P_m u_0;\ \theta_m(0) = P_m \theta_0. \tag{4.6}$$

The proof of existence of a local (in time) solution to the approximating problem is standard (indeed, one has to solve a system of ODE's). Furthermore, we can then write the equivalent of (3.16) and 3.55) (with (u, θ) replaced by (u_m, θ_m); this is now fully justied and no longer formal), which yields that this solution is actually global. Finally, the passage to the limit is based on classical (Aubin- Lions type) compactness results (indeed, we have, in particular, u_m bounded in $L^\infty(0, T; (H^2(\Omega) \cap H_0^1(\Omega)), \theta_m$ bounded in $L^\infty(0, T; H_0^1(\Omega))$ and $\frac{du_m}{dt}$ bounded in $L^2(0, T; L^2(\Omega)))$, independently of m, which yields that (at least for a subsequence which we do not relabel) u_m converges to, say, u almost everywhere and in $\mathcal{C}([0, T], H^{1-q}(\Omega)),\ q > 0$. □

Given the existence of the solution being established, the following result provides us with the uniqueness of the solution. Hence, the theorem

Theorem 4.1. *We assume that the assumptions of theorem 3.1 hold. Then, a solution corresponding* $(u_i(t), \theta_i(t))$ *to (1.9)-(1.12) with initial data* $(u_{0,i}(t), \theta_{0,i}(t)) \in \Psi$*,* $i = 1, 2$*, verifies the following estimate*

$$\begin{aligned} &\|(u_1 - u_2)(t)\|^2_{H^1(\Omega)} + \|(\theta_1 - \theta_2)(t)\|^2 \\ &+ \int_0^t \left(\left\|\frac{\partial u(\tau)}{\partial t}\right\|^2_{-1} + \epsilon\left\|\frac{\partial u(\tau)}{\partial t}\right\|^2 + \|\nabla\theta(\tau)\|^2 \right) d\tau \\ &\le e^{Qt}(\|u_{0,1} - u_{0,2}\|^2_{H^1(\Omega)} + \|\theta_{0,1} - \theta_{0,2}\|^2),\ \forall t \ge 0. \end{aligned} \tag{4.7}$$

Proof. We define $(u(t), \theta(t)) = (u_1(t), \theta_1(t)) - (u_2(t), \theta_2(t))$ and $(u_0, \theta_0) = (u_{1,0}, \theta_{1,0}) - (u_{2,0}, \theta_{2,0})$. Then, the solution $(u(t), \theta(t))$ satisfies the following problem

$$\frac{\partial}{\partial t}(u + \epsilon(-\Delta)u) + \Delta^2 u - \Delta(f(u_1) - f(u_2)) = -\Delta\theta \tag{4.8}$$

$$\frac{\partial \theta}{\partial t} - \Delta\theta = -\frac{\partial u}{\partial t} \tag{4.9}$$

$$\theta = \Delta u = 0 \text{ on } \Gamma, \tag{4.10}$$

$$u(0,x) = u_0(x);\ \theta(0,x) = \theta_0(x),\ \forall x \in \Omega. \tag{4.11}$$

Multiplying equation (4.8) by $(-\Delta)^{-1}\frac{\partial u}{\partial t}$ and equation (4.9) by θ, and integrating over Ω, we can obtain the following expressions, summing the two results

$$\frac{d}{dt}\Gamma_3 + 2\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2 + 2\epsilon\left\|\frac{\partial u}{\partial t}\right\|^2 + 2\|\nabla\theta\|^2 + 2\left(f(u_1) - f(u_2), \frac{\partial u}{\partial t}\right) = 0, \tag{4.12}$$

where

$$\Gamma_3 := \|\nabla u\|^2 + \|\theta\|^2,$$

which satisfies the inequality

$$c(\|u\|_{H^1(\Omega)}^2 + \|\theta\|^2) \le \Gamma_3 \le c'(\|u\|_{H^1(\Omega)}^2 + \|\theta\|^2),\ c, c' > 0. \tag{4.13}$$

We can observe that

$$\left|\left(f(u_1) - f(u_2), \frac{\partial u}{\partial t}\right)\right| \le \int_\Omega |\nabla(f(u_1) - f(u_2))| \left|(-\Delta)^{-\frac{1}{2}}\frac{\partial u}{\partial t}\right| dx.$$

