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DIFFERENTIABILITY IN THE SENS OF FRÉCHET OF A FUNCTIONAL RELATED TO A NONLINEAR HYPERBOLIC PROBLEM AND DEPENDING ON COMMANDS

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ABSTRACT. In this paper we demonstrate the result of the differentiability with commands in the sense of Fréchet of a functional related to a nonlinear hyperbolic problem.

1. INTRODUCTION

Consider the following problem:

$$(1.1)\qquad \begin{cases} \dfrac{\partial^2 u(x;t)}{\partial t^2} - \Delta u(x;t) \\ \qquad +|u(x;t)|^\rho u(x;t) + \dfrac{\partial u(x;t)}{\partial t} = f(x,t) & \text{with } \rho > 0 \\ u(x,t)\Big|_{t=0} = u_0(x) & x \in \Omega \\ \dfrac{\partial u(x,t)}{\partial t}\Big|_{t=0} = u_1(x) & x \in \Omega \\ \dfrac{\partial u}{\partial \overrightarrow{n}}\Big|_{\partial\Omega} = 0 & t \in]0,T[. \end{cases}$$

in a cylinder $D = \{(x;t)/x \in \Omega; t \in]0;T[\}$ where Ω is a bounded open set of $\mathbb{R}^n$ of differentiable boundary $\partial\Omega$, $\overrightarrow{n}$ the unit normal vector to $\partial\Omega$.

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2. Theorems

2.1. Existence theorem of the solution.

Theorem 2.1. *Let be* $f(x;t) \in L^2(D), u_0(x) \in L^P(\Omega) \cap H^1(\Omega)$ *with* $p = \rho + 2 > 0$ *and* $u_1(x) \in L^2(\Omega)$ *the given functions.The problem* (1.1) – (1.1) *admits a solution u(x;t)satisfying to the following:*

$$u(x;t) \in L^\infty\Big(]0;T[;L^p(\Omega) \cap H^1(\Omega)\Big),$$
$$\frac{\partial u(x;t)}{\partial t} \in L^\infty\big(]0;T[;L^2(\Omega)\big).$$

2.2. Uniqueness theorem of the solution.

Theorem 2.2. *Let* $f(x;t) \in L^2(D)$,$u_0(x) \in H^1(\Omega) \cap L^p(\Omega)$, $u_1(x) \in L^2(\Omega)$ *the given functions. Then the solution* $u(x;t)$ *obtained at the theorem* 1 *is unique.*

Remark 2.1. *The theorems* 1 *and* 2 *are demonstrate in [1].*

3. Hypotheses

The function $u(x;t)$ is the unique solution of the problem (1.1)–(1.1) corresponding to the commands $f(x;t) \in Y \subset L^2(D)$, $u_0(x) \in X \subset H^1(\Omega)$, $u_1(x) \in W \subset L^2(\Omega)$ where Y, X, W are convex sets.

Let V be the set of distributional solutions of the problem (1.1)–(1.1). When $(f(x,t), u_0(x;t); u_1(x;t))$ traverses $Y \times X \times W$. The set

$$V \subset \left\{u(x;t)/u(x;t) \in L^\infty((]0;T[\,;H^1\Omega) \cap L^p(\Omega)); \frac{\partial u(x;t)}{\partial t} \in L^\infty(]0;T[\,;L^2(\Omega))\right\}$$

is endowed with the norm $\|u(x;t)\|_{H^1(D)} = \|f(x;t)\|_{L^2(D)} + \|u_0(x)\|_{H^1(\Omega)} + \|u_1(x)\|_{L^2(\Omega)}$ and checks the condition of Lipschitz, we obtain:

$$\|u(x;t)\|_{H^1(D)} \leq C(T)(\|f(x;t)\|_{L^2(D)} + \|u_0(x)\|_{H^1(\Omega)} + \|u_1(x)\|_{L^2(\Omega)}).$$

The variations $f^\varepsilon(x;t), u_0^\varepsilon(x), u_1^\varepsilon(x)$ commands $f^0(x;t), u_0^0(x), u_1^0(x)$ along the directions $f^\varepsilon(x;t) - f^0(x;t)$; $u_0^\varepsilon(x) - u_0^0(x)$; $u_1^\varepsilon(x) - u_1^0(x)$ are defined as follows:

$$\begin{cases} f^\varepsilon(x;t) = f^0(x;t) + \theta(f^\varepsilon(x;t) - f^0(x;t)) \\ \delta f = f^\varepsilon(x;t) - f^0(x;t) \end{cases} \qquad \begin{cases} u_0^\varepsilon(x) = u_0^0(x) + \theta(u_0^\varepsilon(x) - u_0^0(x)) \\ \delta u_0 = u_0^\varepsilon(x) - u_0^0(x), \end{cases}$$

$$\begin{cases} u_1^\varepsilon(x) = u_1^0(x) + \theta(u_1^\varepsilon(x) - u_1^0(x)) \\ \delta u_1 = u_1^\varepsilon(x) - u_1^0(x), \end{cases} \qquad \begin{cases} u^\varepsilon(x;t) = u^0(x;t) + \theta(u^\varepsilon(x;t) - u^0(x;t)) \\ \delta u = u^\varepsilon(x;t) - u^0(x;t). \end{cases}$$

where $u^\varepsilon(x;t)$ is the unique solution of the problem (1.1)–(1.1) corresponding the functions $f^\varepsilon(x;t), u_0^\varepsilon(x), u_1^\varepsilon(x)$ and $u^0(x;t)$ is the unique solution of the problem (1.1)–(1.1) correspondig the commands $f^0(x;t), u_0^0(x), u_1^0(x)$ that δu has the norm $\|\delta u\|_{H^1(D)} = \|\delta f\|_{L^2(D)} + \|\delta u_0\|_{H^1(\Omega)} + \|\delta u_1\|_{L^2(\Omega)}$. Of all these variations $\theta \in]0;1[$ and that $f^0(x;t)$, $f^\varepsilon(x;t) \in Y$; $u_0^0(x)$, $u_0^\varepsilon(x) \in X$; $u_1^0(x)$, $u_1^\varepsilon(x) \in W$.

4. Functional differentiability theorem

Theorem 4.1. *Let be the functional:*

$$J(f(x;t);u_0(x);u_1(x)) = \int_D D(u(x;t);f(x;t);u_0(x);u_1(x))dxdt \tag{4.1}$$

verifying the following conditions:

i) *The function D and all its derivatives D_u, D_f D_{u_0}, D_{u_1} satisfy the Lipschitz conditions relative the variable $u(x;t); f(x;t); u_0(x);\ u_1(x)$.*

ii) *The function $u^0(x;t)$ is the unique solution of the problem* (1.1)–(1.1) *corresponding to commands $f^0(x;t) \in Y$, $u_0^0(x) \in X$, $u_1^0(x) \in W$. Then the functional* (4.1) *is differentiable in the sense of Fréchet relative the variables $f(x;t);u_0(x);u_1(x)$ and its partial derivatives at the point $(f^0(x;t);u_0^0(x);$ $u_1^0(x))$ are defined as follows:*

$$\begin{cases} J_f(f^0(x;t);u_0^0(x);u_1^0(x)) = \displaystyle\int_D \frac{\partial H^{(1)}}{\partial f}(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x);\Phi(x;t)dxdt \\ J_{u_0}(f^0(x;t);u_0^0(x);u_1^0(x)) = \displaystyle\int_D \frac{\partial H^{(2)}}{\partial u_0}(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x);\Phi(x;0))dxdt \\ J_{u_1}(f^0(x;t);u_0^0(x);u_1^0(x)) = \displaystyle\int_D \frac{\partial H^{(3)}}{\partial u_1}(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x);\Phi(x;0)dxdt. \end{cases}$$

