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DIFFERENTIABILITY IN THE FRECHET SENSE OF A FUNCTIONAL RELATED TO A HYPERBOLIC PROBLEM WITH POLYNOMIAL NONLINEARITY

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ABSTRACT. In this work, we construct a functional related to a hyperbolic problem with polynomial non linearity and homogeneous Neuwman conditions and its differentiability in the Frechet sense.

1. INTRODUCTION

Consider the following problem:

$$u'' - \Delta u + u^3 + \frac{\partial u}{\partial t} = f(x,t), \quad x \in \Omega, \quad t \in]0,T[\tag{1.1}$$

$$\frac{\partial u}{\partial \vec{n}}(x,t)|_{\partial\Omega} = 0 \tag{1.2}$$

$$u(x,0) = u_0(x), \quad u'(x,0) = \frac{\partial u}{\partial t}(x,0) = u_1(x) \tag{1.3}$$

in a cylinder $Q_T = \Omega \times (0,T) \subset \mathbb{R}^{n+1}$.

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2. Theorems

Theorem 2.1. *Assume that* $f(x,t)$*,* $u_0(x)$ *and* $u_1(x)$ *are given functions such that* $f(x,t) \in L^2(Q_T)$*,* $u_0(x) \in H^1(\Omega) \cap L^4(\Omega)$*,* $u_1(x) \in L^2(\Omega)$*. Then the problem (1.1)-(1.3) has a solution* $u(x,t)$ *satisfactory:*

$$\begin{cases} \dfrac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \\ u(x,t) \in L^\infty(0,T;H^1(\Omega) \cap L^4(\Omega)). \end{cases}$$

Theorem 2.2. *Let* u *and* v *be in* $L^\infty(0,T;H^1(\Omega) \cap L^4(\Omega))$*, two solutions of the problem (1.1)-(1.3) under the assumptions of Theorem 2.1. Then the resulting solution of the problem is a unique solution.*

Remark 2.1. *Theorems 2.1 and 2.2 are proved in [1]*

3. Problem Position

In the problem (1.1)-(1.3), we assume that the functions $f(x,t)$, $u_0(x)$, $u_1(x)$ are commands and

$$f(x,t) \in Y \subset L^2(Q_T), \quad u_0(x) \in X \subset H^1(\Omega), \quad u_1(x) \in W \subset L^2(\Omega)$$

where Y,X,W are convex sets.

Now consider functional of the form

$$J_k(f,u_0,u_1) = \int_{Q_T} D_k(u(x,t), f(x,t), u_0(x), u_1(x))dxdt, \quad k = \overline{0, s_1+s_2} \tag{3.1}$$

(Here $u(x,t)$ is the unique solution of the problem (1.1)-(1.3) corresponding to the data f, u_0, u_1).

We need to find such orders $f^0(x,t) \in Y$, $u_0^0(x) \in X$, $u_1^0(x) \in W$ so that, for the solution $u^0(x,t)$ of problem (1.1)-(1.3) corresponding to orders (f^0, u_0^0, u_1^0), inequality constraints are verified,

$$J_k(f,u_0,u_1) \le 0, \quad k = \overline{1, s_1} \tag{3.2}$$

equality constraints,

$$J_k(f,u_0,u_1) = 0, \quad k = \overline{s_1+1, s_1+s_2} \tag{3.3}$$

and the functional $J_0(f, u_0, u_1)$ takes the smallest possible value

$$J_0(f^0, u_0^0, u_1^0) = \underset{Y\times X\times W}{inf} J_0(f, u_0, u_1). \tag{3.4}$$

Such orders (f^0, u_0^0, u_1^0) will be said to be optimal.

4. Differentiability of the functional

Proposition 4.1. *Now consider functional of the form*

$$J(f, u_0, u_1) = \int_{Q_T} D(u(x,t), f(x,t), u_0(x), u_1(x))dxdt, \tag{4.1}$$

where $u(x,t)$ is the unique solution of problem (1.1)-(1.3) corresponding to the data f, u_0, u_1. Assume that the following conditions are met:

a) *the function $D(u, f, u_0, u_1)$ and all its partial derivatives $D_u, D_f, D_{u_0}, D_{u_1}$ satisfy the Lipschitz conditions for the arguments (u, f, u_0, u_1)*
b) *$f^0(x,t) \in Y$, $u_0^0(x) \in X$, $u_1^0(x) \in W$ are given commands, and $u^0(x,t)$ their corresponding solution to the problem (1.1)-(1.3).*

Then, the functional (2.1) defined on $L^2(Q_T)\times H^1(\Omega)\times L^2(\Omega)$ is Fréchet differentiable at point(f^0, u_0^0, u_1^0).

Proof. Note that if $f^0(x,t) \in Y, \quad u_0^0(x) \in X, \quad u_1^0(x) \in W$ are given commands and $f(x,t) \in Y, \quad u_0(x) \in X, \quad u_1(x) \in W$ are arbitrary commands, then the variation $f^\varepsilon(x,t)$ of the command $f^0(x,t)$ along the direction $f - f^0$ is defined as follows:

$$f^\varepsilon(x,t) = f^0(x,t) + \varepsilon(f(x,t) - f^0(x,t)) \tag{4.2}$$

$\delta f = f^\varepsilon - f^0 = \varepsilon(f - f^0), \quad \varepsilon \geq 0$

In the same way

$${u_0}^\varepsilon(x,t) = {u_0}^0(x,t) + \varepsilon(u_0(x,t) - {u_0}^0(x,t)), \tag{4.3}$$

with $\delta u_0 = {u_0}^\varepsilon - {u_0}^0 = \varepsilon(u_0 - {u_0}^0)$, $\varepsilon \geq 0$,

$${u_1}^\varepsilon(x,t) = {u_1}^0(x,t) + \varepsilon(u_1(x,t) - {u_1}^0(x,t)) \tag{4.4}$$

with $\delta u_1 = {u_1}^\varepsilon - {u_1}^0 = \varepsilon(u_1 - {u_1}^0)$, $\varepsilon \geq 0$.

In all these variations ε is the same and $f^\varepsilon(x,t) \in Y$, $u_0^\varepsilon(x) \in X$, $u_1^\varepsilon(x) \in W$.

This is accomplished, for example, for $\varepsilon \in [0,1]$, as Y, X, W are convex sets.

Let $u^\varepsilon(x,t)$ be the solution of problem (1.1)-(1.3) corresponding to the data, $f^\varepsilon, u_0^\varepsilon, u_0^\varepsilon$.

Let's determine the formula for the increase ∇f of the functional $J(f, u_0, u_1)$ at the point f^0, u_0^0, u_1^0:

$$
\begin{aligned}
& \Delta J(f^0, u_0^0, u_1^0) = J(f^\varepsilon, u_0^\varepsilon, u_1^\varepsilon) - J(f^0, u_0^0, u_1^0) \\
= & \int_{Q_T} \big[D(u^\varepsilon, f^\varepsilon, u_0^\varepsilon, u_1^\varepsilon) - D(u^0, f^0, u_0^0, u_1^0)\big] dxdt \\
& D(u^\varepsilon, f^\varepsilon, u_0^\varepsilon, u_1^\varepsilon) - D(u^0, f^0, u_0^0, u_1^0) \\
\text{(4.5)} = & D(u^\varepsilon, f^\varepsilon, u_0^\varepsilon, u_1^\varepsilon) - D(u^0, f^\varepsilon, u_0^\varepsilon, u_1^\varepsilon) + D(u^0, f^\varepsilon, u_0^\varepsilon, u_1^\varepsilon)0 \\
- & D(u^0, f^0, u_0^\varepsilon, u_0^\varepsilon) + D(u^0, f^0, u_0^\varepsilon, u_0^\varepsilon) - D(u^0, f^0, u_0^0, u_0^\varepsilon) \\
+ & D(u^0, f^0, u_0^0, u_0^\varepsilon) - D(u^0, f^0, u_0^0, u_1^0) \\
= & \int_0^1 D_u(u^0 + \theta\delta u, f^\varepsilon, u_0^\varepsilon, u_1^\varepsilon)\delta u d\theta + \int_0^1 D_f(u^0, f^0 + \theta\delta f, u_0^\varepsilon, u_1^\varepsilon)\delta f d\theta \\
+ & \int_0^1 D_{u_0}(u^0, f^0, u_0^0 + \theta\delta u_0, u_1^\varepsilon)\delta u_0 d\theta + \int_0^1 D_{u_1}(u^0, f^0, u_0^0, u_1^0 + \theta\delta u_1)\delta u_1 d\theta \\
= & D_u(u^0, f^0, u_0^0, u_1^0)\delta u + \int_0^1 \big[D_u(\hat{u}, f^\varepsilon, u_0^\varepsilon, u_1^\varepsilon) - D_u(u^0, f^0, u_0^0, u_1^0)\big]\delta u d\theta \\
+ & D_f(u^0, f^0, u_0^0, u_1^0)\delta f + \int_0^1 \Big[D_f(u^0, \hat{f}, u_0^\varepsilon, u_1^\varepsilon) - D_u(u^0, f^0, u_0^0, u_1^0)\Big]\delta f d\theta \\
+ & D_{u_0}(u^0, f^0, u_0^0, u_1^0)\delta u_0 + \int_0^1 \big[D_{u_0}(u^0, f^0, \hat{u_0}, u_1^\varepsilon) - D_u(u^0, f^0, u_0^0, u_1^0)\big]\delta u_0 d\theta \\
+ & D_{u_1}(u^0, f^0, u_0^0, u_1^0)\delta u_1 + \int_0^1 \big[D_{u_1}(u^0, f^0, u_0^0, \hat{u_1}) - D_u(u^0, f^0, u_0^0, u_1^0)\big]\delta u_1 d\theta \\
= & D_u(u^0, f^0, u_0^0, u_1^0)\delta u + D_f(u^0, f^0, u_0^0, u_1^0)\delta f + D_{u_0}(u^0, f^0, u_0^0, u_1^0)\delta u_0 \\
+ & D_{u_1}(u^0, f^0, u_0^0, u_1^0)\delta u_1 + r_1 + r_2 + r_3 + r_4,
\end{aligned}
$$

with $\hat{u} = u^0 + \theta\delta u, \quad \delta u = u^\varepsilon - u^0$ where

$$
r_1 = \int_0^1 \big[D_u(\hat{u}, f^\varepsilon, u_0^\varepsilon, u_1^\varepsilon) - D_u(u^0, f^0, u_0^0, u_1^0)\big]\delta u d\theta,
$$

$$r_2 = \int_0^1 \Big[D_f(u^0, \hat{f}, u_0^\varepsilon, u_1^\varepsilon) - D_u(u^0, f^0, u_0^0, u_1^0)\Big]\delta f d\theta$$
$$r_3 = \int_0^1 \big[D_{u_0}(u^0, f^0, \hat{u_0}, u_1^\varepsilon) - D_u(u^0, f^0, u_0^0, u_1^0)\big]\delta u_0 d\theta$$
$$r_4 = \int_0^1 \big[D_{u_1}(u^0, f^0, u_0^0, \hat{u_1}) - D_u(u^0, f^0, u_0^0, u_1^0)\big]\delta u_1 d\theta$$

Equality (4.5) becomes:

$$\begin{aligned} \Delta J &= \int_{Q_T} [D_f(u^0, f^0, u_0^0, u_1^0)\delta f + D_{u_0}(u^0, f^0, u_0^0, u_1^0)\delta u_0 \\ &+ D_{u_1}(u^0, f^0, u_0^0, u_1^0)\delta u_1]dxdt + \int_{Q_T} D_u(u^0, f^0, u_0^0, u_1^0)\delta u dxdt \\ &+ \int_{Q_T}\left(\sum_{i=1}^4 r_i\right)dxdt. \end{aligned} \tag{4.6}$$

Consider the function $\eta(x,t) \in H^1(Q_T)$ such that $\eta(x,T) = 0$ and $\eta(x,0) \neq 0$. Multiplying (1.1) by $\eta(x,t)$ and integrating in Q_T, we obtain:

$$\begin{aligned} &\int_{Q_T} \frac{\partial^2 u}{\partial t^2}\eta(x,t)dxdt - \int_{Q_T} \Delta u\eta(x,t)dxdt + \int_{Q_T} u^3\eta(x,t)dxdt \\ &+ \int_{Q_t} \frac{\partial u}{\partial t}\eta(x,t)dxdt = \int_{Q_T} f(x,t)\eta(x,t)dxdt. \end{aligned} \tag{4.7}$$

Integrating by parts the first and second terms of the first member of (4.7), we obtain

$$\begin{aligned} &- \int_\Omega u_1(x)\eta(x,0)dx - \int_{Q_T} u_t.\eta_t dxdt + \int_{Q_T} \nabla u.\nabla\eta dxdt \\ &+ \int_{Q_T} (u)^3\eta(x,t)dxdt + \int_{Q_T} \frac{\partial u}{\partial t}\eta(x,t)dxdt = \int_{Q_T} f(x,t)\eta(x,t)dxdt. \end{aligned} \tag{4.8}$$

Note that the function $u^\varepsilon(x,t)$ satisfies equality (4.8)

$$\begin{aligned} &- \int_\Omega u_1^\varepsilon(x)\eta(x,0)dx - \int_{Q_T} u_t^\varepsilon.\eta_t dxdt + \int_{Q_T} \nabla u^\varepsilon.\nabla\eta dxdt \\ &+ \int_{Q_T} (u^\varepsilon)^3\eta(x,t)dxdt + \int_{Q_T} \frac{\partial u^\varepsilon}{\partial t}\eta(x,t)dxdt = \int_{Q_T} f^\varepsilon(x,t)\eta(x,t)dxdt. \end{aligned} \tag{4.9}$$

In the same way $u^0(x,t)$ verify (4.8):

$$
\begin{aligned}
&- \int_{\Omega} u_1^0(x)\eta(x,0)dx - \int_{Q_T} u_t^0.\eta_t dxdt + \int_{Q_T} \nabla u^0.\nabla\eta dxdt \\
\text{(4.10)}\ &+ \int_{Q_T} (u^0)^3\eta(x,t)dxdt + \int_{Q_T} \frac{\partial u^0}{\partial t}\eta(x,t)dxdt = \int_{Q_T} f^0(x,t)\eta(x,t)dxdt.
\end{aligned}
$$

Subtracting (4.10) from equality (4.9), we obtain:

$$
\begin{aligned}
&\int_{\Omega} \delta u_1(x)\eta(x,0)dx + \int_{Q_T} \delta u_t.\eta_t dxdt - \int_{Q_T} \nabla(\delta u).\nabla\eta dxdt \\
\text{(4.11)} \quad &- \int_{Q_T} \frac{\delta\partial u}{\partial t}\eta(x,t)dxdt - \int_{\Omega} \left[(u^\varepsilon)^3 - (u^0)^3\right]\eta(x,t)dxdt \\
&+ \int_{Q_T} \delta f(x,t)\eta(x,t)dxdt = 0.
\end{aligned}
$$

Let's put $\eta(x,t) = \Phi^0(x,t)$:

$$
\begin{aligned}
&\int_{\Omega} \delta u_1(x)\Phi^0(x,0)dx + \int_{Q_T} \delta u_t.\Phi_t^0 dxdt - \int_{Q_T} \nabla(\delta u).\nabla\Phi^0 dxdt \\
\text{(4.12)} \quad &- \int_{Q_T} \frac{\delta\partial u}{\partial t}\Phi^0(x,t)dxdt - \int_{Q_T} \left[(u^\varepsilon)^3 - (u^0)^3\right]\Phi^0(x,t)dxdt \\
&+ \int_{Q_T} \delta f(x,t)\Phi^0(x,t)dxdt = 0.
\end{aligned}
$$