Consequently

$$\begin{aligned}
&\left|\left(f(u_1) - f(u_2), \frac{\partial u}{\partial t}\right)\right| \\
&\le c\int_\Omega \Big(|u_1|^{2p-2} + |u_2|^{2p-2} + 1\Big)|\nabla u| \left|(-\Delta)^{-\frac{1}{2}}\frac{\partial u}{\partial t}\right| dx \\
&+ c\int_\Omega \Big(|u_1|^{2p-2} + |u_2|^{2p-2} + 1\Big)\big(|\nabla u_1| + |\nabla u_2|\big)|u| \left|(-\Delta)^{-\frac{1}{2}}\frac{\partial u}{\partial t}\right| dx \\
&\le c\Big(\|u_1\|_{L^\infty(\Omega)}^{2p-2} + \|u_2\|_{L^\infty(\Omega)}^{2p-2} + 1\Big)\|\nabla u\| \left\|\frac{\partial u}{\partial t}\right\|_{-1} \\
&+ c\Big(\|u_1\|_{L^\infty(\Omega)}^{2p-2} + \|u_2\|_{L^\infty(\Omega)}^{2p-2} + 1\Big)\big(\|\nabla u_1\|_{L^4(\Omega)} + \|\nabla u_2\|_{L^4(\Omega)}\big)\|u\|_{L^4(\Omega)} \left\|\frac{\partial u}{\partial t}\right\|_{-1}.
\end{aligned}$$

From the continuous embedding $H^2(\Omega) \subset C(\overline{\Omega})$, we can deduce that

$$\begin{aligned} &\left|\left(f(u_1) - f(u_2), \frac{\partial u}{\partial t}\right)\right| \\ &\leq Q(\|u_{1,0}\|_{H^2}, \|u_{2,0}\|_{H^2}, \|\theta_{1,0}\|_{H^1}, \|\theta_{2,0}\|_{H^1}) \|\nabla u\| \left\|\frac{\partial u}{\partial t}\right\|_{-1} \\ &+ Q(\|u_{1,0}\|_{H^2}, \|u_{2,0}\|_{H^2}, \|\theta_{1,0}\|_{H^1}, \|\theta_{2,0}\|_{H^1}) \\ &\times \left(\|\nabla u_1\|_{L^4(\Omega)} + \|\nabla u_2\|_{L^4(\Omega)}\right) \|u\|_{L^4(\Omega)} \left\|\frac{\partial u}{\partial t}\right\|_{-1}. \end{aligned} \tag{4.14}$$

Now, employing the continuous embedding $H^1(\Omega) \subset L^4(\Omega)$, along with Young's inequality and (4.14), we obtain

$$\begin{aligned} &\left|\left(f(u_1) - f(u_2), \frac{\partial u}{\partial t}\right)\right| \\ &\leq Q(\|u_{1,0}\|_{H^2}, \|u_{2,0}\|_{H^2}, \|\theta_{1,0}\|_{H^1}, \|\theta_{2,0}\|_{H^1}) \|u\|^2_{H^1(\Omega)} + \frac{1}{2} \left\|\frac{\partial u}{\partial t}\right\|^2_{-1}. \end{aligned} \tag{4.15}$$

Inserting (4.15) into (4.12), we arrive at the following inequality

$$\frac{d}{dt}\Gamma_3 + c\left(\left\|\frac{\partial u}{\partial t}\right\|^2_{-1} + \epsilon\left\|\frac{\partial u}{\partial t}\right\|^2 + \|\nabla\theta\|^2\right) \leq Q\Gamma_3, \tag{4.16}$$

where Q is given by

$$Q = Q(\|u_{1,0}\|_{H^2}, \|u_{2,0}\|_{H^2}, \|\theta_{1,0}\|_{H^1}, \|\theta_{2,0}\|_{H^1}).$$

Considering (4.16), we can apply Gronwall's lemma, and as a consequence of (4.13), we obtain

$$\begin{aligned} &\|(u_1 - u_2)(t)\|^2_{H^1(\Omega)} + \|(\theta_1 - \theta_2)(t)\|^2 \\ &+ \int_0^t \left(\left\|\frac{\partial u(\tau)}{\partial t}\right\|^2_{-1} + \epsilon\left\|\frac{\partial u(\tau)}{\partial t}\right\|^2 + \|\nabla\theta(\tau)\|^2\right) d\tau \\ &\leq e^{Qt}(\|u_{0,1} - u_{0,2}\|^2_{H^1(\Omega)} + \|\theta_{0,1} - \theta_{0,2}\|^2), \ \forall t \geq 0. \end{aligned} \tag{4.17}$$

The uniqueness of the solution for the system (1.9)-(1.12) is immediately established by the estimate (4.17), thereby confirming the validity of the theorem. □

Thanks to Theorem 4.1, we are able to state the following theorem

Theorem 4.2. *For $\epsilon < 1$, under the assumptions of Theorem 4.1 being satisfied, the problem (1.9)-(1.12) defines a continuous semigroup $S(t)$ on Ψ given by the expression*

$$\begin{aligned} S(t) : \Psi &\longrightarrow \Psi, \\ (u_0, \theta_0) &\longrightarrow (u(t), \theta(t)), \end{aligned}$$

(that is, $S(0) = I$ (identity operator) and $S(t+s) = S(t) \circ S(s), \forall\, t, s \geq 0$), where $(u(t), \theta(t))$ is the unique solution of (1.9)-(1.12).