Proof. The increment of the functional J at the point $(f^0(x;t);u_0^0(x);u_1^0(x))$ is defined as follows:

$$\Delta J(f^0(x;t);u_0^0(x);u_1^0(x)) = J(f^\varepsilon(x;t);u_0^\varepsilon(x);u_1^\varepsilon(x)) - J(f^0(x;t);u_0^0(x);u_1^0(x))$$

with

$$J(f^{\varepsilon}(x;t);u_0^{\varepsilon}(x);u_1^{\varepsilon}(x))=\int_D D(u^{\varepsilon}(x,t);f^{\varepsilon}(x;t);u_0^{\varepsilon}(x);u_1^{\varepsilon}(x))dxdt$$

(4.2)
$$\Delta J(f^0(x;t);u_0^0(x);u_1^0(x))=\int_D [D(u^{\varepsilon}(x;t);f^{\varepsilon}(x;t);u_0^{\varepsilon}(x);u_1^{\varepsilon}(x)) \\ -D(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x))]dxdt$$

Additionally

$$\begin{aligned} &D(u^{\varepsilon}(x;t),f^{\varepsilon}(x;t);u_0^{\varepsilon}(x);u_1^{\varepsilon}(x))-D(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x))\\ &=D(u^{\varepsilon}(x;t);f^{\varepsilon}(x;t);u_0^{\varepsilon}(x);u_1^{\varepsilon}(x))-D(u^0(x;t);f^{\varepsilon}(x;t);u_0^{\varepsilon}(x);u_1^{\varepsilon}(x))\\ &+D(u^0(x;t);f^{\varepsilon}(x;t);u_0^{\varepsilon}(x);u_1^{\varepsilon}(x))-D(u^0(x;t);f^0(x;t);u_0^{\varepsilon}(x);u_1^{\varepsilon}(x))\\ &+D(u^0(x;t);f^0(x;t);u_0^{\varepsilon}(x);u_1^{\varepsilon}(x))-D(u^0(x;t);f^0(x;t);u_0^0(x);u_1^{\varepsilon}(x))\\ &+D(u^0(x;t);f^0(x;t);u_0^0(x);u_1^{\varepsilon}(x))-D(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x)). \end{aligned}$$

Note that,

$$\begin{aligned} &D(u^{\varepsilon}(x;t);f^{\varepsilon}(x;t);u_0^{\varepsilon}(x);u_1^{\varepsilon}(x))-D(u^0(x;t);f^{\varepsilon}(x;t);u_0^{\varepsilon}(x);u_1^{\varepsilon}(x))\\ &=\int_0^1 [D_u(u^0(x;t)+\theta\delta u;f^{\varepsilon}(x;t);u_0^{\varepsilon}(x);u_1^{\varepsilon}(x))\\ &-D_u(u^0(x;t),f^0(x;t);u_0^0(x);u_1^0(x))]\delta u d\theta\\ &+D_u(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x))\delta u \qquad (4.3)\\ &=r_1+D_u(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x))\delta u, \end{aligned}$$

with

$$r_1=\int_0^1 \big[D_u(u^0(x;t)+\theta\delta u;f^{\varepsilon}(x;t);u_0^{\varepsilon}(x);u_1^{\varepsilon}(x)) \\ -D_u(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x))\big]\,\delta u d\theta.$$

Next,

$$\begin{aligned} &D(u^0(x;t);f^{\varepsilon}(x;t);u_0^{\varepsilon}(x);u_1^{\varepsilon}(x))-D(u^0(x;t);f^0(x;t);u_0^{\varepsilon}(x);u_1^{\varepsilon}(x))\\ &=r_2+D_f(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x))\delta f, \qquad (4.4) \end{aligned}$$

with

$$r_2 = \int_0^1 \Big[D_f(u^0(x;t); f^0(x;t) + \theta\delta f; u_0^\varepsilon(x); u_1^\varepsilon(x)) \\ - D_f(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x))\Big]\delta f d\theta.$$

Since

$$D(u^0(x;t); f^0(x;t); u_0^\varepsilon(x); u_1^\varepsilon(x;t)) - D(u^0(x;t); f^0(x;t); u_0^0(x); u_1^\varepsilon(x)) \\ = r_3 + D_{u_0}(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x))\delta u_0 \tag{4.5}$$

and

$$r_3 = \int_0^1 \Big[D_{u_0}(u^0(x;t); f^0(x;t); u_0^0(x) + \theta\delta u_0; u_1^\varepsilon(x;t)) \\ - D_{u_0}(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x))\Big]\delta u_0 d\theta.$$

Finally,

$$D(u^0(x;t); f^0(x;t); u_0^0(x); u_1^\varepsilon(x) - D(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)) \\ = r_4 + D_{u_1}(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x))\delta u_1. \tag{4.6}$$

with

$$r_4 = \int_0^1 \Big[D_{u_1}(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x) + \theta\delta u_1) \\ - D_{u_1}(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x))\Big]\delta u_1 d\theta.$$

By introducing the relations (4.3)–(4.6) in (4.2), we obtain:

$$\Delta J(f^0(x;t); u_0^0(x); u_1^0(x)) = \int_D \Big[D_u(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x))\delta u \\ + D_f(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x))\delta f + Du_0(u^0(x;t); f^0(x;t); u_0^0(x); \\ u_1^0(x))\delta u_0 + D_{u_1}(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x))\delta u_1\Big] dxdt + \int_D \sum_{i=1}^4 r_i dxdt. \tag{4.7}$$

Consider the function $\Phi(x;t)$ whose trace $t = T$ is equal to 0.

By multiplying the equation (1.1) by $\Phi(x;t)$ and integrating over D, we obtain:

$$\int_\Omega (u_1(x)+u_0(x))\Phi(x;0)dx + \int_D \Big[\frac{\partial u(x;t)}{\partial t}.\frac{\partial \Phi(x;t)}{\partial t} + f(x;t)\Phi(x;t) + u(x;t)\frac{\partial \Phi(x;t)}{\partial t} - \big(\nabla u(x;t);\nabla\Phi(x;t)\big) - u(x;t)|u(x;t)|^\rho\Phi(x;t)\Big]dxdt = 0. \tag{4.8}$$

The function $u^\varepsilon(x;t)$ is the unique solution of the problem (1.1)–(1.1) corresponding to the commands $f^\varepsilon(x;t), u_0^\varepsilon(x), u_1^\varepsilon(x)$ and verifies the integral equality (4.8). we then obtain:

$$\begin{aligned}&\int_\Omega (u_0^\varepsilon(x)+u_1^\varepsilon(x))\Phi(x;0)dx + \int_D \Big[f^\varepsilon(x;t)\Phi(x;t) + u^\varepsilon(x;t)\frac{\partial \Phi(x;t)}{\partial t}\\ &+ \frac{\partial u^\varepsilon(x;t)}{\partial t}\frac{\partial \Phi(x;t)}{\partial t} - (\nabla u^\varepsilon(x;t);\nabla\Phi(x;t))\\ &- |u^\varepsilon(x;t)|^\rho u^\varepsilon(x;t)\Phi(x;t)\Big]dxdt = 0.\end{aligned} \tag{4.9}$$

Similarly $u^0(x;t)$ is the unique solution of the problem (1.1)–(1.1) corresponding to the commands $f^0(x;t), u_0^0(x); u_1^0(x)$ and verifies the integral equality (4.8). We obtain:

$$\begin{aligned}&\int_\Omega (u_0^0(x)+u_1^0(x))\Phi(x;0)dx + \int_D \Big[f^0(x;t)\Phi(x;t) + u^0(x;t)\frac{\partial \Phi(x;t)}{\partial t}\\ &+ \frac{\partial u^0(x;t)}{\partial t}\frac{\partial \Phi(x;t)}{\partial t} - (\nabla u^0(x;t);\nabla\Phi(x;t))\\ &- |u^0(x;t)|^\rho u^0(x;t)\Phi(x;t)\Big]dxdt = 0.\end{aligned} \tag{4.10}$$