Transforming the fourth term of (4.9) by virtue of immersion compactness $H^1(Q_T) \subset L^2(Q_T)$, we obtain the following expression:

$$
\int_{Q_T} \left[(u^\varepsilon)^3 - (u^0)^3\right]\Phi^0(x,t)dxdt = 3\int_{Q_T} (u^0)^2\Phi^0(x,t)\delta u dxdt + o(\|\delta u\|_{H^1(Q_T)}).
$$

In fact

$$
\begin{aligned}
I &= \int_{Q_T} \left[(u^\varepsilon)^3 - (u^0)^3\right]\Phi^0(x,t)dxdt \\
&= \int_{Q_T} \left[\Big((u^\varepsilon)^2 - (u^0)^2\Big)u^\varepsilon + (u^0)^2\big(u^\varepsilon - u^0\big)\right]\Phi^0(x,t)dxdt \\
&= \int_{Q_T} \left[\Big((u^\varepsilon)^2 - (u^0)^2\Big)u^0 + \Big((u^\varepsilon)^2 - (u^0)^2\Big)(u^\varepsilon - u^0) + (u^0)^2\big(u^\varepsilon - u^0\big)\right]\Phi^0(x,t)dxdt \\
&= \int_{Q_T} \left[\Big((u^\varepsilon)^2 - (u^0)^2\Big)u^0 + \Big((u^\varepsilon)^2 - (u^0)^2\Big)\delta u + (u^0)^2\delta u\right]\Phi^0(x,t)dxdt
\end{aligned}
$$

$$= \int_{Q_T} \Big((u^\varepsilon)^2 - (u^0)^2\Big) u^0 \Phi^0 dxdt + \int_{Q_T} \Big((u^\varepsilon)^2 - (u^0)^2\Big) \delta u \Phi^0 dxdt$$
$$+ \int_{Q_T} (u^0)^2 \delta u \Phi^0 dxdt = I_1 + I_2 + I_3.$$

By transforming I_1, we obtain:

$$\begin{aligned} I_1 &= \int_{Q_T} \Big((u^\varepsilon)^2 - (u^0)^2\Big) u^0 \Phi^0(x,t) dxdt \\ &= \int_{Q_T} \Big(\int_0^1 2(u^0 + \theta\delta u).u^0.\delta u.d\theta\Big) \Phi^0 dxdt \\ &= \int_{Q_T} \Big(\int_0^1 2((u^0 + \theta\delta u) - u^0).u^0.\delta u.d\theta\Big) \Phi^0 dxdt + 2\int_{Q_T}\int_0^1 u^0.u^0 \delta u \Phi^0 d\theta dxdt. \end{aligned}$$

Let be

$$q_k = \int_{Q_T}\int_0^1 ((u^0 + \theta\delta u_k) - u^0).u^0\Phi^0\delta u_k d\theta dxdt = \int_{Q_T}\int_0^1 (\theta\delta u_k).u^0\Phi^0\delta u_k d\theta dxdt,$$

where $\delta u_k = u^{\varepsilon_k} - u^0$, $\varepsilon_k = \frac{1}{k}$, $\delta u_k \to 0$ and $\varepsilon_k \to 0$ when $k \to \infty$.

From Hölder's inequality, it follows that:

$$\begin{aligned} |q_k| = \left|\int_{Q_T}\int_0^1 (\theta\delta u_k).u^0\Phi^0\delta u_k d\theta dxdt\right| &\leq \int_{Q_T}\int_0^1 |\theta\delta u_k||u^0||\Phi^0\delta u_k| d\theta dxdt \\ |q_k| &\leq const\|\delta u_k\|_{H^1(q_T)}.\int_0^1 \Big\||\theta\delta u_k||u^0||\Phi^0|\Big\| d\theta \\ \frac{|q_k|}{\|\delta u_k\|_{H^1(q_T)}} &\leq const\Big\||\delta u_k||u^0||\Phi^0|\Big\|. \end{aligned}$$

By stretching k to infinity, we find $q_k = o(\|\delta u\|_{H^1(Q_T)})$. Therefore

$$I_1 = 2\int_{Q_T} (u^0)^2 \delta u \Phi^0 dxdt + o(\|\delta u\|_{H^1(Q_T)}).$$

By analogy, by transforming I_2, we obtain:

$$I_2 = 2\int_{Q_T} u^0\Phi^0(\delta u)^2 dxdt = o(\|\delta u\|_{H^1(Q_T)}).$$

Therefore

$$I_2 = o(\|\delta u\|_{H^1(Q_T)}).$$

Using the results of I_1 and I_2 and using I_3, we obtain the following expression for I:

$$\begin{aligned} I &= I_1 + I_2 + I_3 \\ I &= 2\int_{Q_T} (u^0)^2 \delta u \Phi^0 dxdt + o(\|\delta u\|_{H^1(Q_T)}) + \int_{Q_T} (u^0)^2 \delta u \Phi^0 dxdt \\ I &= 3\int_{Q_T} (u^0)^2 \delta u \Phi^0 dxdt + o(\|\delta u\|_{H^1(Q_T)}) \end{aligned}$$

Equality (4.12) takes the form:

$$\begin{aligned} &\int_{\Omega} \delta u_1(x)\Phi^0(x,0)dx + \int_{Q_T} \delta u_t.\Phi_t^0 dxdt - \int_{Q_T} \nabla(\delta u).\nabla\Phi^0 dxdt \\ -\ & 3\int_{Q_T} (u^0)^2 \delta u \Phi^0 dxdt - \int_{Q_T} \frac{\delta\partial u}{\partial t}\Phi^0(x,t)dxdt + \int_{Q_T} \delta f(x,t)\Phi^0(x,t)dxdt = 0. \end{aligned}$$

Then formula (4.6) for the ΔJ increase in the functional becomes:

$$\begin{aligned} \Delta J &= \int_{Q_T} [D_f \delta f + D_{u_0}\delta u_0 \\ &+ D_{u_1}\delta u_1]dxdt + \int_{Q_T} D_u \delta u dxdt + \int_{\Omega} \delta u_1(x)\Phi^0(x,0)dx \\ &+ \int_{Q_T} \delta u_t.\Phi_t^0 dxdt - \int_{Q_T} \nabla(\delta u).\nabla\Phi^0 dxdt - 3\int_{Q_T} (u^0)^2 \delta u \Phi^0 dxdt \\ &- \int_{Q_T} \frac{\delta\partial u}{\partial t}\Phi^0(x,t)dxdt + \int_{Q_T} \delta f(x,t)\Phi^0(x,t)dxdt \\ &+ \int_{Q_T} \left(\sum_{i=1}^{4} r_i\right) dxdt + o(\|\delta u\|_{H^1(Q_T)}). \end{aligned} \tag{4.13}$$

Taking the approximation of the remaining terms by virtue of the fact that D and D_u satisfy the Lipschitz conditions and (4.13), we can show that:

$$\int_{Q_T} \left(\sum_{i=1}^{4} r_i\right) dxdt = o\left(\|\delta f\|_{L^2(Q_T)} + \|\delta u_0\|_{H^1(\Omega)} + \|\delta u_1\|_{L^2(\Omega)}\right).$$

For example,

$$\left|\int_{Q_T} r_1 dxdt\right| = \left|\int_{Q_T}\int_0^1 \left[D_u(\hat{u}, f^{\varepsilon}, u_0^{\varepsilon}, u_1^{\varepsilon}) - D_u(u^0, f^0, u_0^0, u_1^0)\right]\delta u d\theta dxdt\right|$$

$$\begin{aligned}
&\leq \int_{Q_T}\int_0^1 \big|D_u(\hat{u}, f^\varepsilon, u_0^\varepsilon, u_1^\varepsilon) - D_u(u^0, f^0, u_0^0, u_1^0)\big||\delta u| d\theta dxdt \\
&\leq \int_{Q_T}\int_0^1 L(|\hat{u} - u^0| + |f^\varepsilon - f^0| + |u_0^\varepsilon - u_0^0| + |u_1^\varepsilon - u_1^0|)|\delta u| d\theta dxdt \\
&= L\left[\frac{1}{2}\int_{Q_T} |\delta u|^2 dxdt + \int_{Q_T} |\delta f||\delta u| dxdt\right. \\
&+ \left.\int_{Q_T} |\delta u_0||\delta u| dxdt + \int_{Q_T} |\delta u_1||\delta u| dxdt\right].
\end{aligned}$$

So,

$$\begin{aligned}
&\left|\int_{Q_T} r_1 dxdt\right| \leq \frac{3}{2} L\|\delta u\|^2_{L^2(Q_T)} \leq \frac{3}{2} LC\|\delta u\|^2_{H^1(Q_T)} \\
\Longrightarrow &\int_{Q_T} r_1 dxdt = o(\|\delta u\|_{H^1(Q_T)}).
\end{aligned}$$

Similarly, we find approximations for the other remaining terms. Formula (4.13) can thus be written as:

$$\begin{aligned}
\Delta J &= \int_{Q_T} [D_f \delta f + D_{u_0}\delta u_0 + D_{u_1}\delta u_1] dxdt + \int_{Q_T} D_u \delta u dxdt \\
&+ \int_\Omega \delta u_1(x)\Phi^0(x,0)dx + \int_{Q_T} \delta u_t \Phi^0 dxdt - 3\int_{Q_T} (u^0)^2 \delta u \Phi^0 dxdt \\
&- \int_{Q_T} \nabla(\delta u).\nabla\Phi^0 dxdt - \int_{Q_T} \frac{\partial \delta u}{\partial t}\Phi^0 dxdt \\
&+ \int_{Q_T} \delta f(x,t)\Phi^0(x,t) dxdt + o(\|\delta u\|_{H^1(Q_T)}).
\end{aligned} \tag{4.14}$$

To express the derivatives of functional (4.1) in a more convenient form, consider the following boundary conjugate problem:

$$\begin{cases}
\dfrac{\partial^2 \Phi}{\partial t^2} - \Delta\Phi + 3(u^0)^2\Phi - \dfrac{\partial \Phi}{\partial t} = D_u \\
\dfrac{\partial \Phi}{\partial \vec{n}}(x,t) \mid_{\partial Q_T} = 0, \\
\Phi(x,t) \mid_{t=T} = \dfrac{\partial \Phi}{\partial t}(x,t) \mid_{t=T} = 0, \quad x \in \Omega
\end{cases} \tag{4.15}$$

□

5. Existence of the solution to the conjugate problem

5.1. **Formulation Variational.** Multiplying the first equation in (4.15) by the function $\eta(x,t)$ and integrating over Q_T, we obtain:

$$\int_{Q_T} \frac{\partial^2 \Phi}{\partial t^2}\eta(x,t)dxdt - \int_{Q_T} \Delta\Phi\eta(x,t)dxdt + 3\int_{Q_T} (u^0)^2\Phi\eta(x,t)dxdt$$

$$- \int_{Q_T} \frac{\partial \Phi}{\partial t}\eta(x,t)dxdt = \int_{Q_T} D_u.\eta(x,t)dxdt. \tag{5.1}$$

Using integration by parts and Green's formula, (5,1) becomes:

$$\int_{\Omega} \frac{\partial \Phi}{\partial t}(x,T)\eta(x,T)dx - \int_{\Omega} \Phi_1(x)\eta(x,0)dx - \int_{Q_T} \Phi_t\eta_t dxdt + \int_{Q_T} (\nabla\Phi.\nabla\eta)dxdt$$

$$+ \; 3\int_{Q_T} (u^o)^2\Phi\eta(x,t)dxdt - \int_{Q_T} \frac{\partial \Phi}{\partial t}\eta(x,t)dxdt = \int_{Q_T} D_u.\eta(x,t)dxdt.$$

Definition 5.1. *Any function $\Phi(x,t)$, equal to $\Phi_0(x)$ for $t = 0$ and satisfying the following integral equality, is called a distributional solution of problem (4.15):*

$$- \int_{\Omega} \Phi_1(x)\eta(x,0)dx - \int_{Q_T} \Phi_t\eta_t dxdt + \int_{Q_T} (\nabla\Phi.\nabla\eta)dxdt$$

$$+ \; 3\int_{Q_T} (u^o)^2\Phi\eta(x,t)dxdt - \int_{Q_T} \frac{\partial \Phi}{\partial t}\eta(x,t)dxdt = \int_{Q_T} D_u.\eta(x,t)dxdt,$$

for all $\eta(x,t)$ whose trace for $t = T$ is equal to 0.

5.2. **A priori estimation of these solutions.** Multiplying the first equation of (4.15) by $\dfrac{\partial \Phi}{\partial t}$ and integrating by parts on Ω:

$$\int_{\Omega} \frac{\partial^2 \Phi}{\partial t^2}\frac{\partial \Phi}{\partial t}dx - \int_{\Omega} \Delta\Phi\frac{\partial \Phi}{\partial t}dx + 3\int_{\Omega} (u^0)^2\Phi\frac{\partial \Phi}{\partial t}dx - \int_{\Omega} \frac{\partial \Phi}{\partial t}\frac{\partial \Phi}{\partial t}dx = \int_{\Omega} D_u.\frac{\partial \Phi}{\partial t}dx. \tag{5.2}$$

Transforming the first member of (5.2), we obtain:

$$\int_{\Omega} \frac{\partial^2 \Phi}{\partial t^2}\frac{\partial \Phi}{\partial t}dx = \frac{1}{2}\frac{\partial}{\partial t}\left\|\frac{\partial \Phi}{\partial t}\right\|^2_{L^2(\Omega)}. \tag{5.3}$$

Applying the Green formula:

$$\int_{\Omega} \Delta\Phi\frac{\partial \Phi}{\partial t}dx = -\frac{1}{2}\frac{\partial}{\partial t}\|\nabla\Phi\|^2_{L^2(\Omega)} \tag{5.4}$$

we have:

$$\int_\Omega (u^0)^2 \Phi \frac{\partial \Phi}{\partial t} dx = \frac{1}{2} \frac{\partial}{\partial t} \|u^0 \Phi\|^2_{L^2(\Omega)}, \tag{5.5}$$

and

$$\int_\Omega \frac{\partial \Phi}{\partial t} \frac{\partial \Phi}{\partial t} dx = \left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)}. \tag{5.6}$$

Replacing (5.3), (5.4), (5.5) and (5.6) in (5.2), we obtain:

$$\frac{1}{2} \frac{\partial}{\partial t} \left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} + \frac{1}{2} \frac{\partial}{\partial t} \|\nabla \Phi\|^2_{L^2(\Omega)} + \frac{3}{2} \frac{\partial}{\partial t} \|u^0 \Phi\|^2_{L^2(\Omega)} - \left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} = \int_\Omega D_u . \frac{\partial \Phi}{\partial t} dx$$

implies

$$\frac{1}{2} \left[\frac{\partial}{\partial t} \left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} + \frac{\partial}{\partial t} \|\nabla \Phi\|^2_{L^2(\Omega)} + 3 \frac{\partial}{\partial t} \|u^0 \Phi\|^2_{L^2(\Omega)} \right] - \left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} = \int_\Omega D_u . \frac{\partial \Phi}{\partial t} dx.$$