A direct consequence of Theorem 3.1 is the following corollary

Corollary 4.1. *Given the aforementioned verified assumptions, the semigroup $S(t)_{t\geq 0}$, generated by (1.9)-(1.12), possesses a bounded absorbing set on Ψ, denoted by*

$$\mathcal{B}_0 := \{(u, \theta) \in \Psi : \ \|(u, \theta)\|_{\Psi}^2 \leq 2c^{'}\},$$

where $c^{'}$ is the constant in (3.63). Moreover, for any $B \in \Psi$, there exists $t_0 = t_0(B) > 0$ such that

$$\|S(t)(u_0, \theta_0)\|_{\Psi}^2 \leq 2c^{'}, \ \forall t \geq t_0. \tag{4.18}$$

To establish further regularity properties of the solution of (1.9)-(1.12), we state the following theorem

Theorem 4.3. *For $\epsilon < 1$, under the above assumptions satisfied, then, there exists positive constant c such that for any fixed bounded set $B \subset \Psi$ there exists a positive time $t_1 = t_1(\|B\|_{\Psi})$ such that, for any $(u_0, \theta_0) \in B$ the solution $S(u_0, \theta_0) = (u(t), \theta(t))$ satisfies enjoys the following estimate*

$$\|u(t)\|_{H^2(\Omega)}^2 + \|\theta(t)\|_{H^2(\Omega)}^2 + \int_t^{t+1} \left\| \nabla \frac{\partial \theta(s)}{\partial t} \right\|^2 ds \leq c, \ t \geq t_1. \tag{4.19}$$

Proof. We multiply (1.10) by $-\Delta \dfrac{\partial \theta}{\partial t}$ and integrate over Ω, resulting in

$$\frac{d}{dt}\|\Delta\theta\|^2 + \left\| \nabla \frac{\partial \theta}{\partial t} \right\|^2 \leq \left\| \nabla \frac{\partial u}{\partial t} \right\|^2. \tag{4.20}$$

Note that it follows from (3.20), (3.37) and Gronwall's uniform lemma, that

$$\int_t^{t+1} \left\| \nabla \frac{\partial u}{\partial t} \right\|^2 ds \leq c, \ t \geq t_0 + 1. \tag{4.21}$$

Setting

$$z_0 = \|\Delta\theta\|^2, \; k_0 = 0 \text{ and } b_0 = \left\|\nabla \frac{\partial u}{\partial t}\right\|^2.$$

We deduce from (4.20) that

$$z_0^{'} \leq k_0 z_0 + b_0, \; t \geq t_0. \tag{4.22}$$

Using (3.50), (3.63), and (4.21), there exist positive constants Φ_1, Φ_2, and Φ_3 (independent of time and initial data) such that

$$\int_t^{t+1} z_0(s)ds \leq \Phi_1; \; \int_t^{t+1} b_0(s)ds \leq \Phi_2; \; \int_t^{t+1} k_0(s)ds \leq \Phi_3, \; t \geq t_0. \tag{4.23}$$

Based on (4.22) and (4.23), and using again the uniform Gronwall lemma (see, for example, [11]), we can conclude that

$$z_0(t+1) \leq (\Phi_3 + \Phi_2)e^{\Phi_1}, \; \forall t \geq t_0,$$

which gives, for every $t \geq t_1 = t_0 + 1$,

$$\|\theta(t)\|^2_{H^2(\Omega)} + \int_t^{t+1} \left\|\nabla \frac{\partial \theta(s)}{\partial t}\right\|^2 ds \leq c. \tag{4.24}$$

Combining (3.63) and (4.24), we obtain the following estimate

$$\|u(t)\|^2_{H^2(\Omega)} + \|\theta(t)\|^2_{H^2(\Omega)} + \int_t^{t+1} \left\|\nabla \frac{\partial \theta(s)}{\partial t}\right\|^2 ds \leq c, \; \forall t \geq t_1. \tag{4.25}$$

The proof is now complete. □

According to (4.18) and (4.25), we have the folllowing corollary

Corollary 4.2. *Under the above verified assumptions, the semigroup $\{S(t)\}_{t\geq 0}$, generated by (1.9)-(1.12) possesses on $H^2(\Omega) \times H^2(\Omega)$ a bounded absorbing set, for instance*

$$\mathcal{B}_1 := \{(u,\theta) \in H^2(\Omega) \times H^2(\Omega) : \; \|(u,\theta)\|^2_{H^2(\Omega)\times H^2(\Omega)} \leq c\},$$

where c is the positive constant in (4.25). Moreover, $B \in \Psi$ there exists $t_1 = t_1(B) > 0$, for which

$$\|S(t)(u_0,\theta_0)\|^2_{H^2(\Omega)\times H^2(\Omega)} \leq c, \; \forall t \geq t_1. \tag{4.26}$$