By subtracting (4.9) and (4.10), we obtain:

$$\begin{aligned}&\int_\Omega (\delta u_0+\delta u_1)\Phi(x;0)dx + \int_D \Big[\delta f\Phi(x;t) + \delta u\frac{\partial \Phi(x;t)}{\partial t} + \frac{\delta u}{\partial t}\frac{\partial \Phi(x;t)}{\partial t}\\ &- (\nabla \delta u;\nabla\Phi(x;t)) - (|u^\varepsilon(x;t)|^\rho u^\varepsilon(x;t)\\ &- |u^0(x;t)|^\rho u^0(x;t))\Phi(x;t)\Big]dxdt = 0.\end{aligned} \tag{4.11}$$

By setting $I = \int_D \big(|u^\varepsilon(x;t)|^\rho u^\varepsilon(x;t) - |u^0(x;t)|^\rho u^0(x;t)\big)\Phi(x;t)dxdt$, the transformation of I leads us to $I = I_1 + I_2 + I_3$ with

$$I_1 = \int_D (|u^\varepsilon(x;t)|^\rho - |u^0(x;t)|^\rho)u^0(x;t)\Phi(x;t))dxdt,$$

$$I_2 = \int_D (|u^\varepsilon(x;t)|^\rho - |u^0(x;t)|^\rho))\delta u\Phi(x;t)dxdt,$$
$$I_3 = \int_D |u^0(x;t)|^\rho \delta u\Phi(x;t)dxdt.$$

□

Remark 4.1.

- *For even* ρ*, we have* $|u^\varepsilon(x;t)|^\rho - |u^0(x;t)|^\rho = \big(u^\varepsilon(x;t)\big)^\rho - \big(u^0(x;t)\big)^\rho$.
- *For odd* ρ*, we have* $|u^\varepsilon(x;t)|^\rho - |u^0(x;t)|^\rho = -\Big[\big(u^\varepsilon(x;t)\big)^\rho - \big(u^0(x;t)\big)^\rho\Big]$.
- *For odd* ρ *and* $u^\varepsilon(x;t) > 0$*,* $u^0(x;t) < 0$*, we have*

$$|u^\varepsilon(x;t)|^\rho - |u^0(x;t)|^\rho = \big(u^\varepsilon(x;t)\big)^\rho + \big(u^0(x;t)\big)^\rho.$$

- *For odd* ρ *and* $u^\varepsilon(x;t) < 0$*,* $u^0(x;t) > 0$*, we have*

$$|u^\varepsilon(x;t)|^\rho - |u^0(x;t)|^\rho = -\Big[\big(u^\varepsilon(x;t)\big)^\rho + \big(u^0(x;t)\big)^\rho\Big].$$

Since the last two cases are not interest, then

$$I_1 = \int_D \varphi\Big[\int_0^1 ((u^0(x;t)) + \theta\delta u)^{\rho-1} - u^0(x;t)^{\rho-1})\delta u d\theta\Big] u^0(x;t)\Phi(x;t)dxdt$$
$$+ \rho\int_D \Big[\int_0^1 u^0(x;t)^\rho \delta u d\theta\Big]\Phi(x;t)dxdt.$$

Let set:

$$q_k = \int_D \rho\Big[\int_0^1 ((u^0(x;t) + \theta\delta u_k)^{\rho-1} - u^0(x;t)^{\rho-1})\delta u_k d\theta\Big] u^0(x;t)\Phi(x;t)dxdt.$$

Choose k such that: $\delta u_k = u^{\varepsilon_k}(x;t) - u^0(x;t)$, $\varepsilon_k = \frac{1}{k}$, $q_k \longrightarrow q$, $\delta u_k \longrightarrow 0$, $\|\delta u_k\|_{H^1(D)} \longrightarrow \|\delta u\|_{H^1(D)}$ when $k \longrightarrow +\infty$.

Consider $\frac{1}{p^*} = \dfrac{1}{2} - \dfrac{1}{n}$ with $1 < q < p^*$ for $n \geq 3$; we then obtain $1 < q < \dfrac{2(n+1)}{n-1}$ for $n \geq 2$. Taking $n = 2(\rho+1)$ with $\dfrac{1}{q} + \dfrac{1}{2} = 1$ and applying Hölder's inequality, we get:

$$
\begin{aligned}
&|q_k| \\
&\leq \int_0^1 \int_D \Big[\big(|\rho((u^0(x;t)+\theta\delta u_k)^{\rho-1} - u^0(x;t)^{\rho-1}\big) u^0(x;t)\Phi(x;t)| \big)^{\frac{2(2\rho+3)}{2\rho+1}} dxdt \Big]^{\frac{2\rho+1}{2(2\rho+3)}} d\theta \\
&\times \Big(\int_D |\delta u_k|^2 dxdt \Big)^{\frac{1}{2}}.
\end{aligned}
$$

Applying the Sobolev injection $H^1(D) \subset L^2(D)$, we obtain:

$$
|q_k| \leq \|\rho((u^0(x;t)+\theta\delta u_k)^{\rho-1} - (u^0(x;t))^{\rho-1})\Phi(x;t)u^0(x;t)\|_{L^{\frac{2(2\rho+3)}{2\rho+1}}(D)} .\|\delta u_k\|_{H^1(D)},
$$

and

$$
\begin{aligned}
& \lim_{k\longrightarrow+\infty} \frac{|q_k|}{\|\delta u_k\|_{H^1(D)}} = 0 \\
\Rightarrow \qquad & q = 0(\|\delta u\|_{H^1(D)}) = 0(\|\delta f\|_{L^2(D)} + \|\delta u_0\|_{H^1(\Omega)} + \|\delta u_1\|_{L^2(\Omega)}).
\end{aligned}
$$

Hence

$$
\begin{aligned}
I_1 &= \rho \int_D u^0(x;t)^\rho \delta u \Phi(x;t) dxdt + 0(\|\delta f\|_{L^2(D)} + \|\delta u_0\|_{H^1(\Omega)} + \|\delta u_1\|_{L^2(\Omega)}) \\
I_2 &= \int_D \Big[\rho \int_0^1 \big(u^0(x;t)+\theta\delta u\big)^{\rho-1} \delta u d\theta\Big] \delta u \Phi(x;t) dxdt.
\end{aligned}
$$

By setting

$$
q_k = \int_D \rho\Big[\int_0^1 \big(u^0(x;t)+\theta\delta u_k\big)^{\rho-1}\delta u_k d\theta\Big]\delta u_k \Phi(x;t) dxdt
$$

and applying Hölder's inequality, we get:

$$
\begin{aligned}
|q_k| \leq & \int_0^1 \int_D \Big[\big(|\rho((u^0(x;t)+\theta\delta u_k)^{\rho-1}\big)\delta u_k \Phi(x;t)| \big)^{\frac{2(2\rho+3)}{2\rho+1}} dxdt \Big]^{\frac{2\rho+1}{2(2\rho+3)}} d\theta \\
& \times \Big(\int_D |\delta u_k|^2 dxdt \Big)^{\frac{1}{2}}.
\end{aligned}
$$

Applying the Sobolev injection $H^1(D) \subset L^2(D)$, we obtain:

$$
|q_k| \leq \|\rho((u^0(x;t)+\theta\delta u_k)^{\rho-1})\delta_k \Phi(x;t)\|_{L^{\frac{2(2\rho+3)}{2\rho+1}}(D)} .\|\delta u_k\|_{H^1(D)},
$$

and

$$\lim_{k \longrightarrow +\infty} \frac{|q_k|}{\|\delta u_k\|_{H^1(D)}} = 0$$
$$\Rightarrow \qquad q = 0(\|\delta u\|_{H^1(D)}) = 0(\|\delta f\|_{L^2(D)} + \|\delta u_0\|_{H^1(\Omega)} + \|\delta u_1\|_{L^2(\Omega)}).$$