Using Hölder and Young's inequalities, the second member of (5.7) gives:

$$\begin{aligned} \int_\Omega D_u \frac{\partial \Phi}{\partial t} dx &\leq \left(\int_\Omega |D_u|^2 dx \right)^{1/2} . \left(\int_\Omega \left| \frac{\partial \Phi}{\partial t} \right|^2 dx \right)^{1/2} \\ &\leq \|D_u\|_{L^2(\Omega)} \left\| \frac{\partial \Phi}{\partial t} \right\|_{L^2(\Omega)} \leq \frac{1}{2} \left[\|D_u\|^2_{L^2(\Omega)} + \left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} \right]. \end{aligned} \tag{5.7}$$

Using (5.8), (5.7) becomes:

$$\begin{aligned} &\frac{\partial}{\partial t} \left[\left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} + \|\nabla \Phi\|^2_{L^2(\Omega)} + 3 \|u^0 \Phi\|^2_{L^2(\Omega)} \right] - 2 \left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} \\ &\leq \quad \|D_u\|^2_{L^2(\Omega)} + \left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)}. \end{aligned} \tag{5.8}$$

Integrating (5.9) over $(0, t)$, we obtain:

$$\begin{aligned} &\left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} + \|\nabla \Phi\|^2_{L^2(\Omega)} + 3|u^0|^2 \|\Phi\|^2_{L^2(\Omega)} - 2 \int_0^t \left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} ds \\ - \quad &\left(\left\| \frac{\partial \Phi(x,0)}{\partial t} \right\|^2_{L^2(\Omega)} + \|\nabla \Phi(x,0)\|^2_{L^2(\Omega)} + 3|u^0|^2 \|\Phi(x,0)\|^2_{L^2(\Omega)} \right) \end{aligned}$$

$$\leq \int_0^t \left(\|D_u\|^2_{L^2(\Omega)} + \left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} \right) ds \tag{5.9}$$

and

$$\begin{aligned} &\left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} + \|\nabla \Phi\|^2_{L^2(\Omega)} + 3|u^0|^2 \|\Phi\|^2_{L^2(\Omega)} - 2\int_0^t \left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} ds \\ - &\left(\|\Phi_1\|^2_{L^2(\Omega)} + \|\nabla \Phi_0\|^2_{L^2(\Omega)} + 3|u^0|^2 \|\Phi_0\|^2_{L^2(\Omega)} \right) \leq \int_0^t \left(\|D_u\|^2_{L^2(\Omega)} + \left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} \right) ds. \end{aligned}$$

Finally, we obtain:

$$\begin{aligned} &\left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} + \|\nabla \Phi\|^2_{L^2(\Omega)} + 3|u^0|^2 \|\Phi\|^2_{L^2(\Omega)} - 2\int_0^t \left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} ds \leq \|\Phi_1\|^2_{L^2(\Omega)} \\ &+\|\nabla \Phi_0\|^2_{L^2(\Omega)} + 3|u^0|^2 \|\Phi_0\|^2_{L^2(\Omega)} + \int_0^t \left(\|D_u\|^2_{L^2(\Omega)} + \left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} \right) ds. \end{aligned} \tag{5.10}$$

Let's add the term $\|\Phi\|^2_{L^2(\Omega)}$ to the first and second members of (5.10):

$$\begin{aligned} &\left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} + \|\Phi\|^2_{H^1(\Omega)} + 3|u^0|^2 \|\Phi\|^2_{L^2(\Omega)} - 2\int_0^t \left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} ds \\ \leq\ & \|\Phi\|^2_{L^2(\Omega)} + \|\Phi_1\|^2_{L^2(\Omega)} + \|\nabla \Phi_0\|^2_{L^2(\Omega)} + 3|u^0|^2 \|\Phi_0\|^2_{L^2(\Omega)} \\ +\ & \int_0^t \left(\|D_u\|^2_{L^2(\Omega)} + \left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} \right) ds. \end{aligned} \tag{5.11}$$

For all $t \in [0, T]$; we have:

$$\begin{aligned} &\left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} + \|\Phi\|^2_{H^1(\Omega)} + 3|u^0|^2 \|\Phi\|^2_{L^2(\Omega)} - 2\int_0^T \left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} dt \\ \leq\ & \|\Phi\|^2_{L^2(\Omega)} + \|\Phi_1\|^2_{L^2(\Omega)} + \|\nabla \Phi_0\|^2_{L^2(\Omega)} + 3|u^0|^2 \|\Phi_0\|^2_{L^2(\Omega)} \\ +\ & \int_0^T \left(\|D_u\|^2_{L^2(\Omega)} + \left\| \frac{\partial \Phi}{\partial t} \right\|^2_{L^2(\Omega)} \right) dt. \end{aligned} \tag{5.12}$$

Let's put: $\Phi(t) = \Phi(x, t)$. In this case:

$$\Phi(t) = \int_0^t \frac{\partial \Phi}{\partial s}(s)ds + \Phi(0)$$

$$|\Phi(t)|^2 = \left|\int_0^t \frac{\partial \Phi}{\partial s}(s)ds + \Phi(0\right|^2 \leq \left|\int_0^t \frac{\partial \Phi}{\partial s}(s)ds + \Phi(0)\right|^2 + \left|\int_0^t \frac{\partial \Phi}{\partial s}(s)ds - \Phi(0)\right|^2$$

(5.13) $\Longrightarrow \quad |\Phi(t)|^2 \leq 2\left[\left(\int_0^t \left|\frac{\partial \Phi}{\partial s}(s)\right|ds\right)^2 + |\Phi(0)|^2\right].$

Using Hölder's inequality, we obtain:

$$\int_0^t \left|\frac{\partial \Phi}{\partial s}(x,t)\right|ds \leq \left(\int_0^t |1|ds\right)^{1/2} . \left(\int_0^t \left|\frac{\partial \Phi}{\partial s}(s)\right|^2 ds\right)^{1/2}$$

$$\Longrightarrow \left(\int_0^t \left|\frac{\partial \Phi}{\partial s}(x,t)\right|ds\right)^2 \leq \left(\int_0^t ds\right) . \left(\int_0^t \left|\frac{\partial \Phi}{\partial s}(s)\right|^2 ds\right) \leq t\int_0^t \left|\frac{\partial \Phi}{\partial s}(s)\right|^2 ds$$

(5.13) becomes:

$$|\Phi(t)|^2 \leq 2\left[t\int_0^t \left|\frac{\partial \Phi}{\partial s}(s)\right|^2 ds + |\Phi(0)|^2\right]$$

(5.14) $\Longleftrightarrow \quad |\Phi(t)|^2 \leq 2t\int_0^t \left|\frac{\partial \Phi}{\partial s}(s)\right|^2 ds + 2|\Phi(0)|^2.$

We take $t \in [0,1] \subset [0,T]$, then $1-t \geq 0$. We can add the term $2(1-t)\int_0^t \left|\frac{\partial \Phi}{\partial s}(s)\right|^2 ds$ to the second member of (5.14)

$$|\Phi(t)|^2 \leq 2t\int_0^t \left|\frac{\partial \Phi}{\partial s}(s)\right|^2 ds + 2|\Phi(0)|^2 + 2(1-t)\int_0^t \left|\frac{\partial \Phi}{\partial s}(s)\right|^2 ds,$$

and finally we get:

$$|\Phi(t)|^2 \leq 2\left[|\Phi(0)|^2 + \int_0^t \left|\frac{\partial \Phi}{\partial s}(s)\right|^2 ds\right].$$

Let's integrate on Ω:

$$\begin{aligned}\int_\Omega |\Phi(t)|^2 dx &\leq 2\int_\Omega \left[|\Phi(0)|^2 + \int_0^t \left|\frac{\partial \Phi}{\partial s}(s)\right|^2 ds\right] dx\\ &\leq 2\left[\int_\Omega |\Phi_0|^2 dx + \int_0^t \left(\int_\Omega \left|\frac{\partial \Phi}{\partial s}\right|^2 dx\right) ds\right],\end{aligned}$$

$\forall t \in [0, T]$,

$$\begin{aligned}\int_\Omega |\Phi(t)|^2 dx &\leq 2\left[\int_\Omega |\Phi_0|^2 dx + \int_0^T \left(\int_\Omega \left|\frac{\partial \Phi}{\partial t}\right|^2 dx\right) dt\right]\\ \Longrightarrow \|\Phi(t)\|^2_{L^2(\Omega} &\leq 2\left[\|\Phi_0\|^2_{L^2(\Omega)} + \int_0^T \left\|\frac{\partial \Phi}{\partial t}\right\|^2_{L^2(\Omega)} dt\right]. \qquad (5.15)\end{aligned}$$

Using (5.15), (5.13) becomes:

$$\begin{aligned}&\left\|\frac{\partial \Phi}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi\|^2_{H^1(\Omega)} + 3|u^0|^2\|\Phi\|^2_{L^2(\Omega)} - 2\int_0^T \left\|\frac{\partial \Phi}{\partial t}\right\|^2_{L^2(\Omega)} dt\\ \leq\ & 2\left[\|\Phi_0\|^2_{L^2(\Omega)} + \int_0^T \left\|\frac{\partial \Phi}{\partial t}\right\|^2_{L^2(\Omega)} dt\right] + \|\Phi_1\|^2_{L^2(\Omega)} + \|\nabla\Phi_0\|^2_{L^2(\Omega)}\\ +\ & 3|u^0|^2\|\Phi_0\|^2_{L^2(\Omega)} + \int_0^T \left(\|D_u\|^2_{L^2(\Omega)} + \left\|\frac{\partial \Phi}{\partial t}\right\|^2_{L^2(\Omega)}\right) dt.\end{aligned}$$

This is equivalent to:

$$\begin{aligned}&\left\|\frac{\partial \Phi}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi\|^2_{H^1(\Omega)} + 3|u^0|^2\|\Phi\|^2_{L^2(\Omega)} \leq 2\|\Phi_0\|^2_{L^2(\Omega)} + \|\Phi_1\|^2_{L^2(\Omega)}\\ +\ & \|\nabla\Phi_0\|^2_{L^2(\Omega)} + 3|u^0|^2\|\Phi_0\|^2_{L^2(\Omega)} + 5\int_0^T \left\|\frac{\partial \Phi}{\partial t}\right\|^2_{L^2(\Omega)} dt + \int_0^T \|D_u\|^2_{L^2(\Omega)} dt. \qquad (5.16)\end{aligned}$$

By adding the term $\|\nabla\Phi_0\|^2_{L^2(\Omega)}$ to the second member, (5.16) becomes:

$$\begin{aligned}&\left\|\frac{\partial \Phi}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi\|^2_{H^1(\Omega)} + 3|u^0|^2\|\Phi\|^2_{L^2(\Omega)} \leq 2\|\Phi_0\|^2_{H^1(\Omega)} + \|\Phi_1\|^2_{L^2(\Omega)}\\ +\ & 3|u^0|^2\|\Phi_0\|^2_{L^2(\Omega)} + 5\int_0^T \left\|\frac{\partial \Phi}{\partial t}\right\|^2_{L^2(\Omega)} dt + \int_0^T \|D_u\|^2_{L^2(\Omega)} dt. \qquad (5.17)\end{aligned}$$

By adding the term $\int_0^T \|\Phi\|^2_{H^1(\Omega)} dt$ to the second member of (5.17), we get:

$$\begin{aligned} & \left\|\frac{\partial\Phi}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi\|^2_{H^1(\Omega)} + 3|u^0|^2\|\Phi\|^2_{L^2(\Omega)} \\ \leq\ & 2\|\Phi_0\|^2_{H^1(\Omega)} + \|\Phi_1\|^2_{L^2(\Omega)} + 3|u^0|^2\|\Phi_0\|^2_{L^2(\Omega)} \\ +\ & 5\int_0^T \left\|\frac{\partial\Phi}{\partial t}\right\|^2_{L^2(\Omega)} dt + \int_0^T \|D_u\|^2_{L^2(\Omega)} dt + \int_0^T \|\Phi\|^2_{H^1(\Omega)} dt. \end{aligned} \tag{5.18}$$

Let's put: $k = 2\|\Phi_0\|^2_{H^1(\Omega)} + \|\Phi_1\|^2_{L^2(\Omega)} + 3|u^0|^2\|\Phi_0\|^2_{L^2(\Omega)} + \int_0^T \|D_u\|^2_{L^2(\Omega)} dt$. Then (5.18) becomes:

$$\left\|\frac{\partial\Phi}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi\|^2_{H^1(\Omega)} + 3|u^0|^2\|\Phi\|^2_{L^2(\Omega)} \leq k + 5\int_0^T \left\|\frac{\partial\Phi}{\partial t}\right\|^2_{L^2(\Omega)} dt + \int_0^T \|\Phi\|^2_{H^1(\Omega)} dt.$$

Then, we have the following relationship:

$$\left\|\frac{\partial\Phi}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi\|^2_{H^1(\Omega)} \leq k + 5\left(\int_0^T \left\|\frac{\partial\Phi}{\partial t}\right\|^2_{L^2(\Omega)} dt + \int_0^T \|\Phi\|^2_{H^1(\Omega)} dt\right). \tag{5.19}$$

Let's put: $E(t) = \left\|\frac{\partial\Phi}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi(x,t)\|^2_{H^1(\Omega)}$.