We can also prove the following lemma, which we will use later on

Lemma 4.1. *Under the above assumptions satisfied , the following estimation is verified*

$$\left\|\frac{\partial u}{\partial t}\right\|^2 + \epsilon\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 + \int_t^{t+1}\left\|\Delta\frac{\partial u(s)}{\partial t}\right\| ds \leq c,\ \forall t \geq t_1. \tag{4.27}$$

Proof. Multiplying (3.26) by $\frac{\partial u}{\partial t}$ and integrate over Ω yields, reasoning as above

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\frac{\partial u}{\partial t}\right\|^2 + \epsilon\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\right) + 2\left\|\Delta\frac{\partial u}{\partial t}\right\|^2 \\
&\leq \left(\Delta f'(u)\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}\right) + \left(\Delta\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}\right) + \left(\Delta^2\theta, \frac{\partial u}{\partial t}\right). \\
&\leq c\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 + \left\|\nabla\frac{\partial u}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial u}{\partial t}\right\|\left\|\Delta\theta\right\|. \\
&\leq c'\left\|\nabla\frac{\partial u}{\partial t}\right\|_{-1}\left\|\Delta\frac{\partial u}{\partial t}\right\| + \frac{1}{2}\left\|\Delta\frac{\partial u}{\partial t}\right\|^2 + c\left\|\Delta\theta\right\|^2. \\
&\leq c'\left\|\nabla\frac{\partial u}{\partial t}\right\|_{-1}^2 + \frac{1}{2}\left\|\Delta\frac{\partial u}{\partial t}\right\|^2 + \frac{1}{2}\left\|\Delta\frac{\partial u}{\partial t}\right\|^2 + c\left\|\Delta\theta\right\|^2.
\end{aligned}$$

$$\leq c'\left\|\frac{\partial u}{\partial t}\right\|^2 + \left\|\Delta\frac{\partial u}{\partial t}\right\|^2 + c\left\|\Delta\theta\right\|^2, \tag{4.28}$$

implies

$$\frac{d}{dt}\left(\left\|\frac{\partial u}{\partial t}\right\|^2 + \epsilon\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\right) + 2\left\|\Delta\frac{\partial u}{\partial t}\right\|^2 \leq c'\left\|\frac{\partial u}{\partial t}\right\|^2 + c\left\|\Delta\theta\right\|^2. \tag{4.29}$$

Assuming

$$z_1 = \left\|\frac{\partial u}{\partial t}\right\|^2,\ k_1 = c \text{ and } b_1 = c'\|\Delta\theta\|^2,$$

we can infer from equation (4.28) that $z_1' \leq k_1 z_1 + b_1,\ t \geq t_0$, where, thanks to the previous estimates, z_1, k_1, and b_1 satisfy the assumptions of the uniform Gronwall lemma (for $t \geq t_0$), which implies that, for ($t \geq t_1 = t_0 + 1$)

$$\left\|\frac{\partial u}{\partial t}\right\|^2 + \epsilon\left\|\nabla\frac{\partial u}{\partial t}\right\|^2 + \int_t^{t+1}\left\|\Delta\frac{\partial u(s)}{\partial t}\right\| ds \leq c,\ \forall t \geq t_1. \tag{4.30}$$

The lemma is thus proven. □

4.1. **Global and Exponential attractors.** The existence of a compact absorbing set is demonstrated in Corollary 4.2. From a well-known result on dynamical systems (referenced as [11], for example), we can deduce the following theorem

Theorem 4.4. *For $\epsilon < 1$, the above asssumptions hold, the dynamical system $(S(t), \Psi)$ possesses the global attractor $\mathcal{A}$ which is bounded in $H^2(\Omega) \times H^2(\Omega)$ and compact in Ψ.*

The existence of the global attractor being established, we now prove that the semigroup $S(t)$, $t \geq 0$, possesses the finite-dimensional global attractors. More precisely, we prove the existence of exponential attractors for the semigroup $S(t)$, $t \geq 0$, associated to the problem (1.9)-(1.12). To do this, we need the semigroup to be Lipschitz continuous, to satisfy the smoothing property and to verify a Hölder continuity with respect to time.