From where

$$I_2 = 0(\|\delta f\|_{L^2(D)} + \|\delta u_0\|_{H^1(\Omega)} + |\delta u_1\|_{L^2(\Omega)}).$$

Taking the case where $u^0(x;t) > 0$, we obtain:

$$I_3 = \int_D \big(u^0(x;t)\big)^\rho \delta u \Phi(x;t) dxdt.$$

Then the expression of I becomes:

$$I = (\rho+1) \int_D \big(u^0(x;t)\big)^\rho \delta u \Phi(x;t) dxdt + 0(\|\delta f\|_{L^2(D)} + \|\delta u_0\|_{H^1(\Omega)} + \|\delta u_1\|_{L^2(\Omega)}).$$

The relation (4.11) becomes

$$\begin{aligned}
&\int_\Omega (\delta u_0 + \delta u_1)\Phi(x;0)dx + \int_D [\delta f \Phi(x;t) + \delta u \frac{\partial \Phi(x;t)}{\partial t} + \frac{\partial \delta u}{\partial t}.\frac{\partial \Phi(x;t)}{\partial t} \\
&- (\nabla \delta u; \nabla \Phi(x;t)) - (\rho+1) \int_D [u^0(x;t)]^\rho \delta u \Phi(x;t)] dxdt - 0(\|\delta f\|_{L^2(D)} \\
&+ \|\delta u_0\|_{H^1(\Omega)} + \|\delta u_1\|_{L^2(\Omega)}) = 0.
\end{aligned} \tag{4.12}$$

The increment of the functional J of (4.7) becomes:

$$\begin{aligned}
&\Delta J(f^0(x;t); u_0^0(x); u_1^0(x)) = \int_D \Big[D_f(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x))\delta f \\
&+ D_{u_0}(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x))\delta u_0 \\
&+ D_{u_1}(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x))\delta u_1\Big] dxdt \\
&+ \int_D \Big[\delta u \frac{\partial \Phi(x;t)}{\partial t} + \delta f \Phi(x;t) + \frac{\partial \delta u}{\partial t}\frac{\partial \Phi(x;t)}{\partial t} \\
&+ D_u(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x))\delta u \\
&- (\rho+1)[(u^0(x;t)]^\rho \Phi(x;t)\delta u - (\nabla \delta u; \nabla \Phi(x;t)) + \sum_{i=1}^4 r_i\Big] dxdt \\
&+ \int_\Omega (\delta u_0 + \delta u_1)\Phi(x;0)dx - (\|\delta f\|_{L^2(D)} + \|\delta u_0\|_{H^1(\Omega)} + \|\delta u_1\|_{L^2(\Omega)}).
\end{aligned} \tag{4.13}$$

By taking the approximation of the term r_1 by virtue of the fact that D_u verifies the Lipschitz condition. By setting

$$\begin{aligned} q_k = &\int_D \Big[\int_0^1 \big(D_u(u^0(x;t)+\theta\delta u_k; f^\varepsilon(x;t); u_0^\varepsilon(x); u_1^\varepsilon(x)) \\ &- D_u(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x))\big) \times \delta u_k d\theta\Big] dxdt \\ |q_k| \leq &\int_D \Big[\int_0^1 \big|D_u(u^0(x;t)+\theta\delta u_k; f^\varepsilon(x;t); u_0^\varepsilon(x); u_1^\varepsilon(x)) \\ &- D_u(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x))\big| \times |\delta u_k| d\theta\Big] dxdt \\ |q_k| \leq &\int_D \Big[L\int_0^1 \theta|\delta u_k|^2 d\theta\Big] dxdt = \frac{1}{2}L\int_D |\delta u_k|^2 dxdt. \end{aligned}$$

Applying the Sobolev injection, $H^1(D) \subset L^2(D)$, we obtain:

$$\frac{|q_k|}{\|\delta u_k\|_{H^1(D)}} \leq \frac{L}{2}\sqrt{\Big(\int_D |\delta u_k|^2 dxdt\Big)^{\frac{1}{2}} + \sum_{i=1}^n \Big(\int_D \Big|\frac{\partial \delta u_k}{\partial x_i}\Big|^2 dxdt\Big)^{\frac{1}{2}}}$$

$$\Rightarrow \qquad \lim_{k\to+\infty} \frac{|q_k|}{\|\delta u_k\|_{H^1(D)}} \leq 0.$$

From where

$$r_1 = 0(\|\delta f\|_{L^2(D)} + \|\delta u_0\|_{H^1(\Omega)} + \|\delta u_1\|_{L^2(\Omega)}).$$

By taking the approximation of r_2 by virtue of the fact that D_f verifies the Lipschitz condition. By setting

$$\begin{aligned} q_k = &\int_D \int_0^1 \Big[D_f\big(u^\varepsilon(x;t), f^0(x;t)+\theta\delta f_k, u_0^\varepsilon(x); u_1^\varepsilon(x)\big) \\ &- D_f\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\Big] \times \delta f_k d\theta dxdt, \\ |q_k| \leq &\int_D \int_0^1 \Big|D_f\big(u^\varepsilon(x;t); f^0(x;t)+\theta\delta f_k, u_0^\varepsilon(x); u_1^\varepsilon(x)\big) \\ &- D_f\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\Big| \times |\delta f_k| d\theta dxdt, \\ |q_k| \leq &\int_0^1 \int_D L\theta|\delta f_k|^2 d\theta dxdt \leq \frac{1}{2}L\big(\|\delta f_k\|_{L^2(D)} + \|\delta u_{0k}\|_{H^1(\Omega)} + \|\delta u_{1k}\|_{L^2(\Omega)}\big)^2 \\ &\leq \frac{1}{2}L\|\delta u_k\|^2_{H^1(D)} \end{aligned}$$

$$\frac{|q_k|}{\|\delta u_k\|_{H^1(D)}} \leq \frac{L}{2}\sqrt{\left(\int_D |\delta u_k|^2 dxdt\right)^{\frac{1}{2}} + \sum_{i=1}^{n}\left(\int_D \left|\frac{\partial \delta u_k}{\partial x_i}\right|^2 dxdt\right)^{\frac{1}{2}}}$$
$$\Rightarrow \lim_{k\to+\infty} \frac{|q_k|}{\|\delta u_k\|_{H^1(D)}} \leq 0.$$

From where

$$r_2 = 0(\|\delta f\|_{L^2(D)} + \|\delta u_0\|_{H^1(\Omega)} + \|\delta u_1\|_{L^2(\Omega)}).$$

By taking the approximation of r_3 by virtue of the fact that D_{u_0} satisfies the Lipschitz condition. By setting

$$q_k = \int_D \int_0^1 \Big[D_{u_0}\big(u^0(x;t); f^0(x;t); \theta\delta u_{0k} + u_0^0(x); u_1^\varepsilon(x)\big)$$
$$- D_{u_0}\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\Big] \times \delta u_{0k} d\theta dxdt$$
$$|q_k| \leq \frac{L}{2}\int_D |\delta u_{0k}|^2 dxdt$$
$$\leq \frac{L}{2}\int_0^T \|\delta u_{0k}\|^2_{L^2(\Omega)} dt.$$

By applying the Sobolev injection $H^1(\Omega) \subset L^2(\Omega)$, we get:

$$|q_k| \leq \frac{L}{2}\int_0^T \big(\|u_{1k}\|_{L^2(\Omega)} + \|\delta f_k\|_{L^2(D)} + \|\delta u_{0k}\|_{H^1(\Omega)}\big)^2 dt$$
$$\frac{|q_k|}{\|\delta u_k\|_{H^1(D)}} \leq \frac{LT}{2}\sqrt{\left(\int_D |\delta u_k|^2 dxdt\right)^{\frac{1}{2}} + \sum_{i=1}^{n}\left(\int_D \left|\frac{\partial \delta u_k}{\partial x_i}\right|^2 dxdt\right)^{\frac{1}{2}}}$$
$$\Rightarrow \lim_{k\to+\infty} \frac{|q_k|}{\|\delta u_k\|_{H^1(D)}} \leq 0$$