We get:

$$E(t) \leq k + 5\int_0^T \left\|\frac{\partial\Phi}{\partial t}(x,t)\right\|^2_{L^2(\Omega)} dt + \int_0^T \|\Phi(x,t)\|^2_{H^1(\Omega)} dt.$$

According to Gronwall

$$\begin{aligned} E(t) &\leq ke^{\int_0^T 5E(s)ds} \\ E(t) &\leq ke^{\int_0^T 5ds} \\ E(t) &\leq ke^{5T} \\ \Rightarrow E(t) &\leq c \quad \text{with} \quad c = ke^{5T} \\ E(t) &= \left\|\frac{\partial\Phi}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi(x,t)\|^2_{H^1(\Omega)} \leq c \quad \text{p.p} \quad \text{on} \quad [0,T]. \end{aligned} \tag{5.20}$$

We have:

$$\begin{cases} \left\|\dfrac{\partial \Phi}{\partial t}\right\|^2_{L^2(\Omega)} & \leq c_1 \\ \|\Phi(x,t)\|^2_{H^1(\Omega)} & \leq c_2 \end{cases}, \quad \text{with} \quad c = c_1 + c_2$$

$$\Longrightarrow \begin{cases} \dfrac{\partial \Phi}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \\ \Phi(x,t) \in L^\infty(0,T;H^1(\Omega)) \end{cases}$$

On the other hand (5.18) gives:

$$\begin{aligned} & \left\|\frac{\partial \Phi}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi\|^2_{H^1(\Omega)} + 3|u^0|^2\|\Phi\|^2_{L^2(\Omega)} \\ \leq \;& 2\|\Phi_0\|^2_{H^1(\Omega)} + \|\Phi_1\|^2_{L^2(\Omega)} + 3|u^0|^2\|\Phi_0\|^2_{L^2(\Omega)} \\ + \;& 5\int_0^T \left\|\frac{\partial \Phi}{\partial t}\right\|^2_{L^2(\Omega)} dt + \int_0^T \|D_u\|^2_{L^2(\Omega)} dt. \end{aligned}$$

Let's adding the term $\int_0^T \|\Phi\|^2_{L^2(\Omega)} dt$ to the second member of (5.18), we get:

$$\begin{aligned} & \left\|\frac{\partial \Phi}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi\|^2_{H^1(\Omega)} + 3|u^0|^2\|\Phi\|^2_{L^2(\Omega)} \\ \leq \;& 2\|\Phi_0\|^2_{H^1(\Omega)} + \|\Phi_1\|^2_{L^2(\Omega)} + 3|u^0|^2\|\Phi_0\|^2_{L^2(\Omega)} + 5\int_0^T \left\|\frac{\partial \Phi}{\partial t}\right\|^2_{L^2(\Omega)} dt \\ + \;& \int_0^T \|D_u\|^2_{L^2(\Omega)} dt + \int_0^T \|\Phi\|^2_{L^2(\Omega)} dt. \end{aligned} \tag{5.21}$$

Let's put:

$$k_1 = 2\|\Phi_0\|^2_{H^1(\Omega)} + \|\Phi_1\|^2_{L^2(\Omega)} + 3|u^0|^2\|\Phi_0\|^2_{L^2(\Omega)} + 5\int_0^T \left\|\frac{\partial \Phi}{\partial t}\right\|^2_{L^2(\Omega)} dt + \int_0^T \|D_u\|^2_{L^2(\Omega)} dt.$$

The inequality becomes:

$$\left\|\frac{\partial \Phi}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi\|^2_{H^1(\Omega)} + 3|u^0|^2\|\Phi\|^2_{L^2(\Omega)} \leq k_1 + \int_0^T \|\Phi\|^2_{L^2(\Omega)} dt.$$

We obtain:

$$3|u^0|^2\|\Phi\|^2_{L^2(\Omega)} \leq k_1 + \int_0^T \|\Phi\|^2_{L^2(\Omega)} dt. \tag{5.22}$$

Let's put: $|u^0|^2 = k_2$, implying

$$\|\Phi\|^2_{L^2(\Omega)} \leq \frac{k_1}{3k_2} + \frac{1}{3k_2}\int_0^T \|\Phi\|^2_{L^2(\Omega)} dt.$$

Let's put: $F(t) = \|\Phi\|^2_{L^2(\Omega)}$ $\quad k_3 = \dfrac{k_1}{3k_2}$ and $k_4 = \dfrac{1}{3k_2}$, we have:

$$F(t) \leq k_3 + k_4 \int_0^T \|\Phi\|^2_{L^2(\Omega)} dt.$$

According to Gronwall

$$\begin{aligned} F(t) &\leq k_3 e^{k_4 \int_0^T F(s)ds} \\ \Longrightarrow F(t) &\leq k_3 e^{k_4 \int_0^T ds} \\ \Longrightarrow F(t) &\leq k_3 e^{k_4(T)} \\ \Longrightarrow F(t) &\leq c_3 \quad \text{with} \quad const = k_3 e^{k_4(T)} \\ \underset{t\in[0,T]}{inf} \|\Phi\|^2_{L^2(\Omega)} &\leq c_3 \\ \Longrightarrow & \qquad \Phi \in L^\infty(0,T,L^2(\Omega)). \end{aligned} \tag{5.23}$$

From the above

$$\begin{cases} \dfrac{\partial \Phi}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \\ \Phi(x,t) \in L^\infty(0,T;H^1(\Omega)) \end{cases}, \text{ and } \quad \Phi \in L^\infty(0,T,L^2(\Omega)).$$

Then

$$\begin{cases} \dfrac{\partial \Phi}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \\ \Phi(x,t) \in L^\infty(0,T;H^1(\Omega) \cap L^2(\Omega)) \end{cases}. \tag{5.24}$$

Hence the following theorem:

Theorem 5.1. *Let' be D_u, $\Phi_0(x)$ and $\Phi_1(x)$ given functions such as: $D_u \in L^2(Q_T)$, $\Phi_0(x) \in H^1(\Omega) \cap L^2(\Omega)$, $\Phi_1(x) \in L^2(\Omega)$. Then problem (1.1)-(1.3) admits a solution $\Phi(x,t)$ satisfying the following conditions:*

$$\begin{cases} \dfrac{\partial \Phi}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \\ \Phi(x,t) \in L^\infty(0,T;H^1(\Omega) \cap L^2(\Omega)) \end{cases}$$

Proof. To prove Theorem (5.1), we will use the Faedo-Galerkin method, which is divided into three steps:

Step 1: Construction of approximate solutions;

Step 2: A priori estimation of these solutions;

Step 3: Passage to the limit.

Step 1: Construction of approximate solutions. Let's pose:

$$V = \{u \in H^1(\Omega) \cap L^2(\Omega)\}.$$

As the space V is a separable Banach space, then it has a basis denoted by $(e_i)_{1\leq i\leq m}$, $i \in \mathbb{N}^*$; $(e_i, e_j) = 0$, $\forall i \neq j$ and $\|e_j\| = 1$, $\forall j$ where functions (e_j) are such that: $(e_j) \in V \ \forall j$.

With homogeneous Neumann conditions, the operator $-\Delta$ has a sequence of eigenvalues $\{\lambda_i\}_{i\geq 0}$ with associated functions (e_j) are its own associated functions.

The problem is to find in any sub-space $V_m = \{e_1, e_2, \ldots, e_m\}$ of V an approximate solution $\Phi_m = \Phi_m(t)$ in the form:

$$\Phi_m(t) = \sum_{i=1}^{m} \Phi_{im}(t) e_i.$$

The Φ_{im} to be determined by the following conditions:

$$\text{(5.25)} \qquad \begin{cases} \dfrac{\partial^2 \Phi_m}{\partial t^2} - \Delta\Phi_m + 3{u^0}^2\Phi_m - \dfrac{\partial u_m}{\partial t} = D_{um} = p_m(D_u) \\ \dfrac{\partial \Phi_m}{\partial \vec{n}} \mid_{\partial\Omega} = 0, \quad t \in (0, T) \\ \Phi_m \mid_{t=0} = \Phi_{0m}(x), \quad x \in \Omega \\ \dfrac{\partial \Phi_m}{\partial t} \mid_{t=0} = \Phi_{1m}(x), \quad x \in \Omega. \end{cases}$$

By posing $g(\Phi_m(t)) = 3(u^0)^2\Phi_m$, for all $e_j \in V$, we obtain:

$$\text{(5.26)} \quad \left(\frac{\partial^2 \Phi_m}{\partial t^2}, e_j\right) - (\Delta\Phi_m, e_j) + (g(\Phi_m(t)), e_j) - \left(\frac{\partial \Phi_m}{\partial t}, e_j\right) = (D_u, e_j).$$

By replacing $u_m(t) = \sum_{i=1}^{m} \Phi_{im}(t) e_i$ in the equation (5.23)

$$
\begin{aligned}
& \left(\frac{\partial^2\Phi_m}{\partial t^2}\sum_{i=1}^{m}\Phi_{im}(t)e_i, e_j\right) - (\Delta\Phi_m\sum_{i=1}^{m}\Phi_{im}(t)e_i, e_j) + (g(\Phi_m(t)), e_j) \\
- & \left(\frac{\partial\Phi_m}{\partial t}\sum_{i=1}^{m}\Phi_{im}(t)e_i, e_j\right) = (D_u, e_j) \\
& \sum_{i=1}^{m}\frac{\partial^2}{\partial t^2}(\Phi_{im}(t)e_i, e_j) + \sum_{i=1}^{m}\Phi_{im}(t)(-\Delta e_i, e_j) + (g(\Phi_m(t))), e_j) \\
- & \sum_{i=1}^{m}\frac{\partial}{\partial t}(\Phi_{im}(t)e_i, e_j) = (D_u, e_j) \\
& \sum_{i=1}^{m}(e_i, e_j)\frac{\partial^2}{\partial t^2}\Phi_{im}(t) + \sum_{i=1}^{m}\Phi_{im}(t)(\lambda_i e_i, e_j) + (g(\Phi_m(t))), e_j) \\
- & \sum_{i=1}^{m}(e_i, e_j)\frac{\partial}{\partial t}\Phi_{im}(t) = (D_u, e_j) \\
& \sum_{i=1}^{m}(e_i, e_j)\frac{\partial^2}{\partial t^2}\Phi_{im}(t) + \sum_{i=1}^{m}(e_i, e_j)\lambda_i\Phi_{im}(t) + (g(\Phi_m(t))), e_j) \\
- & \sum_{i=1}^{m}(e_i, e_j)\frac{\partial}{\partial t}\Phi_{im}(t) = (D_u, e_j).
\end{aligned}
$$

The base $(e_i)_{1\le i\le m}$ being orthonormal, then: $(e_i, e_j) = \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$.

Then, we have:

$$
\frac{\partial^2}{\partial t^2}\Phi_{im}(t) + \lambda_i\Phi_{im}(t) + (g(\Phi_m(t), e_j) - \frac{\partial}{\partial t}\Phi_{im}(t) = (D_u, e_j).
$$

Let's pose: $\frac{\partial^2}{\partial t^2}\Phi_{im}(t) = \Phi_{im}^{''}$ and $\frac{\partial}{\partial t}\Phi_{im}(t) = \Phi_{im}^{'}$,

$$
\begin{aligned}
& \Phi_{im}^{''} + \lambda_i\Phi_{im}(t) + (g(\Phi_m(t), e_j) - \Phi_{im}^{'} = (D_u, e_j) \\
\Longrightarrow\ & \Phi_{im}^{''} + \lambda_i\Phi_{im}(t) + (g(\Phi_m(t), e_j) - \Phi_{im}^{'} - (D_u, e_j) = 0 \\
\Longrightarrow\ & \Phi_{im}^{''} + \lambda_i\Phi_{im}(t) - \Phi_{im}^{'} + (g(\Phi_m(t)) - D_u, e_j) = 0. \qquad (5.27)
\end{aligned}
$$

The equation (5.24) is a system of nonlinear differential equations completed by the following initial conditions:

$$\text{(5.28)} \qquad \Phi_m(0) = \Phi_{0m},\ \Phi_{0m} = \sum_{i=1}^{m} \alpha_{im} e_i \underset{m\to\infty}{\longrightarrow} \Phi_0 \quad \text{in} \quad H^1(\Omega)\cap L^2(\Omega)$$

$$\text{(5.29)} \qquad \Phi_m^{'}(0) = \Phi_{1m},\ \Phi_{1m} = \sum_{i=1}^{m} \beta_{im} e_i \underset{m\to\infty}{\longrightarrow} \Phi_1 \quad \text{in} \quad L^2(\Omega).$$

Equation (5.24) is a system of nonlinear differential equations written in matrix form as follows:

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} \Phi''_{1m} \\ \Phi''_{2m} \\ . \\ . \\ . \\ \Phi''_{mm} \end{pmatrix} - \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} \Phi'_{1m} \\ \Phi'_{2m} \\ . \\ . \\ . \\ \Phi'_{mm} \end{pmatrix}$$

$$+ \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 0 & \dots & \lambda_m \end{pmatrix} \begin{pmatrix} \Phi_{1m} \\ \Phi_{2m} \\ . \\ . \\ . \\ \Phi_{mm} \end{pmatrix} + \begin{pmatrix} (g(\Phi_m(t)) - D_u, e_1) \\ (g(\Phi_m(t)) - D_u, e_2) \\ . \\ . \\ . \\ (g(\Phi_m(t)) - D_u, e_m) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}$$

$$I_m X''(t) - I_m X'(t) + A_m X(t) + B_m = 0,$$

where

$$I_m = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 0 & \dots & 1 \end{pmatrix},\ A_m = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 0 & \dots & \lambda_m \end{pmatrix},\ B_m = \begin{pmatrix} (g(\Phi_m(t)) - D_u, e_1) \\ (g(\Phi_m(t)) - D_u, e_2) \\ . \\ . \\ . \\ (g(\Phi_m(t)) - D_u, e_m) \end{pmatrix}$$

$$X(t) = \begin{pmatrix} \Phi_{1m} \\ \Phi_{2m} \\ . \\ . \\ . \\ \Phi_{mm} \end{pmatrix}, X''(t) = \begin{pmatrix} \Phi''_{1m} \\ \Phi''_{2m} \\ . \\ . \\ . \\ \Phi''_{mm} \end{pmatrix} \text{ and } X'(t) = \begin{pmatrix} \Phi'_{1m} \\ \Phi'_{2m} \\ . \\ . \\ . \\ \Phi'_{mm} \end{pmatrix}.$$

Since det A_m=$1 \neq 0$, the matrix A_m is invertible.
So the system has a unique solution defined in the interval $[0, t_m]$.

Step 2: A priori estimates. Multiply the first equation of (5.22) of index j by $Phi'_{jm}(t)$ and sum to j. It follows that

$$\left(\frac{\partial^2 \Phi_m}{\partial t^2}, \Phi'_{jm}(t)\right) - (\Delta \Phi_m, \Phi'_{jm}(t)) + (g(\Phi_m(t)), \Phi'_{jm}(t)) - \left(\frac{\partial \Phi_m}{\partial t}, \Phi'_{jm}(t)\right) = (D_u, \Phi'_{jm}(t)) \tag{5.30}$$

$$\left(\frac{\partial^2 \Phi_m}{\partial t^2}, \frac{\partial \Phi_m}{\partial t}\right) - \left(\Delta \Phi_m, \frac{\partial \Phi_m}{\partial t}\right) + \left(\Phi_m^2, \frac{\partial \Phi_m}{\partial t}\right) - \left(\frac{\partial \Phi_m}{\partial t}, \frac{\partial \Phi_m}{\partial t}\right) = \left(D_u, \frac{\partial \Phi_m}{\partial t}\right). \tag{5.31}$$

From Green's formula and using Hölder 's and Young' s inequalities, we obtain:

$$\frac{1}{2}\left[\frac{\partial}{\partial t}\left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} + \frac{\partial}{\partial t}\|\nabla \Phi_m\|^2_{L^2(\Omega)} + 3\frac{\partial}{\partial t}\|(u^0)^2 \Phi_m\|^2_{L^2(\Omega)}\right] - \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)}$$
$$\leq \frac{1}{2}\left[\|D_u\|^2_{L^2(\Omega)} + \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)}\right]$$
$$\Rightarrow \quad \frac{1}{2}\left[\frac{\partial}{\partial t}\left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} + \frac{\partial}{\partial t}\|\nabla \Phi_m\|^2_{L^2(\Omega)} + 3\frac{\partial}{\partial t}\|(u^0)^2 \Phi_m\|^2_{L^2(\Omega)}\right] - \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)}$$
$$\leq \frac{1}{2}\left[\|D_u\|^2_{L^2(\Omega)} + \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)}\right].$$

Then, we have:

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} + \|\nabla\Phi_m\|^2_{L^2(\Omega)} + \frac{3}{2}|u^0|^2\|\Phi_m\|^2_{L^2(\Omega)}\right] - 2\left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)}\\ \leq\ & \|p_m(D_u)\|^2_{L^2(\Omega)} + \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)}\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} + \|\nabla\Phi_m\|^2_{L^2(\Omega)} + \frac{3}{2}|u^0|^2\|\Phi_m\|^2_{L^2(\Omega)}\right] - 2\left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)}\\ \leq\ & \|D_u\|^2_{L^2(\Omega)} + \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)}\end{aligned}$$

because $\|p_m(D_u)\|^2_{L^2(\Omega)} \leq \|D_u\|^2_{L^2(\Omega)}$. Let's integrate on $(0,T)$:

$$\begin{aligned}&\int_0^t \frac{\partial}{\partial s}\left[\left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} + \|\nabla\Phi_m\|^2_{L^2(\Omega)} + \frac{3}{2}|u^0|^2\|\Phi_m\|^2_{L^2(\Omega)}\right]ds - 2\int_0^t \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} ds\\ \leq\ & \int_0^t \left(\|D_u\|^2_{L^2(\Omega)} + \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)}\right)ds\end{aligned}$$

we obtain:

$$\begin{aligned}&\left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} + \|\nabla\Phi_m\|^2_{L^2(\Omega)} + \frac{3}{2}|u^0|^2\|\Phi_m\|^2_{L^2(\Omega)} - 2\int_0^t \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} ds\\ -\ & \left(\left\|\frac{\partial \Phi_{0m}}{\partial t}\right\|^2_{L^2(\Omega)} + \|\nabla\Phi_{0m}\|^2_{L^2(\Omega)} + \frac{3}{2}|u^0|^2\|\Phi_m\|^2_{L^2(\Omega)}\right)\\ \leq\ & \int_0^t \left(\|D_u\|^2_{L^2(\Omega)} + \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)}\right)ds.\end{aligned}$$

With $\dfrac{\partial \Phi_{0m}}{\partial t} = \Phi_{1m}$, we have:

$$\begin{aligned}&\left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} + \|\nabla\Phi_m\|^2_{L^2(\Omega)} + \frac{3}{2}|u^0|^2\|\Phi_m\|^2_{L^2(\Omega)} - 2\int_0^t \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} ds\\ -\ & \left(\|\Phi_{1m}\|^2_{L^2(\Omega)} + \|\nabla\Phi_{0m}\|^2_{L^2(\Omega)} + \frac{3}{2}|u^0|^2\|\Phi_m\|^2_{L^2(\Omega)}\right)\end{aligned}$$

$$\leq \int_0^t \left(\|D_u\|^2_{L^2(\Omega)} + \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)}\right) ds$$

$$\left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} + \|\nabla \Phi_m\|^2_{L^2(\Omega)} + \frac{3}{2}|u^0|^2 \|\Phi_m\|^2_{L^2(\Omega)} - 2\int_0^t \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} ds$$

$$\leq \|\Phi_{1m}\|^2_{L^2(\Omega)} + \|\nabla \Phi_{0m}\|^2_{L^2(\Omega)} + \frac{3}{2}|u^0|^2 \|\Phi_m\|^2_{L^2(\Omega)} + \int_0^t \left(\|D_u\|^2_{L^2(\Omega)} + \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)}\right) ds. \tag{5.32}$$

Let's add member by member inequality (5.28) by $\|\Phi_m\|^2_{L^2(\Omega)}$, we obtain:

$$\left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi_m\|^2_{L^2(\Omega)} + \|\nabla \Phi_m\|^2_{L^2(\Omega)} + \frac{3}{2}|u^0|^2 \|\Phi_m\|^2_{L^2(\Omega)} - 2\int_0^t \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} ds$$

$$\leq \|\Phi_m\|^2_{L^2(\Omega)} + \|\Phi_{1m}\|^2_{L^2(\Omega)} + \|\nabla \Phi_{0m}\|^2_{L^2(\Omega)} + \frac{3}{2}|u^0|^2 \|\Phi_m\|^2_{L^2(\Omega)} + \int_0^t \left(\|D_u\|^2_{L^2(\Omega)} + \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)}\right) ds.$$

This gives us:

$$\left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi_m\|^2_{H^1(\Omega)} + \frac{3}{2}|u^0|^2 \|\Phi_m\|^2_{L^2(\Omega)} - 2\int_0^t \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} ds$$

$$\leq \|\Phi_m\|^2_{L^2(\Omega)} + \|\Phi_{1m}\|^2_{L^2(\Omega)} + \|\nabla \Phi_{0m}\|^2_{L^2(\Omega)} + \frac{3}{2}|u^0|^2 \|\Phi_m\|^2_{L^2(\Omega)} + \int_0^t \left(\|D_u\|^2_{L^2(\Omega)} + \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)}\right) ds. \tag{5.33}$$

For all $t \in [0, T]$, we have:

$$\left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi_m\|^2_{H^1(\Omega)} + \frac{3}{2}|u^0|^2 \|\Phi_m\|^2_{L^2(\Omega)} - 2\int_0^T \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} ds$$

$$\leq \|\Phi_m\|^2_{L^2(\Omega)} + \|\Phi_{1m}\|^2_{L^2(\Omega)} + \|\nabla \Phi_{0m}\|^2_{L^2(\Omega)} + \frac{3}{2}|u^0|^2 \|\Phi_m\|^2_{L^2(\Omega)} + \int_0^T \left(\|D_u\|^2_{L^2(\Omega)} + \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)}\right) dt. \tag{5.34}$$

Let's put: $\Phi_m(t) = \Phi_m(x, t)$. Then,

$$\Phi_m(t) = \int_0^t \frac{\partial \Phi_m}{\partial s}(s)ds + \Phi_m(0) = \int_0^t \frac{\partial \Phi_m}{\partial s}(s)ds + \Phi_{0m}$$

because $\Phi_m(0) = \Phi_{0m}$. Also,

$$\begin{aligned}
|\Phi_m(t)| &= \left|\int_0^t \frac{\partial \Phi_m}{\partial s}(s)ds + u_{0m}\right| \Longleftrightarrow |\Phi_m(t)|^2 = \left|\int_0^t \frac{\partial \Phi_m}{\partial s}(s)ds + \Phi_{0m}\right|^2 \\
|\Phi_m(t)|^2 &\leq \left|\int_0^t \frac{\partial \Phi_m}{\partial s}(s)ds + \Phi_{0m}\right|^2 + \left|\int_0^t \frac{\partial \Phi_m}{\partial s}(s)ds - \Phi_{0m}\right|^2 \\
&\leq 2\left[\left|\int_0^t \frac{\partial \Phi_m}{\partial s}(s)ds\right|^2 + |\Phi_{0m}^2|\right] \\
|\Phi_m(t)|^2 &\leq 2\left[\left(\int_0^t \left|\frac{\partial \Phi_m}{\partial s}(s)\right|ds\right)^2 + |\Phi_{0m}|^2\right].
\end{aligned}$$

According to Hölder:

$$\begin{aligned}
\int_0^t \left|\frac{\partial \Phi_m}{\partial s}(x,t)\right|ds &\leq \left(\int_0^t |1|ds\right)^{1/2}.\left(\int_0^t \left|\frac{\partial \Phi_m}{\partial s}(s)\right|^2 ds\right)^{1/2} \\
\left(\int_0^t |\frac{\partial \Phi_m}{\partial s}(x,t)|ds\right)^2 &\leq \left(\int_0^t ds\right).\left(\int_0^t \left|\frac{\partial \Phi_m}{\partial s}(s)\right|^2 ds\right) \leq t\int_0^t \left|\frac{\partial \Phi_m}{\partial s}(s)\right|^2 ds.
\end{aligned}$$

We obtain:

$$|\Phi_m(t)|^2 \leq 2\left[t\int_0^t \left|\frac{\partial \Phi_m}{\partial s}(s)\right|^2 ds\right] + |\Phi_{0m}|^2 \leq 2t\int_0^t \left|\frac{\partial \Phi_m}{\partial s}(s)\right|^2 ds + 2|\Phi_{0m}|^2.$$

By taking $t \in [0,1] \subset [0,T]$, then $1-t \geq 0$. We can add the term $2(1-t)\int_0^t \left|\frac{\partial \Phi_m}{\partial s}(s)\right|^2 ds$ to the second member:

$$|\Phi_m(t)|^2 \leq 2t\int_0^t \left|\frac{\partial \Phi_m}{\partial s}(s)\right|^2 ds + 2|\Phi_{0m}|^2 + 2(1-t)\int_0^t \left|\frac{\partial \Phi_m}{\partial s}(s)\right|^2 ds.$$

Finally we obtain:

$$|\Phi_m(t)|^2 \leq 2\left[|\Phi_{0m}|^2 + \int_0^t \left|\frac{\partial \Phi_m}{\partial s}(s)\right|^2 ds\right],$$

Integrate on Ω,

$$\begin{aligned}\int_\Omega |\Phi_m(t)|^2 dx &\leq 2\int_\Omega \left[|\Phi_{0m}|^2 + \int_0^t \left| \frac{\partial \Phi_m}{\partial s}(s) \right|^2 ds \right] dx \\ &\leq 2\left[\int_\Omega |\Phi_{0m}|^2 dx + \int_0^t \left(\int_\Omega \left| \frac{\partial \Phi_m}{\partial s} \right|^2 dx \right) ds \right]\end{aligned}$$

$\forall t \in [0, T]$,

$$\int_\Omega |\Phi_m(t)|^2 dx \leq 2\left[\int_\Omega |\Phi_{0m}|^2 dx + \int_0^T \left(\int_\Omega \left| \frac{\partial \Phi_m}{\partial t} \right|^2 dx \right) dt \right]$$

$$\|\Phi_m(t)\|^2_{L^2(\Omega} \leq 2\left[\|\Phi_{0m}\|^2_{L^2(\Omega)} + \int_0^T \left\| \frac{\partial \Phi_m}{\partial t} \right\|^2_{L^2(\Omega)} dt \right]. \tag{5.35}$$

By using (5.31), (5.30) becomes:

$$\begin{aligned}&\left\| \frac{\partial \Phi_m}{\partial t} \right\|^2_{L^2(\Omega)} + \|\Phi_m\|^2_{H^1(\Omega)} + \frac{3}{2}|u^0|^2 \|\Phi_m\|^2_{L^2(\Omega)} - 2\int_0^T \left\| \frac{\partial \Phi_m}{\partial t} \right\|^2_{L^2(\Omega)} dt \\ \leq\ & 2\|\Phi_{0m}\|^2_{L^2(\Omega)} + 3\int_0^T \left\| \frac{\partial \Phi_m}{\partial t} \right\|^2_{L^2(\Omega)} dt + \|\Phi_{1m}\|^2_{L^2(\Omega)} + \|\nabla \Phi_{0m}\|^2_{L^2(\Omega)} \\ +\ & \frac{3}{2}|u^0|^2 \|\Phi_m\|^2_{L^2(\Omega)} + \int_0^T \|D_u\|^2_{L^2(\Omega)} dt.\end{aligned}$$

By adding the terme $\|\nabla \Phi_{0m}\|^2_{L^2(\Omega)}$ to the second member of (5.32), we obtain:

$$\begin{aligned}&\left\| \frac{\partial \Phi_m}{\partial t} \right\|^2_{L^2(\Omega)} + \|\Phi_m\|^2_{H^1(\Omega)} + \frac{3}{2}|u^0|^2 \|\Phi_m\|^2_{L^2(\Omega)} \leq 2\|\Phi_{0m}\|^2_{H^1(\Omega)} \\ +\ & 5\int_0^T \left\| \frac{\partial \Phi_m}{\partial t} \right\|^2_{L^2(\Omega)} dt + \|\Phi_{1m}\|^2_{L^2(\Omega)} + \frac{3}{2}|u^0|^2 \|\Phi_m\|^2_{L^2(\Omega)} + \int_0^T \|D_u\|^2_{L^2(\Omega)} dt.\end{aligned} \tag{5.36}$$

By adding the term $\int_0^T \|\Phi_m\|^2_{H^1(\Omega)} dt$ to the second member of (5.33), we obtain:

$$\begin{aligned}&\left\| \frac{\partial \Phi_m}{\partial t} \right\|^2_{L^2(\Omega)} + \|\Phi_m\|^2_{H^1(\Omega)} + \frac{3}{2}|u^0|^2 \|\Phi_m\|^2_{L^2(\Omega)} \\ \leq\ & 2\|\Phi_{0m}\|^2_{H^1(\Omega)} + 5\int_0^T \left\| \frac{\partial \Phi_m}{\partial t} \right\|^2_{L^2(\Omega)} dt + \|\Phi_{1m}\|^2_{L^2(\Omega)}\end{aligned}$$

$$+\frac{3}{2}|u^0|^2\|\Phi_m\|^2_{L^2(\Omega)} + \int_0^T \|D_u\|^2_{L^2(\Omega)}dt + \int_0^T \|\Phi_m\|^2_{H^1(\Omega)}dt.$$

Let's put: $k_5 = 2\|\Phi_{0m}\|^2_{H^1(\Omega)} + \|\phi_{1m}\|^2_{L^2(\Omega)} + \frac{3}{2}|u^0|^2\|\Phi_m\|^2_{L^2(\Omega)} + \int_0^T \|D_u\|^2_{L^2(\Omega)}dt$. Then (5.34) becomes:

$$\begin{aligned} &\left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi_m\|^2_{H^1(\Omega)} + \frac{3}{2}|u^0|^2\|\Phi_m\|^2_{L^2(\Omega)} \\ \leq\ & k_5 + 5\int_0^T \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} dt + \int_0^T \|\Phi_m\|^2_{H^1(\Omega)}dt \\ &\left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi_m\|^2_{H^1(\Omega)} \\ \leq\ & k_5 + 5\left(\int_0^T \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} dt + \int_0^T \|\Phi_m\|^2_{H^1(\Omega)}dt\right). \end{aligned}$$

Let's put: $I_m(t) = \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi_m\|^2_{H^1(\Omega)}$.