Lemma 4.2. *Let $\mathcal{B}_1$ be the set defined in Corollary 4.2, and let $\pi_1, \pi_2 \in \mathcal{B}_1$, where $\pi_1 = (u_{0,1}, \theta_{0,1})$ and $\pi_2 = (u_{0,2}, \theta_{0,2})$ are initial data for two solutions $(u_1(t), \theta_1(t))$ and $(u_2(t), \theta_2(t))$ of (1.9)-(1.12), respectively. Then, the corresponding solutions of the problem (1.9)-(1.12) satisfy the following estimate:*

$$\|S(t)\pi_1 - S(t)\pi_2\|^2_{H^2(\Omega)\times H^2(\Omega)} \leq \frac{c^{'}(\nu+1)}{\nu} e^{ct} \|\pi_1 - \pi_2\|^2_{\Psi}, \ \forall t > t_2,$$

where $\nu := t - t_2$, while c and $c^{'}$ are positive constants which only depend on the norms of the initial data in Ψ, on Ω and on the other structural parameters of the problem.

Proof. Let us define $(u(t), \theta(t)) = S(t)\pi_1 - S(t)\pi_2$ and $\pi_1 - \pi_2 = (u_{0,1}, \theta_{0,1}) - (u_{0,2}, \theta_{0,2})$. Next, differentiating equation (4.8) with respect to time and using (4.9), we obtain the following expression

$$(4.31)\quad (-\Delta)^{-1} \frac{\partial}{\partial t}\frac{\partial u}{\partial t} + \frac{\partial}{\partial t}\frac{\partial u}{\partial t} - \Delta \frac{\partial u}{\partial t} + f^{'}(u_1)\frac{\partial u}{\partial t} + (f^{'}(u_1) - f^{'}(u_2))\frac{\partial u_2}{\partial t} = \Delta\theta - \frac{\partial u}{\partial t}.$$

Multiplying (4.31) by $(t-t_2)\frac{\partial u}{\partial t}$ and integrate over Ω, we obtain, due to (1.16),

$$\begin{aligned}&\frac{d}{dt}\left((t-t_2)\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\epsilon(t-t_2)\left\|\frac{\partial u}{\partial t}\right\|^2\right)+2(t-t_2)\left\|\nabla\frac{\partial u}{\partial t}\right\|^2\\&\le\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\left\|\frac{\partial u}{\partial t}\right\|^2+2c_3(t-t_2)\left\|\frac{\partial u}{\partial t}\right\|^2+2(t-t_2)\left\|\frac{\partial u}{\partial t}\right\|^2\\&+2(t-t_2)\left|\left((f'(u_1)-f'(u_2))\frac{\partial u_2}{\partial t},\frac{\partial u}{\partial t}\right)\right|+2(t-t_2)\left(\Delta\theta,\frac{\partial u}{\partial t}\right).\end{aligned}\tag{4.32}$$

Now, multiplying (4.9) by $(t-t_2)\Delta^2\theta$ and integrate over Ω, we obtain, using the Young's inequality on the right-hand side

$$\frac{d}{dt}\left((t-t_2)\|\Delta\theta\|^2\right)+\frac{(t-t_2)}{2}\|\nabla\Delta\theta\|^2\le\frac{2(t-t_2)}{3}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2+\|\Delta\theta\|^2.\tag{4.33}$$

Finally, summing (4.32) and (4.33), we conclude that

$$\begin{aligned}&\frac{d}{dt}\left((t-t_2)\Gamma_4\right)+\frac{4(t-t_2)}{3}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2+\frac{(t-t_2)}{2}\|\nabla\Delta\theta\|^2\\&\le\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\left\|\frac{\partial u}{\partial t}\right\|^2+\|\Delta\theta\|^2+2c_3(t-t_2)\left\|\frac{\partial u}{\partial t}\right\|^2+2(t-t_2)\left\|\frac{\partial u}{\partial t}\right\|^2\\&+2(t-t_2)\left|\left((f'(u_1)-f'(u_2))\frac{\partial u_2}{\partial t},\frac{\partial u}{\partial t}\right)\right|+2(t-t_2)\left(\Delta\theta,\frac{\partial u}{\partial t}\right),\end{aligned}\tag{4.34}$$

for all $t>t_2$, where $\Gamma_4(t):=\left\|\frac{\partial u(t)}{\partial t}\right\|_{-1}^2+\epsilon\left\|\frac{\partial u(t)}{\partial t}\right\|^2+\|\Delta\theta(t)\|^2$.

Notice that, owing to the continuous embedding $H^2(\Omega)\subset C(\overline{\Omega})$, we obtain

$$\begin{aligned}\left|\left((f'(u_1)-f'(u_2))\frac{\partial u_2}{\partial t},\frac{\partial u}{\partial t}\right)\right| &\le \int_\Omega|f'(u_1)-f'(u_2)|\left|\frac{\partial u_2}{\partial t}\right|\left|\frac{\partial u}{\partial t}\right|dx\\&\le\int_\Omega\left|\int_0^1 f''(u_1+s(u_2-u_1)dsu\right|\left|\frac{\partial u_2}{\partial t}\right|\left|\frac{\partial u}{\partial t}\right|dx\\&\le\left\|\int_0^1 f''(u_1+s(u_2-u_1)dsu\right\|\left\|\frac{\partial u_2}{\partial t}\right\|\left\|\frac{\partial u}{\partial t}\right\|\\&\le c\|u\|_{L^4(\Omega)}\left\|\frac{\partial u_2}{\partial t}\right\|\left\|\frac{\partial u}{\partial t}\right\|_{L^4(\Omega)}.\end{aligned}$$