From where

$$r_3 = 0(\|\delta f\|_{L^2(D)} + \|\delta u_0\|_{H^1(\Omega)} + \|\delta u_1\|_{L^2(\Omega)}).$$

By taking the approximation of r_4 by virtue of the fact that D_{u_1} satisfies the Lipschitz condition. By setting

$$\begin{aligned}
q_k &= \int_D \int_0^1 \Big[D_{u_1}\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x) + \theta\delta u_{1k}\big) \\
&\quad - D_{u_1}\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\Big] \times \delta u_{1k} d\theta dx dt \\
|q_k| &\leq \frac{L}{2}\int_D |\delta u_{1k}|^2 dx dt \leq \frac{L}{2}\int_0^T \|\delta u_{1k}\|^2_{L^2(\Omega)} dt \\
|q_k| &\leq \frac{L}{2}\int_0^T \big(\|u_{1k}\|_{L^2(\Omega)} + \|\delta f_k\|_{L^2(D)} + \|\delta u_{0k}\|_{H^1(\Omega)}\big)^2 dt \\
\frac{|q_k|}{\|\delta u_k\|_{H^1(D)}} &\leq \frac{LT}{2}\sqrt{\Big(\int_D |\delta u_k|^2 dx dt\Big)^{\frac{1}{2}} + \sum_{i=1}^n \Big(\int_D \Big|\frac{\partial \delta u_k}{\partial x_i}\Big|^2 dx dt\Big)^{\frac{1}{2}}} \\
&\Rightarrow \qquad \lim_{k\to+\infty} \frac{|q_k|}{\|\delta u_k\|_{H^1(D)}} \leq 0.
\end{aligned}$$

From where

$$r_4 = 0(\|\delta f\|_{L^2(D)} + \|\delta u_0\|_{H^1(\Omega)} + \|\delta u_1\|_{L^2(\Omega)}).$$

Because $r_1 = r_2 = r_3 = r_4 = 0(\|\delta f\|_{L^2(D)} + \|\delta u_0\|_{H^1(\Omega)} + \|\delta u_1\|_{L^2(\Omega)})$. The relation (4.13) becomes:

$$\begin{aligned}
\Delta J(f^0(x;t); u_0^0(x); u_1^0(x)) &= \int_D \Big[D_f(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x))\delta u \\
&+ D_{u_0}(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x))\delta u_0 \\
&+ D_{u_1}(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x))\delta u_1 + \delta f \Phi(x;t)\Big] dx dt \\
&+ \int_D \Big[\delta u \frac{\partial \Phi(x;t)}{\partial t} + \frac{\partial \delta u}{\partial t}.\frac{\partial \Phi(x;t)}{\partial t} + D_u(u^0(x;t) f^0(x;t); u_0^0(x); u_1^0(x))\delta u \\
&- (\nabla \delta u; \nabla \Phi(x;t)) - (\rho+1) \times \big[u^0(x;t)\big]^\rho \Phi(x;t)\delta u\Big] dx dt \\
&+ \int_\Omega (\delta u_0 + \delta u_1)\Phi(x;0) dx + 0(\|\delta f\|_{L^2(D)} + \|\delta u_0\|_{H^1(\Omega)} + \|\delta u_1\|_{L^2(\Omega)}).
\end{aligned} \tag{4.14}$$

To express the writing of the derivatives of the functional J of the relation (4.14) in a convenient form, it is necessary that:

$$\int_D \Big[-\big(\nabla\delta u; \nabla\Phi(x;t)\big) + \frac{\partial\delta u}{\partial t}\frac{\partial\Phi(x;t)}{\partial t} + \delta u\frac{\partial\Phi(x;t)}{\partial t} - (\rho+1)[u^0(x;t)]^\rho\Phi(x;t)\delta u$$
$$+ D_u\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\delta u\Big] dxdt = 0,$$
$$\int_\Omega \Big[\delta u\frac{\partial\Phi(x;t)}{\partial t}\Big]_0^T dx - \int_D \Big[\Big(\frac{\partial^2\Phi(x;t)}{\partial t^2} - \frac{\partial\Phi(x;t)}{\partial t} - \Delta\Phi(x;t) + (\rho+1)[u^0(x;t)]^\rho$$
$$\cdot\Phi(x;t) - D_u\big(u^0(x;t), f^0(x;t), u_0^0(x), u_1^0(x)\big)\delta u\Big)\Big] dxdt = 0.$$

Consider the function $\Phi(x;t)$ whose trace $t = T$ is equal to 0,

$$\Phi(x;t)\big|_{t=T} = \frac{\partial\Phi(x;t)}{\partial t}\big|_{t=T} = 0.$$

Moreover $u^\varepsilon(x;t)$, $u^0(x;t)$ being the unique solution of the problem (1.1)–(1.1). $u^\varepsilon(x;t) - u^0(x;t)\big|_{t=0} = 0$. We get the following conjugate problem:

$$\begin{cases} \dfrac{\partial^2\Phi(x;t)}{\partial t^2} - \Delta\Phi(x;t) + (\rho+1)[u^0(x;t)]^\rho\Phi(x;t) - \frac{\partial\Phi(x;t)}{\partial t} = \\ \qquad D_u\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big),\ \rho > 0 \\ \Phi(x;t)|_{t=0} = \Phi_0(x),\ x\in\Omega \\ \dfrac{\partial\Phi(x;t)}{\partial t}\Big|_{t=0} = \Phi_1(x),\ x\in\Omega \\ \dfrac{\partial\Phi(x;t)}{\partial\overrightarrow{n}}|_{\partial\Omega} = 0,\ t\in]0;T[\end{cases},$$

of the nonlinear hyperbolic problem (1.1)–(1.1) in a cylinder $D = \Omega\times]0;T[$ where Ω is a bounded open set from $\mathbb{R}^n$, with differentiable boundary $\partial\Omega$, $\overrightarrow{n}$ the unit normal vector to $\partial\Omega$.

Let us show existence and uniqueness of the solution of the conjugate problem.

4.1. **Existence of the solution of the conjugate problem.**

4.1.1. *Variational formulation.* By multiplying the equation (4.15) by the function $v(x;t)$ and by integrating over D, we obtain:

$$\int_D \frac{\partial^2\Phi(x;t)}{\partial t^2}v(x;t)dxdt - \int_D \Delta\Phi(x;t)v(x;t)dxdt - \int_D \frac{\partial\Phi(x;t)}{\partial t}v(x;t)dxdt$$

$$+(\rho+1)\int_D \big[u^0(x,t)\big]^\rho \Phi(x;t)v(x;t)dxdt = \int_D D_u\big(u^0(x;t); f^0(x;t); u_0^0(x);$$
$$u_1^0(x)\big)v(x;t)dxdt. \tag{4.15}$$

Let's examine the first three terms of the equality(4.15)

$$\int_D \frac{\partial^2\Phi(x;t)}{\partial t^2}v(x;t)dxdt = \int_\Omega \Big(\frac{\partial\Phi(x;T)}{\partial t}v(x;T) - \Phi_1(x)v(x;0)\Big)dx$$
$$-\int_D \frac{\partial\Phi(x;t)}{\partial t}\frac{\partial v(x;t)}{\partial t}dxdt. \tag{4.16}$$

According to Green's formula

$$\int_D \Delta\Phi(x;t)v(x;t)dxdt = -\int_D \big(\nabla\Phi(x;t)\nabla v(x;t)\big)dxdt, \tag{4.17}$$

and

$$\int_D \frac{\partial\Phi(x;t)}{\partial t}v(x;t)dxdt = \int_\Omega \big(\Phi(x;T)v(x;T) - \Phi_0(x)v(x;0)\big)dx \tag{4.18}$$
$$-\int_D \Phi(x;t)\frac{\partial v(x;t)}{\partial t}dxdt. \tag{4.19}$$