This implies

$$I_m(t) \leq k_5 + \int_0^T 5I_m(s)ds.$$

According to Gronwall, we have:

$$I_m(t) \leq k_5 e^{\int_0^T 5ds} \quad \Longrightarrow \quad I_m(t) \leq k_5 e^{5T} \quad \Longrightarrow \quad I_m(t) \leq c_4$$

with $c_4 = k_5 e^{5T}$. Then we obtain:

$$\left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi_m\|^2_{H^1(\Omega)} \leq c_4, \quad \text{and} \quad \begin{cases} \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} & \leq c_5 \\ \|\Phi_m\|^2_{H^1(\Omega)} & \leq c_6 \end{cases},$$

with $c_4 = c_5 + c_6$, and

(5.37)
$$\Longrightarrow \begin{cases} \frac{\partial \Phi_m}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \\ u_m \in L^\infty(0,T;H^1(\Omega)) \end{cases}.$$

Let's add the term $\int_0^T \|\Phi_m\|^2_{L^2(\Omega)}dt$ to the second member of (5.33), we obtain:

$$\begin{aligned} & \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi_m\|^2_{H^1(\Omega)} + \frac{3}{2}|u^0|^2\|\Phi_m\|^2_{L^2(\Omega)} \\ \leq & \ 2\|\Phi_{0m}\|^2_{H^1(\Omega)} + 5\int_0^T \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} dt \\ + & \ \|\Phi_{1m}\|^2_{L^2(\Omega)} + \frac{3}{2}|u^0|^2\|\Phi_m\|^2_{L^2(\Omega)} \\ + & \int_0^T \|D_u\|^2_{L^2(\Omega)}dt + \int_0^T \|\Phi_m\|^2_{L^2(\Omega)}dt. \end{aligned} \tag{5.38}$$

Let's put: $k_6 = 2\|\Phi_{0m}\|^2_{H^1(\Omega)} + 5\int_0^T \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} dt + \|\Phi_{1m}\|^2_{L^2(\Omega)} + \frac{3}{2}|u^0|^2\|\Phi_m\|^2_{L^2(\Omega)} + \int_0^T \|D_u\|^2_{L^2(\Omega)}dt$. Then (5.37) becomes:

$$\left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi_m\|^2_{H^1(\Omega)} + \frac{3}{2}|u^0|^2\|\Phi_m\|^2_{L^2(\Omega)} \leq k_6 + \int_0^T \|\Phi_m\|^2_{L^2(\Omega)}dt$$

$$\begin{aligned} \frac{3}{2}|u^0|^2\|\Phi_m\|^2_{L^2(\Omega)} & \leq \left\|\frac{\partial \Phi_m}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi_m\|^2_{H^1(\Omega)} + \frac{3}{2}|u^0|^2\|\Phi_m\|^2_{L^2(\Omega)} \\ & \leq k_6 + \int_0^T \|\Phi_m\|^2_{L^2(\Omega)}dt \end{aligned}$$

$$\begin{aligned} \Longrightarrow \frac{3}{2}|u^0|^2\|\Phi_m\|^2_{L^2(\Omega)} & \leq k_6 + \int_0^T \|\Phi_m\|^2_{L^2(\Omega)}dt \\ \Longrightarrow \|\Phi_m\|^2_{L^2(\Omega)} & \leq k_7 + k_8\int_0^T \|\Phi_m\|^2_{L^2(\Omega)}dt \\ \Longrightarrow \|\Phi_m\|^2_{L^2(\Omega)} & \leq k_7 + k_8\int_0^T \|\Phi_m\|^2_{L^2(\Omega)}dt. \end{aligned} \tag{5.39}$$

According to Gronwall, we have:

$$\begin{aligned} \|\Phi_m\|^2_{L^2(\Omega)} & \leq k_7 e^{k_8\int_0^T dt} \\ \|\Phi_m\|^2_{L^2(\Omega)} & \leq k_7 e^{k_8(T)} \end{aligned}$$

$$\Longrightarrow \|\Phi_m\|^2_{L^2(\Omega)} \leq c_7 \text{ with } c_7 = k_7 e^{k_8(T)}$$

$$\underset{t\in[0,T]}{inf}\ \|\Phi_m\|^2_{L^2(\Omega)} \leq c_7$$

$$\Longrightarrow \Phi_m \in L^\infty(0,T,L^2(\Omega)). \tag{5.40}$$

According (5.36) and (5.39) we have:

$$\begin{cases} \dfrac{\partial \Phi_m}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \\ \Phi_m \in L^\infty(0,T;H^1(\Omega)\cap L^2(\Omega)). \end{cases} \tag{5.41}$$

When $m \to \infty$, Φ_m is still a bounded set of $L^\infty(0,T;H^1(\Omega)\cap L^2(\Omega))$ and $\dfrac{\partial \Phi_m}{\partial t}$ of $L^\infty(0,T;L^2(\Omega))$

Step 3: Passage to the limit. The sequence (Φ_n) is bounded in $L^\infty(0,T;H^1(\Omega)\cap L^2(\Omega))$, so it is bounded in $L^2(0,T;H^1(\Omega)\cap L^2(\Omega))$. Since $L^\infty(0,T;H^1(\Omega)\cap L^2(\Omega))$ is a separable Banach space, then there exists a sub-sequence (Φ_ξ) extracted from (Φ_n) such that:

$$\begin{cases} \Phi_\xi \overset{*}{\to} \Phi \quad \text{in} \quad L^\infty(0,T;H^1(\Omega)\cap L^2(\Omega)) \\ \dfrac{\partial \Phi_\xi}{\partial t} \overset{*}{\to} \dfrac{\partial \Phi}{\partial t} \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)) \end{cases}. \tag{5.42}$$

So,

$$\begin{cases} \Phi_\xi \to \Phi \quad \text{in} \quad L^\infty(0,T;H^1(\Omega)\cap L^2(\Omega)) \\ \dfrac{\partial \Phi_\xi}{\partial t} \to \dfrac{\partial \Phi}{\partial t} \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)) \end{cases}. \tag{5.43}$$

According to the problem (1.1)-(1.3),

$$\frac{\partial^2 \Phi}{\partial t^2} - \Delta\Phi + 3{(u^0)}^2\Phi - \frac{\partial \Phi}{\partial t} = D_u$$

$$\Longrightarrow \frac{\partial^2 \Phi}{\partial t^2} = \Delta u - 3{(u^0)}^2\Phi + \frac{\partial \Phi}{\partial t} + D_u.$$

As $\Delta : H^1(\Omega) \longrightarrow H^{-1}(\Omega) \Longrightarrow \Delta \in \mathcal{L}(H^1(\Omega), H^{-1}(\Omega))$ and $\Phi \in L^\infty(0,T;H^1(\Omega)) \Longrightarrow \Delta\Phi \in L^\infty(0,T;H^{-1}(\Omega))$, we will have $\Phi \in L^\infty(0,T;L^2(\Omega))$. This results in:

$$\frac{\partial^2 \Phi}{\partial t^2} \in L^\infty(0,T;H^{-1}(\Omega)) \cup L^\infty(0,T;L^2(\Omega)) \cup L^\infty(0,T;L^2(\Omega)) \cup L^2(0,T;L^2(\Omega))$$

and
$$\frac{\partial^2\Phi}{\partial t^2} \in L^\infty(0,T;H^{-1}(\Omega)\cup L^2(\Omega)\cup L^2(\Omega))\cup L^2(0,T;L^2(\Omega)).$$
In particular $\frac{\partial^2\Phi}{\partial t^2} \in L^2(0,T;H^{-1}(\Omega)\cup L^2(\Omega))$.

As Φ_m is bounded in $L^\infty(0,T;L^2(\Omega))$ then $\Phi_\xi \xrightarrow{*} w$ in $L^\infty(0,T;L^2(\Omega))$ then $\Phi_\xi \xrightarrow{*} w$ in $L^\infty(0,T;L^2(\Omega))$. □

Lemma 5.1. *Let's be Ω a bounded open of $\mathbb{R}^n_x$, h_ξ and h are two functions of $L^q(\Omega)$, $1<q<\infty$, such that:*
$$\|h_\xi\|_{L^q(\Omega)} \leq c, \quad h_\xi \longrightarrow h \quad \textit{p.p in} \quad \Omega.$$
Then $h_\xi \longrightarrow h$ in $L^q(\Omega)$ weak. We apply the following lemma:
$$h_\xi = \Phi_\xi, \quad q=2,$$
which means that
$$h_\xi \longrightarrow w \quad \textit{in} \quad L^\infty(0,T;L^2(\Omega)).$$
According to the lemma (5.1) $w=h=\Phi$. Hence
$$g(\Phi_\xi) = 3(u^0)^2\Phi_\xi \longrightarrow g(\Phi) = 3(u^0)^2\Phi.$$

We also have:
$$Du_m = p_m(D_u)$$
and
$$\|D_u m\|_{L^2(\Omega)} = \|p_m(D_u)\|_{L^2(\Omega)} \leq \|D_u\|_{L^2(\Omega)},$$
$D_u m$ is bounded in $L^2(Q_T)$. We can extract a sub-sequence Du_ξ of f_m such that $Du_\xi \longrightarrow f$ in $L^2(Q_T)$.

Now we show that Φ satisfies all the conditions of (5.22).

First note that for j fixed, we have:
$$\left(\frac{\partial^2\Phi_\xi}{\partial t^2}, e_j\right) - (\Delta\Phi_\xi, e_j) + (g(\Phi_\xi(t)), e_j) - \left(\frac{\partial\Phi_\xi}{\partial t}, e_j\right) = (Du_\xi, e_j). \tag{5.44}$$
Passing to the limit in (5.43), we obtain
$$\frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial t}, e_j\right) - (\Delta\Phi, e_j) - \left(\frac{\partial\Phi}{\partial t}, e_j\right) + (g(\Phi), e_j) = (f, e_j),$$

where V_m being dense in V. Then, for all $v \in V$, $e_j \longrightarrow v$ when $j \longrightarrow \infty$, we have:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial t}, v\right) - (\Delta\Phi, v) - \left(\frac{\partial \Phi}{\partial t}, v\right) + (g(\Phi), v) &= (f, v),\\
\left(\frac{\partial^2 \Phi}{\partial t^2}, v\right) - (\Delta\Phi, v) - \left(\frac{\partial \Phi}{\partial t}, v\right) + (g(\Phi), v) &= (f, v),\\
\left(\frac{\partial^2 \Phi}{\partial t^2} - \Delta\Phi - \frac{\partial \Phi}{\partial t} + g(\Phi), v\right) &= (f, v).
\end{aligned}$$

Hence,

$$\frac{\partial^2 \Phi}{\partial t^2} - \Delta\Phi - \frac{\partial \Phi}{\partial t} + g(\Phi) = f.$$

Let's verify $\frac{\partial \Phi}{\partial t}(x,0) = \Phi_1(x)$ and $\Phi(x,0) = \Phi_0(x)$.

– Let's verify $\frac{\partial \Phi}{\partial t}(x,0) = \Phi_1(x)$.

Let $\theta(x,t) \in L^\infty(0,T; H^1(\Omega) \cap L^4(\Omega))$ be a function whose trace for $t = T$ is $\theta(x,T) = 0$ and $\theta(x,0) \neq 0$.

As $(\Phi_{0m}(x))_{m\geq 1}$ is bounded in the space $H^1(\Omega) \cap L^4(\Omega)$, we can extract a sub-sequence $(\Phi_{0\xi}(x))_{\xi\geq 1}$ of $(\Phi_{0m}(x))_{m\geq 1}$ such that $\Phi_{0\xi}(x) \longrightarrow \Phi_0(x)$ in $H^1(\Omega) \cap L^4(\Omega)$. As the same $(\Phi_{1m}(x))_{m\geq 1}$ is bounded in $L^2(\Omega)$, we can extract a sub-sequence $(\Phi_{1\xi}(x))_{\xi\geq 1}$ of $(\Phi_{1m}(x))_{m\geq 1}$ such that $\Phi_{1\xi}(x) \longrightarrow \Phi_1(x)$ in $L^2(\Omega)$.

Multiply the first equation in (2.24) by $\theta(x,t)$ and integrate on $(0,T)$:

$$\begin{aligned}
&\int_0^T \frac{\partial^2 \Phi_m}{\partial t^2}\theta(x,t)dt - \int_0^T \Delta\Phi_m \theta(x,t)dt - \int_0^T \frac{\partial \Phi_m}{\partial t}\theta(x,t)dt + \int_0^T g(\Phi_m)\theta(x,t)dt\\
= &\int_0^T f_m \theta(x,t)dt.
\end{aligned}$$

Using integration by parts, we obtain:

$$\begin{aligned}
&\left[\frac{\partial \Phi_m}{\partial t}\theta(x,t)\right]_0^T - \int_0^T \frac{\partial \Phi_m}{\partial t}\theta'(x,t)dt - \int_0^T \Delta\Phi_m \theta(x,t)dt + \int_0^T g(\Phi_m)\theta(x,t)dt\\
- &\int_0^T \frac{\partial \Phi_m}{\partial t}\theta(x,t)dt = \int_0^T f_m \theta(x,t)dt,
\end{aligned}$$

$$\frac{\partial \Phi_m}{\partial t}\theta(x,T) - \frac{\partial \Phi_m}{\partial t}\theta(x,0) - \int_0^T \frac{\partial \Phi_m}{\partial t}\theta'(x,t)dt - \int_0^T \Delta\Phi_m \theta(x,t)dt$$
$$+ \int_0^T g(\Phi_m)\theta(x,t)dt - \int_0^T \frac{\partial \Phi_m}{\partial t}\theta(x,t)dt = \int_0^T f_m \theta(x,t)dt,$$

$$- \int_0^T \frac{\partial \Phi_m}{\partial t}\theta'(x,t)dt - \int_0^T \Delta\Phi_m \theta(x,t)dt + \int_0^T g(\Phi_m)\theta(x,t)dt$$
$$- \int_0^T \frac{\partial \Phi_m}{\partial t}\theta(x,t)dt = \int_0^T f_m \theta(x,t)dt + \frac{\partial \Phi_m}{\partial t}\theta(x,0),$$

with m being fixed and for $m = \xi$, we have:

$$- \int_0^T \frac{\partial \Phi_\xi}{\partial t}\theta'(x,t)dt - \int_0^T \Delta\Phi_\xi \theta(x,t)dt + \int_0^T g(\Phi_\xi)\theta(x,t)dt - \int_0^T \frac{\partial \Phi_\xi}{\partial t}\theta(x,t)dt$$
$$= \int_0^T f_\xi \theta(x,t)dt + \frac{\partial \Phi_\xi}{\partial t}\theta(x,0).$$

Passing to the limit, we have:

$$- \int_0^T \frac{\partial \Phi}{\partial t}\theta'(x,t)dt - \int_0^T \Delta\Phi \theta(x,t)dt + \int_0^T g(\Phi)\theta(x,t)dt$$
$$- \int_0^T \frac{\partial \Phi}{\partial t}\theta(x,t)dt = \int_0^T f.\theta(x,t)dt + \frac{\partial \Phi}{\partial t}\theta(x,0). \tag{5.45}$$

On the other hand, consider the equation:

$$\frac{\partial^2 \Phi}{\partial t^2} - \Delta\Phi - \frac{\partial \Phi}{\partial t} + g(\Phi) = f,$$

and by carrying out the same calculations, we obtain:

$$- \int_0^T \frac{\partial \Phi}{\partial t}\theta'(x,t)dt - \int_0^T \Delta\Phi \theta(x,t)dt + \int_0^T g(\Phi)\theta(x,t)dt$$
$$- \int_0^T \frac{\partial \Phi}{\partial t}\theta(x,t)dt = \int_0^T f.\theta(x,t)dt + \Phi_1(x)\theta(x,0). \tag{5.46}$$

Differentiating (5.45) and (5.44), we have:

$$\frac{\partial \Phi}{\partial t}(x,0)\theta(x,0) - \Phi_1(x)\theta(x,0) = 0,$$

$$\left(\frac{\partial \Phi}{\partial t}(x,0) - \Phi_1(x)\right)\theta(x,0) = 0,$$

$$\frac{\partial \Phi}{\partial t}(x,0) = \Phi_1(x), \quad \text{car} \quad \theta(x,0) \neq 0.$$

$\star$ Let's verify $\Phi(x,0) = \Phi_0(x)$.