As $H^1(\Omega)$ embeds into $L^4(\Omega)$,

$$\left|\left((f'(u_1)-f'(u_2))\frac{\partial u_2}{\partial t},\frac{\partial u}{\partial t}\right)\right|\le c\|u\|_{H^1(\Omega)}\left\|\frac{\partial u_2}{\partial t}\right\|\left\|\frac{\partial u}{\partial t}\right\|_{H^1(\Omega)}.\tag{4.35}$$

Inserting the estimates (4.35) into (4.34), we are led to

$$\frac{d}{dt}\left((t-t_2)\Gamma_4(t)\right)+\frac{(t-t_2)}{3}\left\|\nabla\frac{\partial u}{\partial t}\right\|^2+(t-t_2)\|\nabla\Delta\theta\|^2$$

$$\leq c(t-t_2)\Gamma_4(t)+\Gamma_4(t)+c(t-t_2)\|\nabla u\|^2\left\|\frac{\partial u_2}{\partial t}\right\|^2, \tag{4.36}$$

where, we have used again the interpolation inequality (3.32).

Now, using the estimates (3.8) and (3.63) (for $(u,\theta)=(u_2,\theta_2)$), we have

$$\int_{t_2}^{t}\left\|\frac{\partial u_2(\tau)}{\partial t}\right\|^2 d\tau\leq c,\ \forall t>t_2. \tag{4.37}$$

Integrating (4.36) over (t_2,t) and using Theorem 4.1 along with the estimate (4.37), we assert that

$$\left\|\frac{\partial u(t)}{\partial t}\right\|_{-1}^2+\epsilon\left\|\frac{\partial u(t)}{\partial t}\right\|^2+\|\theta(t)\|_{H^2(\Omega)}^2$$

$$+c\int_{t_2}^{t}(s-t_2)\left(\|\theta(\tau)\|_{H^3(\Omega)}^2+\left\|\frac{\partial u(\tau)}{\partial t}\right\|_{H^1(\Omega)}^2\right)d\tau$$

$$\leq\frac{c'(\nu+1)}{\nu}e^{ct}\|\pi_1-\pi_2\|_{\Psi}^2,\ \forall t>t_2. \tag{4.38}$$

We rewrite (4.8) in the form

$$-\Delta u = \tilde{h}_u(t), \tag{4.39}$$

$$u = 0 \text{ on } \Gamma, \tag{4.40}$$

for a fixed $t>t_2$, where

$$\tilde{h}_u(t)=-(-\Delta)^{-1}\frac{\partial u}{\partial t}-\epsilon\frac{\partial u}{\partial t}--(f(u_1)-f(u_2))+\theta. \tag{4.41}$$

Multiplying (4.41) by $\tilde{h}_u(t)$ and integrate over Ω implies

$$\|\tilde{h}_u(t)\|^2\leq c\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\epsilon\left\|\frac{\partial u}{\partial t}\right\|^2+\|\nabla\theta\|^2\right)+c''\|f(u_1)-f(u_2)\|^2.$$

Using the above estimates, we conclude that

$$\|\tilde{h}_u(t)\|^2\leq c\left(\left\|\frac{\partial u}{\partial t}\right\|_{-1}^2+\left\|\frac{\partial u}{\partial t}\right\|^2+\|\nabla\theta\|^2+\|\nabla u\|^2\right). \tag{4.42}$$

The use of estimate (4.38) and Theorem 4.1, results in

$$\|\tilde{h}_u(t)\|^2 \leq \frac{c^{'}(\nu+1)}{\nu} e^{ct}\|\pi_1 - \pi_2\|_{\Psi}^2, \ \forall t > t_2. \tag{4.43}$$

Another multiplication of (4.39) by $-\Delta u$ and integrate over Ω yields

$$\|\Delta u\|^2 \leq \|\tilde{h}_u(t)\| \|\Delta u\|.$$

Applying the Young's inequality, we find

$$\|\Delta u\|^2 \leq \|\tilde{h}_u(t)\|^2,$$

which leads, through (4.43) to

$$\|u(t)\|_{H^2(\Omega)}^2 \leq \frac{c^{'}(\nu+1)}{\nu} e^{ct}\|\pi_1 - \pi_2\|_{\Psi}^2, \ \forall t > t_2. \tag{4.44}$$