By introducing the relations (4.16), (4.17), (4.18) in (4.15), we obtain:

$$\int_\Omega \Big(-\Phi(x;T)v(x;T) + \frac{\partial\Phi(x;T)}{\partial t}v(x;T) - \Phi_1(x)v(x;0) + \Phi_0(x)v(x;0)\Big)dx$$
$$+\int_D \big(\nabla\Phi(x;t);\nabla v(x;t)\big)dxdt - \int_D \frac{\partial\Phi(x;t)}{\partial t}\frac{\partial v(x;t)}{\partial t}dxdt + \int_D \Phi(x;t)\frac{\partial v(x;t)}{\partial t}dxdt$$
$$+(\rho+1)\int_D \big[u^0(x,t)\big]^\rho\Phi(x;t)v(x;t)dxdt = \int_D D_u\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)$$
$$\times v(x;t)dxdt.$$

Definition 4.1. *We call distributional solution of the problem* (4.15)–(4.15)*, any function* $\Phi(x;t) \in H^1(D)$ *equal to* $\Phi_0(x)$ *for* $t = 0$ *and satisfying the following integral equality:*

$$\int_\Omega \Big(-\Phi_1(x)v(x;0) + \Phi_0(x)v(x;0)\Big)dx + \int_D \big(\nabla\Phi(x;t);\nabla v(x;t)\big)dxdt$$

$$
\begin{aligned}
&-\int_D \frac{\partial \Phi(x;t)}{\partial t}\frac{\partial v(x;t)}{\partial t}dxdt + \int_D \Phi(x;t)\frac{\partial v(x;t)}{t}dxdt \\
&+ (\rho+1)\int_D \big[u^0(x,t)\big]^{\rho}\Phi(x;t)v(x;t)dxdt \\
&= \int_D D_u\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)v(x;t)dxdt,
\end{aligned}
$$

for all $v(x;t) \in H^1(D)$ whose the trace for $t = T$ is equal to 0.

4.1.2. *A priori estimate.* Multiply the equation (4.15) by a function $\dfrac{\partial \Phi(x;t)}{\partial t}$ and integrate over Ω.

$$
\begin{aligned}
&\int_\Omega \frac{\partial^2 \Phi(x;t)}{\partial t^2}\frac{\partial \Phi(x;t)}{\partial t}dx - \int_\Omega \Delta\Phi(x;t)\frac{\partial \Phi(x;t)}{\partial t}dx \\
&-\int_\Omega \frac{\partial \Phi(x;t)}{\partial t}\frac{\partial \Phi(x;t)}{\partial t}dx + +(\rho+1)\int_\Omega \big[u^0(x,t)\big]^{\rho}\Phi(x;t)\frac{\partial \Phi(x;t)}{\partial t}dx \\
&= \int_\Omega D_u\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\frac{\partial \Phi(x;t)}{\partial t}dx.
\end{aligned}
\tag{4.20}
$$

By applying Green's formula

$$
\begin{aligned}
\int_\Omega \Delta\Phi(x;t)\frac{\partial \Phi(x;t)}{\partial t}dx &= -\frac{1}{2}\frac{d}{dt}\int_\Omega \big(\nabla\Phi(x;t); \nabla\Phi(x;t)\big)dx \\
&= -\frac{1}{2}\frac{d}{dt}\|\nabla\Phi(x;t)\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{4.21}
$$

By introducing the relation (4.21) in (4.20), we obtain:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\left\|\frac{\partial \Phi(x;t)}{\partial t}\right\|^2_{L^2(\Omega)} + \|\nabla\Phi(x;t)\|^2_{L^2(\Omega)}\right] \\
&+ (\rho+1)\int_\Omega \big[u^0(x,t)\big]^{\rho}\Phi(x;t)\frac{\partial \Phi(x;t)}{\partial t}dx - \int_\Omega \left|\frac{\partial \Phi(x;t)}{\partial t}\right|^2 dx \\
&= \int_\Omega D_u\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\frac{\partial \Phi(x;t)}{\partial t}dx.
\end{aligned}
\tag{4.22}
$$

By integrating the relation (4.22) from 0 to t, we obtain:

$$
\begin{aligned}
&\Big\|\frac{\partial\Phi(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)} + \|\nabla\Phi(x;t)\|^2_{L^2(\Omega)} + 2(\rho+1)\int_\Omega \\
&\cdot\int_0^t \big[u^0(x,s)\big]^\rho \Phi(x;s)\frac{\partial\Phi(x;s)}{\partial s}dxds - 2\int_0^t\int_\Omega\Big|\frac{\partial\Phi(x;s)}{\partial s}\Big|^2 dxds \qquad (4.23)\\
&= 2\int_0^t\int_\Omega \frac{\partial\Phi(x;s)}{\partial s}D_u\big(u^0(x;s);f^0(x;s);u_0^0(x);u_1^0(x)\big)dxds \\
&+ \|\Phi_1(x)\|^2_{L^2(\Omega)} + \|\nabla\Phi_0(x)\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Thus, for all $t\in]0;T[$ we obtain:

$$
\begin{aligned}
&\Big\|\frac{\partial\Phi(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)} + \|\nabla\Phi(x;t)\|^2_{L^2(\Omega)} \\
&\le 2(\rho+1)\int_\Omega\int_0^T \big|\big[(u^0(x,t)\big]^\rho\big||\Phi(x;t)|\Big|\frac{\partial\Phi(x;t)}{\partial t}\Big|dxdt \\
&+ 2\int_0^T\int_\Omega\Big|\frac{\partial\Phi(x;t)}{\partial t}\Big|^2 dxdt \qquad (4.24)\\
&+ 2\int_0^T\int_\Omega\Big|\frac{\partial\Phi(x;t)}{\partial t}\Big||D_u\big(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x)\big)|dxdt \\
&+ \|\Phi_1(x)\|^2_{L^2(\Omega)} + \|\nabla\Phi_0(x)\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Considering $\frac{1}{2}+\frac{1}{n}+\frac{1}{q}=1$ where $q=\frac{\rho+2}{\rho+1}$, $q>0$ and applying Hölder's inequality to the term $2\int_\Omega |[u^0(x;t)]^\rho||\Phi(x;t)||\frac{\partial\Phi(x;t)}{\partial t}|dx$, we obtain:

$$
\begin{aligned}
&2\int_\Omega \big|\big[(u^0(x,t)\big]^\rho\big|\Big|\frac{\partial\Phi(x;t)}{\partial t}\Big|\big|\Phi(x;t)\big|dx \\
&\le 2\|\big[(u^0(x,t)\big]^\rho\|_{L^n(\Omega)}\Big\|\frac{\partial\Phi(x;t)}{\partial t}\Big\|_{L^2(\Omega)}\|\Phi(x;t)\|_{L^q(\Omega)}.
\end{aligned}
$$

Since $\big[(u^0(x,t)\big]^\rho$ is bounded in $L^n(\Omega)$ and $L^2(\Omega)\subset L^q(\Omega)$ we will have:

$$
2\int_\Omega \big|\big[(u^0(x,t)\big]^\rho\big|\Big|\frac{\partial\Phi(x;t)}{\partial t}\Big||\Phi(x;t)|dx \le 2c_{18}\Big\|\frac{\partial\Phi(x;t)}{\partial t}\Big\|_{L^2(\Omega)}\|\Phi(x;t)\|_{L^2(\Omega)}. \qquad (4.25)
$$

By applying Young's inequality to the second member of the equation (4.25), we obtain:

$$
\begin{aligned}
&2\int_\Omega \big|\big[(u^0(x,t)\big]^\rho\big|\Big|\frac{\partial\Phi(x;t)}{\partial t}\Big||\Phi(x;t)|dx\\
&\le c_{18}\Big(\Big\|\frac{\partial\Phi(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)}+\|\Phi(x;t)\|^2_{L^2(\Omega)}\Big).
\end{aligned}
\tag{4.26}
$$