In the same way, multiply the first equation in (5.22) by $\theta(x,t)$ and integrate by parts over $(0,T)$

$$\int_0^T \frac{\partial^2 \Phi_m}{\partial t^2}\theta(x,t)dt - \int_0^T \Delta\Phi_m \theta(x,t)dt - \int_0^T \frac{\partial \Phi_m}{\partial t}\theta(x,t)dt + \int_0^T g(\Phi_m)\theta(x,t)dt$$
$$= \int_0^T f_m \theta(x,t)dt.$$

Using integration by parts, we obtain:

$$\int_0^T \frac{\partial^2 \Phi_m}{\partial t^2}\theta(x,t)dt - \int_0^T \Delta\Phi_m \theta(x,t)dt - \Phi_m\theta(x,T) + u_m(x,0)\theta(x,0)$$
$$+ \int_0^T \Phi_m \theta'(x,t)dt + \int_0^T g(\Phi_m)\theta(x,t)dt = \int_0^T f_m\theta(x,t)dt$$

$$\int_0^T \frac{\partial^2 \Phi_m}{\partial t^2}\theta(x,t)dt - \int_0^T \Delta\Phi_m \theta(x,t)dt + \int_0^T \Phi_m \theta'(x,t)dt$$
$$+ \int_0^T g(\Phi_m)\theta(x,t)dt = \int_0^T f_m\theta(x,t)dt - \Phi_m(x,0)\theta(x,0),$$

with m being fixed and $m = \xi$:

$$\int_0^T \frac{\partial^2 \Phi_\xi}{\partial t^2}\theta(x,t)dt - \int_0^T \Delta\Phi_\xi \theta(x,t)dt + \int_0^T \Phi_\xi \theta'(x,t)dt$$
$$+ \int_0^T g(\Phi_\xi)\theta(x,t)dt = \int_0^T f_\xi\theta(x,t)dt - \Phi_\xi(x,0)\theta(x,0).$$

Passing to the limit, we have:

$$\int_0^T \frac{\partial^2 \Phi}{\partial t^2}\theta(x,t)dt - \int_0^T \Delta\Phi \theta(x,t)dt + \int_0^T \Phi \theta'(x,t)dt$$
$$+ \int_0^T g(\Phi)\theta(x,t)dt = \int_0^T f\theta(x,t)dt + \Phi(x,0)\theta(x,0). \tag{5.47}$$

On the other hand, considering the equation:

$$\frac{\partial^2\Phi}{\partial t^2} - \Delta\Phi - \frac{\partial\Phi}{\partial t} + g(\Phi) = f,$$

and using the same calculations, we obtain:

$$\int_0^T \frac{\partial^2\Phi}{\partial t^2}\theta(x,t)dt - \int_0^T \Delta\Phi\theta(x,t)dt + \int_0^T \Phi.\theta'(x,t)dt$$

$$(5.48) \qquad + \int_0^T g(\Phi)\theta(x,t)dt = \int_0^T f\theta(x,t)dt - \Phi_0(x)\theta(x,0).$$

Differentiating (5.47) and (5.46), we obtain

$$\begin{aligned} \Phi(x,0)\theta(x,0) - \Phi_0(x)\theta(x,0) &= 0 \\ (\Phi(x,0) - \Phi_0(x))\theta(x,0) &= 0 \\ \Phi(x,0) = \Phi_0(x), \quad \text{because} \quad \theta(x,0) &\neq 0. \end{aligned}$$

Hence, Φ is the solution of problem (1.1)-(1.3).

6. Uniqueness of the solution

Theorem 6.1. *Let be Φ and $\Psi \in L^\infty(0,T;H^1(\Omega) \cap L^2(\Omega))$ two solutions of the problem (1.1)-(1.3) under the assumptions of Theorem (5.1). Then the solution of the problem obtained in the theorem is unique.*

Proof. Knowing that Φ and Ψ are two solutions of the problem, then let put $\Upsilon = \Phi - \Psi$. We have:

$$(6.1) \qquad \begin{cases} \dfrac{\partial^2\Upsilon}{\partial t^2} - \Delta\Upsilon - \dfrac{\partial\Upsilon}{\partial t} + g(\Phi) - g(\Psi) = 0 \\ \dfrac{\partial\Upsilon}{\partial\vec{n}} |_{\partial\Omega} = 0, \quad t \in (0,T) \\ \Upsilon_0(x) = \Phi_0(x) - \Psi_0(x), \quad x \in \Omega \\ \Upsilon_1(x) = \Phi_1(x) - \Psi_1(x), \quad x \in \Omega. \end{cases}$$

Let be $s \in (0,T)$, consider the auxiliary function defined on $\Omega\times]0,T[$:

$$(6.2) \qquad \lambda(x,t) = \begin{cases} -\displaystyle\int_t^s \Upsilon(\sigma)d\sigma & \text{if} \quad 0 < t \leq s \\ 0 & \text{if} \quad s < t \leq T \end{cases}.$$

Such as $\dfrac{\partial\lambda(x,t)}{\partial t} = \Upsilon(x,t) = \Upsilon(t)$,

$$\Upsilon_1(x,t) = \int_0^t \Upsilon(\sigma)d\sigma, \quad \text{so that} \quad \lambda(x,t) = \Upsilon_1(x,t) - \Upsilon_1(x,s) \quad \text{if} \quad t \leq s.$$

Multiply the first equation in (3.1) by $\lambda(x,t) = \lambda(t)$ and integrate on $]0,s[$, we have:

$$\int_0^s \frac{\partial^2\Upsilon}{\partial t^2}\lambda(t)dt - \int_0^s \Delta\Upsilon\lambda(t)dt - \int_0^s \frac{\partial\Upsilon}{\partial t}\lambda(t)dt + \int_0^s (g(\Phi) - g(\Psi))\phi(t)dt = 0$$

and

$$\int_0^s \frac{\partial^2\Upsilon}{\partial t^2}\lambda(t)dt - \int_0^s \Delta\Upsilon\lambda(t)dt - \int_0^s \frac{\partial\Upsilon}{\partial t}\lambda(t)dt = \int_0^s (g(\Psi) - g(\Phi))\lambda(t)dt. \tag{6.3}$$

By integrating by parts the first and last terms of the first member of (6.3), we obtain:

$$\begin{aligned} &\frac{\partial\Upsilon}{\partial t}(x,s)\lambda(x,s) - \frac{\partial\Upsilon}{\partial t}(x,0)\lambda(x,0) - \int_0^s \frac{\partial\Upsilon}{\partial t}\frac{\partial\lambda}{\partial t}dt - \int_0^s \Delta\Upsilon\lambda(t)dt \\ &\Upsilon(x,s)\lambda(x,s) + \Upsilon(x,0)\lambda(x,0) + \int_0^s \Upsilon(t)\frac{\partial\lambda(t)}{\partial t}dt = \int_0^s (g(\Psi) - g(\Phi))\phi(t)dt \end{aligned} \tag{6.4}$$

or $\lambda(x,s) = -\displaystyle\int_s^s \Upsilon(\sigma)d\sigma = 0$:

$$\begin{aligned} &- \frac{\partial\Upsilon}{\partial t}(x,0)\lambda(x,0) - \int_0^s \frac{\partial\Upsilon}{\partial t}\frac{\partial\lambda}{\partial t}dt + \int_0^s \Delta\Upsilon\lambda(t)dt + \Upsilon(x,0)\lambda(x,0) \\ &+ \int_0^s \Upsilon(t)\frac{\partial\lambda(t)}{\partial t}dt = \int_0^s (g(\Psi) - g(\Phi))\lambda(t)dt. \end{aligned}$$

Integrate on Ω, we obtain:

$$\begin{aligned} &\int_\Omega \frac{\partial\Upsilon}{\partial t}(x,0)\lambda(x,0)dx - \int_\Omega\int_0^s \frac{\partial\Upsilon}{\partial t}\frac{\partial\lambda}{\partial t}dtdx - \int_\Omega\int_0^s \Delta\Upsilon\lambda(t)dtdx \\ + &\int_\Omega \Upsilon(x,0)\lambda(x,0)dx + \int_\Omega\int_0^s \Upsilon(t)\frac{\partial\lambda(t)}{\partial t}dtdx = \int_\Omega\int_0^s (g(\Psi) - g(\Phi))\lambda(t)dtdx. \end{aligned} \tag{6.5}$$

On the one hand:

$$\int_\Omega \frac{\partial\Upsilon}{\partial t}(x,0)\lambda(x,0)dx = \int_\Omega \Upsilon_1(x)\lambda(x,0)dx = \int_\Omega (\Phi_1(x) - \Psi_1(x))\lambda(x,0)dx,$$

with

$$\lambda(x,0) = \Upsilon_1(x,0) - \Upsilon_1(x,s); \quad \Upsilon_1(x,s) = \int_0^s \Upsilon(\sigma)d\sigma$$

and

$$\Upsilon_1(x,0) = \int_0^0 \Upsilon(\sigma)d\sigma = 0,$$

implying

$$\int_\Omega \frac{\partial \Upsilon}{\partial t}(x,0)\lambda(x,0)dx = \int_\Omega (\Phi_1(x) - \Psi_1(x))\Upsilon_1(x,s)dx. \tag{6.6}$$

The same work is repeated for $\int_\Omega \Upsilon(x,0)\lambda(x,0)dx$. We obtain:

$$\int_\Omega \Upsilon(x,0)\lambda(x,0)dx = \int_\Omega (\Phi_0(x) - \Psi_0(x))\Upsilon_1(x,s)dx. \tag{6.7}$$

On the other hand:

$$\int_\Omega \int_0^s \frac{\partial \Upsilon}{\partial t}\frac{\partial \lambda}{\partial t}dtdx = \int_\Omega \int_0^s \frac{\partial \Upsilon}{\partial t}\frac{\partial}{\partial t}\left(-\int_t^s \Upsilon(x,\sigma)d\sigma\right)dtdx$$

$$\Longrightarrow \quad \int_\Omega \int_0^s \frac{\partial \Upsilon}{\partial t}\frac{\partial \lambda}{\partial t}dtdx = \int_\Omega \int_0^s \frac{\partial \Upsilon}{\partial t}\left(-\frac{\partial}{\partial t}\int_t^s \Upsilon(\sigma)d\sigma\right)dtdx$$

$$\Longrightarrow \quad \int_\Omega \int_0^s \frac{\partial \Upsilon}{\partial t}\frac{\partial \lambda}{\partial t}dtdx = \frac{1}{2}\int_\Omega \left[\Upsilon^2(x,t)\right]_0^s dx = \frac{1}{2}\int_\Omega \left[\Upsilon^2(x,s) - \Upsilon^2(x,0)\right]dx,$$

with $\Upsilon(x,0) = \Upsilon_0(x) = \Phi_0(x) - \Psi_0(x)$. Then,

$$\int_\Omega \int_0^s \frac{\partial \Upsilon}{\partial t}\frac{\partial \lambda}{\partial t}dtdx = \frac{1}{2}\int_\Omega \left[\Upsilon^2(x,s) - (\Phi_0(x) - \Psi_0(x))^2\right]dx$$

$$\Longrightarrow \quad \int_\Omega \int_0^s \frac{\partial \Upsilon}{\partial t}\frac{\partial \lambda}{\partial t}dtdx = \frac{1}{2}\|\Upsilon(x,s)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\Phi_0(x) - \Psi_0(x)\|^2_{L^2(\Omega)}. \tag{6.8}$$

The work is reproduced for even $\int_\Omega \int_0^s \Upsilon(t)\frac{\partial \lambda(t)}{\partial t}dtdx$. We obtain:

$$\int_\Omega \int_0^s \Upsilon(t)\frac{\partial \lambda(t)}{\partial t}dtdx = \int_\Omega \int_0^s \Upsilon^2 dtdx = \int_0^s \|\Upsilon\|^2_{L^2(\Omega)}dt. \tag{6.9}$$

According to Green's formula:

$$\int_\Omega \int_0^s \Delta\Upsilon\lambda(t)dtdx = \int_\Omega \int_0^s \nabla\Upsilon.\nabla\lambda dtdx = \int_\Omega \int_0^s \nabla\Upsilon.\nabla\left(-\int_t^s \Upsilon(\sigma)d\sigma\right)dtdx$$

$$\Longrightarrow \int_\Omega \int_0^s \Delta\Upsilon\lambda(t)dtdx = \int_\Omega \int_0^s \nabla\Upsilon.\left(-\int_t^s \nabla\Upsilon(\sigma)d\sigma\right)dtdx$$

We know that:

$$\begin{aligned}\lambda(x,t) &= -\int_t^s \Upsilon(\sigma)d\sigma \quad \text{if} \quad t \leq s; \quad \frac{\partial\lambda(x,t)}{\partial t} \\ &= \frac{\partial}{\partial t}\left(-\int_t^s \Upsilon(\sigma)d\sigma\right) = \Upsilon(x,t) = \Upsilon(t)\end{aligned}$$

implying

$$\begin{aligned}\nabla\Upsilon(t) &= \nabla\left[\frac{\partial}{\partial t}\left(-\int_t^s \Upsilon(\sigma)d\sigma\right)\right] = \frac{\partial}{\partial t}\left(-\nabla\int_t^s \Upsilon(\sigma)d\sigma\right) \\ &= \frac{\partial}{\partial t}\left(-\int_t^s \nabla\Upsilon(\sigma)d\sigma\right). \qquad (6.10)\end{aligned}$$

We finally have

$$\begin{aligned}\int_\Omega\int_0^s \nabla\Upsilon\nabla\lambda dtdx &= \int_\Omega\int_0^s \frac{\partial}{\partial t}\left(-\int_t^s \nabla\Upsilon(\sigma)d\sigma\right)\left(-\int_t^s \nabla\Upsilon(\sigma)d\sigma\right)dtdx \\ &= \frac{1}{2}\int_\Omega\int_0^s \frac{\partial}{\partial t}\left(\int_t^s \nabla\Upsilon(\sigma)d\sigma\right)^2 dtdx = -\frac{1}{2}\int_\Omega |\nabla\Upsilon_1(x,s)|^2 dx,\end{aligned}$$

implying

$$\int_\Omega\int_0^s \nabla\Upsilon\nabla\lambda dtdx = -\frac{1}{2}\|\nabla\Upsilon_1(x,s)\|^2_{L^2(\Omega)}. \qquad (6.11)$$

Using (6.6), (6.7), (6.8), (6.9) and (6.10), then (6.5) becomes:

$$\begin{aligned}&- \int_\Omega (\Phi_1(x)-\Psi_1(x))\Upsilon_1(x)dx - \frac{1}{2}\|\Upsilon(x,s)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Phi_0(x)-\Psi_0(x)\|^2_{L^2(\Omega)} \\ &+ \int_0^s \|\Upsilon\|^2_{L^2(\Omega)}dt + \int_\Omega (\Phi_0(x)-\Psi_0(x))\Upsilon_1(x)dx + \frac{1}{2}\|\nabla\Upsilon(x,s)\|^2_{L^2(\Omega)} \qquad (6.12) \\ &= \int_\Omega\int_0^s (g(\Psi)-g(\Phi))\lambda(t)dtdx. \qquad (6.13)\end{aligned}$$

We have the following relationship:

$$g(\Psi)-g(\Phi) = 3(u^0)^2\Psi - 3(u^0)^2\Phi.$$

Integrate on $(0,s)$, we get

$$\int_0^s g(\Psi)-g(\Phi)dt = \int_0^s 3(u^0)^2\Psi - 3(u^0)^2\Phi dt,$$

and

$$\left|\int_0^s (g(\Psi)-g(\Phi))\lambda(t)dt\right| = \left|\int_0^s (3{(u^0)}^2\Psi - 3{(u^0)}^2\Phi)\lambda(t)dt\right|$$

$$\leq \int_0^s |(3{(u^0)}^2\Psi - 3{(u^0)}^2\Phi)||\lambda(t)|dt \tag{6.14}$$

We obtain

$$\int_0^s |(g(\Psi)-g(\Phi))||\lambda(t)|dt \leq 3\int_0^s |{(u^0)}^2||\Upsilon||\lambda(t)|dt$$

with ${(u^0)}^2 \in L^\infty(0,T;L^n(\Omega))$, $\Upsilon \in L^\infty(0,T;L^2(\Omega))$ and $\lambda \in L^\infty(0,T;L^q(\Omega))$.