Finally, we deduce from (4.38) and (4.44) that

$$\|S(t)\pi_1 - S(t)\pi_2\|_{H^2(\Omega)\times H^2(\Omega)}^2 \leq \frac{c^{'}(\nu+1)}{\nu} e^{ct}\|\pi_1 - \pi_2\|_{\Psi}^2, \ \forall t > t_2. \tag{4.45}$$

Hence, the proof. □

Lemma 4.3. *The semigroup* $\{S(t)\}$ *is Holder continuous in time on* $[t_2, 2t_2] \times \mathcal{B}_1$. *In other words, there exists* $c(t_2) > 0$ *such that for any* $z \in \mathcal{B}_1$ *and any* $t, t^{'} \in [t_2, 2t_2]$ *the following inequality holds*

$$\|S(t)\pi_1 - S(t^{'})\pi_2\|_{\Psi} \leq c(t_2)|t^{'} - t|^{\frac{1}{2}}.$$

Proof. The lemma regarding Lipschitz continuity with respect to the initial conditions follows directly from Theorem 4.1. To establish the Holder continuity with respect to time, we integrate (4.28) over $(t_2, 2t_2)$ using (4.26) and (4.30), resulting in the following expression

$$\int_{t_2}^{2t_2} \left\|\frac{\partial u(\tau)}{\partial t}\right\|_{H^2(\Omega)}^2 d\tau \leq c(t_2). \tag{4.46}$$

Similarly, we integrate (4.20) and using (4.26) and (4.46), we find

$$\int_{t_2}^{2t_2} \left\|\frac{\partial \theta(\tau)}{\partial t}\right\|_{H^1(\Omega)}^2 d\tau \leq c(t_2). \tag{4.47}$$

Consequently,

$$
\begin{aligned}
& \|(u(t)-u(t^{'}),\theta(t)-\theta(t^{'}))\|_{\Psi} \\
\leq\ & c(\|u(t)-u(t^{'})\|_{H^2(\Omega)}+\|\theta(t)-\theta(t^{'})\|_{H^1(\Omega)}) \\
\leq\ & c\left(\left\|\int_t^{t^{'}}\frac{\partial u(\tau)}{\partial t}d\tau\right\|_{H^2(\Omega)}+\left\|\int_t^{t^{'}}\frac{\partial \theta(\tau)}{\partial t}d\tau\right\|_{H^1(\Omega)}\right) \\
\leq\ & c|t^{'}-t|^{\frac{1}{2}}\left|\int_t^{t^{'}}\left(\left\|\frac{\partial u(\tau)}{\partial t}\right\|^2_{H^2(\Omega)}+\left\|\frac{\partial \theta(\tau)}{\partial t}\right\|^2_{H^1(\Omega)}\right)d\tau\right|^{\frac{1}{2}} \\
\leq\ & c(t_2)|t^{'}-t|^{\frac{1}{2}}, \qquad (4.48)
\end{aligned}
$$

and Lemma 4.3 is thus proven. □

Hence, the subsequent outcome can be deduced from Theorem 4.1, along with Lemma 4.2 and Lemma 4.3 (as demonstrated in, for instance, [10, 16, 17]).

Theorem 4.5. *The semigroup $\{S(t)\}$ admits an exponential attractor $\mathcal{M}$ in Ψ, i.e.,*

(i) *$\mathcal{M}$ is compact in Ψ;*
(ii) *$\mathcal{M}$ is positively invariant, $S(t)\mathcal{M}\subset\mathcal{M}$, $\forall t\geq 0$;*
(iii) *$\mathcal{M}$ has finite fractal dimension in Ψ;*
(iv) *$\mathcal{M}$ attracts exponentially fast the bounded subsets of Ψ*

$$\forall B\in\Psi \text{ bounded},\ dist_{\Psi}(S(t)B,\mathcal{M})\leq Q(\|B\|_{\Psi})e^{-ct}, c>0,\ t\geq 0,$$

where the constant c is independent of B and $dist_{\Psi}$ denotes the Hausdorff semidistance between sets defined by

$$dist_{\Psi}(A,B)=\{\sup_{a\in A}\inf_{b\in B}\|a-b\|_{\Psi}\}.$$

Since $\mathcal{M}$ is a compact attracting set, we deduce from Theorem 4.5 and standard results (see, for example, [1, 15]) the following corollary

Corollary 4.3. *The semigroup $S(t)$ possesses the finite-dimensional global attractor $\mathcal{A}\subset\mathcal{B}_0$.*

REFERENCES

[1] A. BABIN, M. VISHIK: *Attractors of Evolution Equations*, North-Holland, Amsterdam, 1992.

[2] P.B.B. MATALA, D. MOUKOKO, M.E.I. GOYAUD: *Global attractor for a hyperbolic Cahn–Hilliard equation with a proliferation term*, Asymptot. Anal., **1** (2021), 1–23.