Applying Hölder's and Young's inequality to the term

$$2\int_D |D_u\big(u^0(x;t);f^0(x;t);u^0_0(x);u^0_1(x)\big)|\Big|\frac{\partial u(x;t)}{\partial t}\Big|dxdt,$$

we get:

$$
\begin{aligned}
&2\int_D \big|D_u\big(u^0(x;t);f^0(x;t);u^0_0(x);u^0_1(x)\big)\big|\Big|\frac{\partial\Phi(x;t)}{\partial t}\Big|dxdt\\
&\le\Big(\|D_u\big(u^0(x;t);f^0(x;t);u^0_0(x);u^0_1(x)\big)\|^2_{L^2(D)}+\Big\|\frac{\partial\Phi(x;t)}{\partial t}\Big\|^2_{L^2(D)}\Big).
\end{aligned}
\tag{4.27}
$$

By introducing the relations (4.26) and (4.27) in (4.24), we obtain:

$$
\begin{aligned}
&\Big\|\frac{\partial\Phi(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)}+\|\nabla\Phi(x;t)\|^2_{L^2(\Omega)}\\
&\le(\rho+1)c_{18}\int_0^T\Big(\|\Phi(x;t)\|^2_{L^2(\Omega)}+\Big\|\frac{\partial\Phi(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)}\Big)dt\\
&+3\int_0^T\Big\|\frac{\partial\Phi(x;t)}{\partial t}\Big\|^2_{L^2(D)}dt+\|D_u\big(u^0(x;t);f^0(x;t);u^0_0(x);u^0_1(x)\big)\|^2_{L^2(D)}\\
&+\|\Phi_1(x)\|^2_{L^2(\Omega)}+\|\nabla\Phi_0(x)\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{4.28}
$$

By adding member to member the term $\|\Phi(x;t)\|^2_{L^2(\Omega)}$ to the relation (4.28), we obtain:

$$
\begin{aligned}
&\Big\|\frac{\partial\Phi(x,t)}{\partial t}\Big\|^2_{L^2(\Omega)}+\|\Phi(x;t)\|^2_{H^1(\Omega)}\\
&\le(\rho+1)c_{18}\int_O^T\Big(\|\Phi(x;t)\|^2_{L^2(\Omega)}+\Big\|\frac{\partial\Phi(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)}\Big)dt\\
&+3\int_0^T\Big\|\frac{\partial\Phi(x;t)}{\partial t}\Big\|^2_{L^2(D)}dt+\|D_u\big(u^0(x;t);f^0(x;t);u^0_0(x);u^0_1(x)\big)\|^2_{L^2(D)}\\
&+\|\Phi_1(x)\|^2_{L^2(\Omega)}+\|\nabla\Phi_0(x)\|^2_{L^2(\Omega)}+\|\Phi(x;t)\|^2_{L^2(\Omega)}.
\end{aligned}
\tag{4.29}
$$

By setting $\Phi(x;t) = \displaystyle\int_0^t \frac{\partial\Phi(x;s)}{\partial s} ds + \Phi(x;0)$,

$$|\Phi(x;t)|^2 \leq |\Phi_0(x)|^2 + 2|\Phi_0(x)| \int_0^t \Big|\frac{\partial\Phi(x;s)}{\partial s}\Big| ds + \left(\int_0^t \Big|\frac{\partial\Phi(x;s)}{\partial s}\Big| ds\right)^2.$$

By applying the Young's inequality to the term $|\Phi_0(x)| \displaystyle\int_0^t \Big|\frac{\partial\Phi(x;s)}{\partial s}\Big| ds$, we get:

$$|\Phi_0(x)| \int_0^t \Big|\frac{\partial\Phi(x;s)}{\partial s}\Big| ds \leq \frac{1}{2}|\Phi_0(x)|^2 + \frac{1}{2}\left(\int_0^t \Big|\frac{\partial\Phi(x;s)}{\partial s}\Big| ds\right)^2.$$

Thus

$$|\Phi(x;t)|^2 \leq 2\left[|\Phi_0(x)|^2 + \left(\int_0^t \Big|\frac{\partial\Phi(x;s)}{\partial s}\Big| ds\right)^2\right].$$

Similarly, applying Hölder's inequality to the term $\displaystyle\int_0^t \Big|\frac{\partial\Phi(x;s)}{\partial s}\Big| ds$, we obtain:

$$\left[\int_0^t \Big|\frac{\partial\Phi(x;s)}{\partial s}\Big| ds\right]^2 \leq \left[\left(\int_0^t ds\right)^{\frac{1}{2}} \left(\int_0^t \Big|\frac{\partial\Phi(x;s)}{\partial s}\Big|^2 ds\right)^{\frac{1}{2}}\right]^2.$$

We will have:

$$|\Phi(x;t)|^2 \leq 2\left[|\Phi_0(x)|^2 + t\int_0^t \Big|\frac{\partial\Phi(x;s)}{\partial s}\Big|^2 ds\right].$$

By taking $t \in]0;1[\subset]0;T[$, we have: $0 < 1-t < 1$ and

$$|\Phi(x;t)|^2 \leq 2\left[|\Phi_0(x)|^2 + t\int_0^t \Big|\frac{\partial\Phi(x;s)}{\partial s}\Big|^2 ds\right] + 2(1-t)\int_0^t \Big|\frac{\partial\Phi(x;s)}{\partial s}\Big|^2 ds$$

$$|\Phi(x;t)|^2 \leq 2\left[|\Phi_0(x)|^2 + \int_0^t \Big|\frac{\partial\Phi(x;s)}{\partial s}\Big|^2 ds\right]. \tag{4.30}$$

By integrating the relation (4.30) over Ω, we obtain:

$$\|\Phi(x;t)\|^2_{L^2(\Omega)} \leq 2\int_\Omega |\Phi_0(x)|^2 dx + 2\int_0^t \left(\int_\Omega \Big|\frac{\partial\Phi(x;s)}{\partial s}\Big|^2 dx\right) ds,$$

$$\|\Phi(x;t)\|^2_{L^2(\Omega)} \leq 2\|\Phi_0(x)\|^2_{L^2(\Omega)} + 2\int_0^t \Big\|\frac{\partial\Phi(x;s)}{\partial s}\Big\|^2_{L^2(\Omega)} ds.$$

Thus, for all $t \in]0;T[$ we obtain:

$$\|\Phi(x;t)\|^2_{L^2(\Omega)} \leq 2\left(\|\Phi_0(x)\|^2_{L^2(\Omega)} + \int_0^T \left\|\frac{\partial\Phi(x;t)}{\partial t}\right\|^2_{L^2(\Omega)} dt\right). \tag{4.31}$$

Introducing the relation (4.31) in (4.29), we obtain:

$$\begin{aligned}
&\left\|\frac{\partial\Phi(x;t)}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi(x;t)\|^2_{H^1(\Omega)} \\
&\leq (\rho+1)c_{18}\int_0^T \left(\|\Phi(x;t)\|^2_{H^1(\Omega)} + \left\|\frac{\partial\Phi(x;t)}{\partial t}\right\|^2_{L^2(\Omega)}\right) dt \\
&+ 5\int_0^T \left\|\frac{\partial\Phi(x;t)}{\partial t}\right\|^2_{L^2(\Omega)} dt + \|D_u\big(u^0(x;t); f^0(x;t); u^0_0(x); u^0_1(x)\big)\|^2_{L^2(D)} \\
&+ 2\|\Phi_0(x)\|^2_{L^2(\Omega)} + \|\Phi_1(x)\|^2_{L^2(\Omega)} + \|\nabla\Phi_0(x)\|^2_{L^2(\Omega)}.
\end{aligned} \tag{4.32}$$