By integrating in Ω, we have:

$$\int_\Omega \left(\int_0^s |(g(\Psi)-g(\Phi))||\lambda(t)|dt\right)dx \leq 3\int_\Omega \left(\int_0^s |{(u^0)}^2||\Upsilon||\lambda(t)|dt\right)dx. \tag{6.15}$$

Applying the Hölder inequality (6.12) becomes:

$$\int_\Omega \left(\int_0^s |(g(\Psi)-g(\Phi))||\lambda(t)|dt\right)dx$$

$$\leq 3\left[\left(\int_\Omega |\Upsilon|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |{(u^0)}^2|^n dx\right)^{\frac{1}{n}}\left(\int_\Omega |\lambda(t)|^q dx\right)^{\frac{1}{q}}\right] \tag{6.16}$$

$$\int_\Omega \left(\int_0^s |(g(\Psi)-g(\Phi))||\lambda(t)|dt\right)dx$$

$$\leq 3\int_0^s \|\Upsilon\|_{L^2(\Omega)}\|{(u^0)}^2\|_{L^n(\Omega)}\|\lambda(t)\|_{L^q(\Omega)}dt, \tag{6.17}$$

where $\frac{1}{2}+\frac{1}{n}+\frac{1}{q}=1$. As $\lambda(x,t)=\Upsilon_1(x,t)-\Upsilon_1(x,s)$, then:

$$\|\lambda(x,t)\|_{L^q(\Omega)} = \|\Upsilon_1(x,t)-\Upsilon_1(x,s)\|_{L^q(\Omega)} \leq \|\Upsilon_1(x,t)\|_{L^q(\Omega)} + \|\Upsilon_1(x,s)\|_{L^q(\Omega)}.$$

As $H^1(\Omega)\subset L^q(\Omega) \Longrightarrow \exists c_7>0$, such as:

$$\|\lambda(x,t)\|_{L^q(\Omega)} \leq c_8\|\lambda(x,t)\|_{H^1(\Omega)},$$

$$\|\lambda(x,t)\|_{L^q(\Omega)} \leq c_8(\|\Upsilon_1(x,t)\|_{H^1(\Omega)} + \|\Upsilon_1(x,s)\|_{H^1(\Omega)}). \tag{6.18}$$

By using (6.13), (6.12) becomes:

$$\int_\Omega \left(\int_0^s |(g(\Psi) - g(\Phi))||\lambda(t)|dt \right) dx$$

$$\leq \; c_9 \int_0^s \left[\|\Upsilon\|_{L^2(\Omega)} \|(u^0)^2\|_{L^n(\Omega)} \times (\|\Upsilon_1(x,t)\|_{H^1(\Omega)} + \|\Upsilon_1(x,s)\|_{H^1(\Omega)}) \right] dt.$$

Note also that:

$$\|(u^0)^2\|_{L^n(\Omega)} \leq c_1 0 \|(u^0)^2\|_{H^1(\Omega)}$$

and $(u^0) \in L^\infty(0,T;H^1(\Omega))$ implies $\|(u^0)^2\|_{H^1(\Omega)} \leq c_{11}$ p.p on Ω. Also,

$$\|(u^0)^2\|_{L^n(\Omega)} \leq c_{12}.$$

Then we have:

$$\int_\Omega \left(\int_0^s |(g(\Psi) - g(\Phi))||\lambda(t)|dt \right) dx$$

(6.19) $$\leq \; c_{13} \int_0^s \|\Upsilon\|_{L^2(\Omega)} \times (\|\Upsilon_1(x,t)\|_{H^1(\Omega)} + \|\Upsilon_1(x,s)\|_{H^1(\Omega)}) dt$$

$$\leq \; c_{13} \int_0^s \|\Upsilon\|_{L^2(\Omega)} \|\Upsilon_1(x,t)\|_{H^1(\Omega)} dt + c_{12} \int_0^s \|\Upsilon\|_{L^2(\Omega)} \|\Upsilon_1(x,s)\|_{H^1(\Omega)} dt,$$

implying

$$\int_\Omega \left(\int_0^s |(g(\Psi) - g(\Phi))||\lambda(t)|dt \right) dx \leq c_{13} \int_0^s \|\Upsilon\|_{L^2(\Omega)} \|\Upsilon_1(x,t)\|_{H^1(\Omega)} dt$$

(6.20) $$+ \; c_{13} \|\Upsilon_1(x,s)\|_{H^1(\Omega)} \int_0^s \|\Upsilon\|_{L^2(\Omega)} dt.$$

Applying Young's inequality (6.14) becomes:

$$\int_\Omega \left(\int_0^s |(g(\Psi) - g(\Phi))||\lambda(t)|dt \right) dx \leq c_{13} \left[\int_0^s \frac{\varepsilon}{2} \|\Upsilon\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon} \|\Upsilon_1(x,t)\|^2_{H^1(\Omega)} dt \right]$$

$$+ \; c_{13} \left[\frac{\varepsilon}{2} T \|\Upsilon_1(x,s)\|^2_{H^1(\Omega)} + \int_0^s \frac{1}{2\varepsilon} \|\Upsilon\|^2_{L^2(\Omega)} dt \right]$$

and

(6.21) $$\int_\Omega \left(\int_0^s |(g(\Psi) - g(\Phi))||\lambda(t)|dt \right) dx$$

$$\leq \; c_{13} \int_0^s \left[\left(\frac{\varepsilon}{2} + \frac{1}{2\varepsilon} \right) \|\Upsilon\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon} \|\Upsilon_1(x,t)\|^2_{H^1(\Omega)} \right] dt + \frac{\varepsilon}{2} T c_{13} \|\Upsilon_1(x,s)\|^2_{H^1(\Omega)}.$$

By using (6.15), (6.11) becomes:

$$\begin{aligned}
&- \int_\Omega (\Phi_1(x)-\Psi_1(x))\Upsilon_1(x)dx - \frac{1}{2}\|\Upsilon(x,s)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Phi_0(x)-\Psi_0(x)\|^2_{L^2(\Omega)} \\
&+ \frac{1}{2}\|\nabla\Upsilon(x,s)\|^2_{L^2(\Omega)} + \int_\Omega (\Phi_0(x)-\Psi_0(x))\Upsilon_1(x)dx + \int_0^s \|\Upsilon\|^2_{L^2(\Omega)}dt \\
&\leq c_{13}\int_0^s \left[\left(\frac{\varepsilon}{2}+\frac{1}{2\varepsilon}\right)\|\Upsilon\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon}\|\Upsilon_1(x,t)\|^2_{H^1(\Omega)}\right]dt + \frac{\varepsilon}{2}Tc_{13}\|\Upsilon_1(x,s)\|^2_{H^1(\Omega)},
\end{aligned}$$

implying

$$\begin{aligned}
&- \frac{1}{2}\|\Upsilon(x,s)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\Phi_0(x)-\Psi_0(x)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla\Upsilon(x,s)\|^2_{L^2(\Omega)} \\
&\leq -\int_0^s \|\Upsilon\|^2_{L^2(\Omega)}dt + c_{13}\int_0^s \left[\left(\frac{\varepsilon}{2}+\frac{1}{2\varepsilon}\right)\|\Upsilon\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon}\|\Upsilon_1(x,t)\|^2_{H^1(\Omega)}\right]dt \\
&+ \frac{\varepsilon}{2}Tc_{13}\|\Upsilon_1(x,s)\|^2_{H^1(\Omega)} + \int_\Omega (\Phi_1(x)-\Psi_1(x))\Upsilon_1(x)dx \\
&- \int_\Omega (\Phi_0(x)-\Psi_0(x))\Upsilon_1(x)dx.
\end{aligned} \tag{6.22}$$

Applying Hölder to the second member of (6.16), we finally have:

$$\begin{aligned}
&- \|\Upsilon(x,s)\|^2_{L^2(\Omega)} + \|\Phi_0(x)-\Psi_0(x)\|^2_{L^2(\Omega)} + \|\nabla\Upsilon(x,s)\|^2_{L^2(\Omega)} \\
&\leq c_{13}\int_0^s \left[\left(\varepsilon+\frac{1}{\varepsilon}-\frac{2}{c_{12}}\right)\|\Upsilon\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\|\Upsilon_1(x,t)\|^2_{H^1(\Omega)}\right]dt \\
&+ \varepsilon Tc_{13}\|\Upsilon_1(x,s)\|^2_{H^1(\Omega)} + 2\|\Phi_1(x)-\Psi_1(x)\|^2_{L^2(\Omega)}\|\Upsilon_1(x,s)\|^2_{L^2(\Omega)} \\
&- 2\|\Phi_0(x)-\Psi_0(x)\|^2_{L^2(\Omega)}\|\Upsilon_1(x,s)\|^2_{L^2(\Omega)}
\end{aligned} \tag{6.23}$$

Reducing (6.17) by appropriate expressions on both sides, we have:

$$\begin{aligned}
\|\Upsilon(x,s)\|^2_{L^2(\Omega)} &\leq c_{12}\int_0^s \left[\left(\varepsilon+\frac{1}{\varepsilon}+\frac{2}{c_{13}}\right)\|\Upsilon\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\|\Upsilon_1(x,t)\|^2_{H^1(\Omega)}\right]dt \\
&+ \varepsilon Tc_{13}\|\Upsilon_1(x,s)\|^2_{H^1(\Omega)}.
\end{aligned} \tag{6.24}$$

By adding the term $\|\Upsilon_1(x,s)\|^2_{H^1(\Omega)}$ to the first and second members of the (6.18) We obtain:

$$
\begin{aligned}
& \|\Upsilon(x,s)\|^2_{L^2(\Omega)} + \|\Upsilon_1(x,s)\|^2_{H^1(\Omega)} \\
(6.25) \quad \leq\ & c_{13}\int_0^s \left[\left(\varepsilon + \frac{1}{\varepsilon} + \frac{2}{c_{13}}\right)\|\Upsilon\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\|\Upsilon_1(x,t)\|^2_{H^1(\Omega)}\right]dt \\
+\ & \varepsilon T c_{13}\|\Upsilon_1(x,s)\|^2_{H^1(\Omega)} + c_{14}\|\Upsilon_1(x,s)\|^2_{H^1(\Omega)}
\end{aligned}
$$

and

$$
\begin{aligned}
& \|\Upsilon(x,s)\|^2_{L^2(\Omega)} + \|\Upsilon_1(x,s)\|^2_{H^1(\Omega)} \\
\leq\ & c_{13}\int_0^s \left[\left(\varepsilon + \frac{1}{\varepsilon} + \frac{2}{c_{13}}\right)\|\Upsilon\|^2_{L^2(\Omega)} + \frac{1}{\varepsilon}\|\Upsilon_1(x,t)\|^2_{H^1(\Omega)}\right]dt \\
+\ & (\epsilon T c_{13} + c_{14})\|\Upsilon_1(x,s)\|^2_{H^1(\Omega)}.
\end{aligned}
$$

By choosing $\varepsilon > 0$ infinitely small ($\varepsilon <<< 1$) such that:

$$\varepsilon + \frac{1}{\varepsilon} + \frac{2}{c_{13}} = const > 0$$

and $1 - (\varepsilon T c_{13} + c_{14}) = const > 0$ with $\|\Upsilon\|^2_{L^2(\Omega)} = \|\Upsilon(x,t)\|^2_{L^2(\Omega)}$. Finally, we have:

$$
\begin{aligned}
(6.26) \quad & \|\Upsilon(x,s)\|^2_{L^2(\Omega)} + \|\Upsilon_1(x,s)\|^2_{H^1(\Omega)} \\
(6.27) \quad \leq\ & const \int_0^s \left[\|\Upsilon(x,t)\|^2_{L^2(\Omega)} + \|\Upsilon_1(x,t)\|^2_{H^1(\Omega)}\right]dt.
\end{aligned}
$$

According to Gronwall, we'll have:

$$\Upsilon = 0 \Longleftrightarrow \Phi - \Psi = 0 \Longleftrightarrow \Phi = \Psi.$$

Hence problem (1.1)-(1.3) has a unique solution.

Since there is a unique solution $\Phi(x,t) \in H^1(Q_T)$ of the conjugate problem, we obtain the increase $DeltaJ$ the following final expression:

$$
\begin{aligned}
\Delta J \ =\ & \int_{Q_T} [D_f(u^0, f^0, u_0^0, u_1^0)\delta f + D_{u_0}(u^0, f^0, u_0^0, u_1^0)\delta u_0 \\
+\ & D_{u_1}(u^0, f^0, u_0^0, u_1^0)\delta u_1]dxdt \\
+\ & \int_\Omega \delta u_1(x)\Phi^0(x,0)dx + \int_{Q_T} \delta f(x,t)\Phi^0(x,t)dxdt \\
+\ & o\left(\|\delta f\|_{L^2(Q_T)} + \|\delta u_0\|_{H^1(\Omega)} + \|\delta u_1\|_{L^2(\Omega)}\right)
\end{aligned}
$$

Consider the following functions:

$$\begin{aligned} H^{(1)}(u,f,u_0,u_1,\Phi) &= D(u,f,u_1) + \Phi f, \\ H^{(2)}(u,f,u_0,u_1) &= D(u,f,u_0,u_1), \\ H^{(3)}(u,f,u_0,u_1,\Phi) &= \int_0^T \left[D(u,f,u_0,u_1) + \frac{1}{T}\Phi(x,0)u_1 \right] dxdt. \end{aligned}$$

Then the formula for the increase in the functional can be written as:

$$\begin{aligned} \Delta J &= \int_{Q_T} \frac{\partial H^{(1)}}{\partial f}(u^0, f^0, u_0^0, u_1^0, \Phi^0)(f^\varepsilon - f^0)dxdt \\ &+ \int_{Q_T} \frac{\partial H^{(2)}}{\partial f}(u^0, f^0, u_0^0, u_1^0)(u_0^\varepsilon - u_0^0)dxdt \\ &+ \int_{Q_T} \frac{\partial H^{(3)}}{\partial f}(u^0, f^0, u_0^0, u_1^0, \Phi^0)(u_1^\varepsilon - u_1^0)dxdt \\ &+ o\left(\|\delta f\|_{L^2(Q_T)} + \|\delta u_0\|_{H^1(\Omega)} + \|\delta u_1\|_{L^2(\Omega)} \right). \end{aligned}$$

□

We have proved the following theorem:

Theorem 6.2. *Assume that the conditions of Proposition 2.1 are satisfied for the function D. Then the functional $J(f, u_0, u_1)$ is Fréchet differentiable with respect to the variables f, u_0, u_1 and its partial derivatives at the point (f^0, u_0^0, u_1^0) are expressed as follows:*

$$\begin{aligned} J_f \delta f &= \int_{Q_T} \frac{\partial H^{(1)}}{\partial f}(u^0, f^0, u_0^0, u_1^0, \Phi^0)\delta f dxdt \\ J_{u_0}\delta u_0 &= \int_{Q_T} \frac{\partial H^{(2)}}{\partial f}(u^0, f^0, u_0^0, u_1^0, \Phi^0)\delta u_0 dxdt \\ J_{u_1}\delta u_1 &= \int_{Q_T} \frac{\partial H^{(3)}}{\partial f}(u^0, f^0, u_0^0, u_1^0, \Phi^0)\delta u_1 dxdt. \end{aligned}$$

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