[3] G. CAGINALP: *An analysis of a phase field model of a free boundary*, Arch. Ration. Mech. Anal., **92** (1986), 205–245.

[4] B. SAOUD: *Attracteurs pour des systemes dissipatifs autonomous et nonautonomous*, These unique, Faculté des Sciences Fondamentales et Appliquées, Université de Poitiers.

[5] G. CAGINALP: *The dynamics of a conserved phase-field system; Stefan-like, Hele-Shaw and Cahn-Hilliard models as asymptotic limits*, IMA J. Appl. Math., **44** (1990), 77–94.

[6] G. CAGINALP: *Conserved-phase field system; implication for kinetic undercooling, Appl. Rev-B*, **38** (1988), 789-791.

[7] J. CAHN, J. HILLIARD: *Free energy of a nonunifor system I. Interfacial free energy*, J.Chem. Phys., **28** (1958), 258-267.

[8] L. CHERFILS, A. MIRANVILLE: *On the Caginalp system with dynamic boundary conditions and singular potentials*, Appl. Anal., **54** (2009), 89-115.

[9] L. CHERFILS, A. MIRANVILLE]: *Some results on the asymptotic behavior of the Caginalp system with singular potentials*, Adv. Math. Sci. Appl., **17** (2007), 107-129.

[10] A. EDEN, C. FOIAS, B. NICOLAENKO, R. TEMAM, : *Exponential attractors for dissipative evolution equations,* Research in Applied Mathematics, **37**, John-Wiley, New York, 1994.

[11] G. GAL, M. GRASSELLI, : *Asyptotic behavior of a Cahn-Hilliard-Navier-Stokes system in 2D*, Ann.I.H.Poincaré-AN, **27** (2010), 401-436.

[12] C. GIORGI, M. GRASSELLI, V. PATA : *Uniform attractors for a phase-field model with memory and quadratic nonlinearity*, Indiana Univ, Math. J., **48** (1996), 1395-1445.

[13] M.E. GURTIN, D.POLIGNONE, J. VINALS: *Two-phase binary fluids and immiscible fluids described by an order parameter*, Math. Models Methods Appli. Sci, **6** (1996), 815-831.

[14] G. GIACOMIN, J. LEBOWITZ: *Phase segregation dynamics in particle systems with long range interaction I, Macroscopic limits*, J. Stat. Phys., **87** (1997), 37–61.

[15] A. MIRANVILLE: *Asymptotic behaviour of a generalized Cahn-Hilliard equation with a proliferation term*, Appl. Anal., **92** (2013), 1308–1321.

[16] A. MIRANVILLE: *On the conserved phase-field model*, J. Math. Anal. Appl., **400** (2013), 143–152.

[17] A. MIRANVILLE: *Asymptotic behavior of a generalized Cahn-Hilliard equation with a proliferation term*, Appl. Anal. , **92** (2013), 1308-1321.

[18] S.F. ZHOU, F.Q. YIN, Z.G. OUYANG: *R*andom attractor for damped nonlinear wave equation with white noise, J. Appl. Dynam. Syst., **4** (2005), 883-903.

[19] M.J.D. DIEU, D. MOUKOKO, D. AMPINI, F. MOUKAMBA: *Existence of Global Attractor for Cahn-Hilliard Perturbed Phase-Field System with Dirichlet Boundary Condition and Regular Potentie*, Journal of Advances in Mathematics and Computer Science, **26**(6) (2018), 1-20.

[20] M.J.D. DIEU, M.E.I. GOYAUD, D. MOUKOKO, : *Pullback attractor for a nonautonomous parabolic Cahn-Hilliard phase-field system*, AIMS Mathematics, **8**(9) (2023), 22037–22066.

[21] A.C. NIMI, F.D.R.LANGA, A.J.P. BISSOUESSE, D.MOUKOKO, M.BATCHI: *Robust exponential attractors for the Cahn-Hilliar-Oono-Navier-Stokes System*, Discrete and Continuous Dynamical Systems-Series S, **16**(9) (2023), 2426-2451.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE,
MARIEN N'GOUABI UNIVERSITY
BRAZZAVILLE,
CONGO.
Email address: gael0ssebi19@gmail.com

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
MARIEN N'GOUABI UNIVERSITY
BRAZZAVILLE,
CONGO.
Email address: mmangoubijean@gmail.com

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
OUAGADOUGOU POLYTECHNIC SCHOOL
OUAGADOUGOU,
BURKINA FASO.
Email address: mahamatmaimos@yahoo.fr

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
MARIEN N'GOUABI UNIVERSITY
BRAZZAVILLE,
CCONGO.
Email address: vitalm28@gmail.com

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
MARIEN N'GOUABI UNIVERSITY
BRAZZAVILLE,
CCONGO.
Email address: tathychristian@yahoo.fr