By adding to the second member of the relation (4.32) the following terms: $5\int_0^T \|\Phi(x;t)\|^2_{H^1(\Omega)} dt$, $\|\nabla\Phi_0(x)\|^2_{L^2(\Omega)}$ and $c_{21}\int_0^T \|\nabla\Phi(x;t)\|^2_{L^2(\Omega)}$ or $c_{19} = (\rho+1)c_{18} + 5$. By setting

$$c_{20} = \|D_u\big(u^0(x;t); f^0(x;t); u^0_0(x); u^0_1(x)\big)\|^2_{L^2(D)} + \|\Phi_1(x)\|^2_{L^2(\Omega)} + 2\|\Phi_0(x)\|_{H^1(\Omega)},$$

we have:

$$\begin{aligned}
&\left\|\frac{\partial u(x;t)}{\partial t}\right\|^2_{L^2(\Omega)} + \|u(x;t)\|^2_{H^1(\Omega)} \\
&\leq c_{20} + c_{19}\int_0^T \left(\|u(x;t)\|^2_{H^1(\Omega)} + \left\|\frac{\partial u(x;t)}{\partial t}\right\|^2_{L^2(\Omega)}\right) dt.
\end{aligned} \tag{4.33}$$

By setting $F(t) = \left\|\frac{\partial u(x;t)}{\partial t}\right\|^2_{L^2(\Omega)} + \|u(x;t)\|^2_{H^1(\Omega)}$,

$$F(t) \leq c_{20} + c_{19}\int_0^T F(s)ds.$$

According to Gronwall's lemma $F(t) \leq c_{20}e^{c_{19}\int_0^T ds}$. Hence $F(t) \leq M \ a.s$ for all $t \in]0;T[$ with $M = c_{20}e^{c_{19}T}$.

$$\left\|\frac{\partial\Phi(x;t)}{\partial t}\right\|^2_{L^2(\Omega)} \leq c_{21} \quad \Rightarrow \quad \sup_{t\in]0;T[} ess\left\|\frac{\partial\Phi(x;t)}{\partial t}\right\|^2_{L^2(\Omega)} \leq c_{22}.$$

Thus we have:

$$\frac{\partial\Phi(x;t)}{\partial t} \in L^{\infty}\Big(]0;T[;L^2(\Omega)\Big), \tag{4.34}$$

$$\Big\|u(x;t)\Big\|^2_{H^1(\Omega)} \leq c_{23} \quad \Rightarrow \quad \sup_{t\in]0;T[} ess\Big\|u(x;t)\Big\|_{H^1(\Omega)} \leq c_{24}.$$

Thus we have:

$$\Phi(x;t) \in L^{\infty}\Big(]0;T[;H^1(\Omega)\Big). \tag{4.35}$$

The relation (4.32) show that $\Phi_0(x) \in H^1(\Omega)$, $\Phi_1(x) \in L^2(\Omega)$, $D_u\big(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x)\big) \in L^2(D)$; the relations (4.34) and (4.35) shows that

$$\Phi(x;t) \in L^{\infty}\Big(]0;T[;H^1(\Omega)\Big),$$
$$\frac{\partial\Phi(x;t)}{\partial t} \in L^{\infty}\big(]0;T[;L^2(\Omega)\big).$$

Theorem 4.2. *Let $D_u\big(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x)\big) \in L^2(D)$, $\Phi_0(x) \in H^1(\Omega)$ and $\Phi_1(x) \in L^2(\Omega)$ the given functions. Then the problem* (4.15)–(4.15) *admits a solution $\Phi(x;t)$ satisfying the following conditions:*

$$\Phi(x;t) \in L^{\infty}\Big(]0;T[;H^1(\Omega)\Big),$$
$$\frac{\partial\Phi(x;t)}{\partial t} \in L^{\infty}\big(]0;T[;L^2(\Omega)\big).$$

Proof. The proof of this theorem requires the use of the Faedo-Galerkin method which takes place in three steps.

Step 1: Approximate solution. Since the Hilbert space $H^1(\Omega)$ is separable, it admits a countable Hilbert basis denoted by $(w_i)_{1\leq i\leq n}$, for all $i \neq j$, $\|w_i\| = 1$ where the functions w_i are regular such that $w_i \in H^1(\Omega)$ for all i.

With the homogeneous Neumann bound condition, the operator Δ has a sequence of eigenvalues $(\lambda_i)_{i\geq 1}$ whose functions w_i are eigenfunctions associated with it. The problem is to find in any subspace V_m of $H^1(\Omega)$ generated by $w_1, w_2, \cdots, w_m$ an approximate solution Φ_m of V_m and $V_m = \{w_1, \cdots, w_m\}$ dense in $H^1(\Omega)$. We will look for $\Phi_m(x,t) = \sum_{i=1}^{m} \Phi_{im}(x;t)w_i$, solution to the problem below.

$$\begin{cases} \dfrac{\partial^2\Phi_m(x;t)}{\partial t^2} - \Delta\Phi_m(x;t) + (\rho+1)\Big[u^0(x;t)\Big]^\rho \Phi_m(x;t) - \dfrac{\partial\Phi_m(x;t)}{\partial t} = \\ \qquad D_u^m\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big);\ \rho > 0 \\ \Phi_m(x;t)\Big|_{t=0} = \Phi_{0m}(x),\ \ x \in \Omega \\ \dfrac{\partial\Phi_m(x,t)}{\partial t}\Big|_{t=0} = \Phi_{1m}(x),\ \ x \in \Omega \\ \dfrac{\partial\Phi_m(x;t)}{\partial \overrightarrow{n}}\Big|_{\partial\Omega} = 0,\ t \in]0;T[. \end{cases}$$

Let us write in the Hilbertian basis the following terms:

$$(\rho+1)u^0(x;t)\Phi_m(x;t) = (\rho+1)\sum_{i=1}^{m}\Big[u^0(x;t)\Big]^\rho \Phi_{im}(x;t)w_i,$$

$$D_u^m\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big) = \sum_{i=1}^{m} D_u^{im}\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)w_i;$$

$$\Phi_m(x;t) = \sum_{i=1}^{m}\Phi_{im}(x;t)w_i$$

Multiply the equation (4.36) by $w_k \in V$, we get:

$$\Big(\frac{\partial^2\Phi_m(x;t)}{\partial t^2}; w_k\Big) - \big(\Delta\Phi_m(x;t); w_k\big) + (\rho+1)\Big[u^0(x;t)\Big]^\rho\big(\Phi_m(x;t); w_k\big)$$
$$- \Big(\frac{\partial\Phi_m(x;t)}{\partial t}; w_k\Big) = \Big(D_u^m\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big); w_k\Big), \tag{4.36}$$

with $\Delta = -\lambda_i$. The system (4.36) is a system of nonlinear ordinary differential equations of order 2 which admits initial conditions:

$$\Phi_m(x;0) = \Phi_{0m}(x), \quad \frac{\partial\Phi_m(x;0)}{\partial t} = \Phi_{1m}(x).$$

The basis $(w_i)_{1\le i\le m}$ being orthonormal then

$$(w_i; w_k) = \delta_{ik} = \begin{cases} 1 \text{ si } i = k \\ 0 \text{ si } i \neq k \end{cases},$$

and

$$\frac{d^2\Phi_{km}(x;t)}{dt^2} - \frac{d\Phi_{km}(x;t)}{dt} + \lambda_k \Phi_{km}(x;t) + (\rho+1)[u^0(x;t)]^\rho\big(\Phi_{km}(x;t)\big) - D_u^{km}\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big) = 0.$$

We will have:

$$\Phi''_{km}(x;t) - \Phi'_{km}(x;t) + \lambda_k \Phi_{km}(x;t) + (\rho+1)[u^0(x;t)]^\rho \Phi_{km}(x;t) - D_u^{km}\big(u^0(x;t); f^0(x;); u_0^0(x); u_1^0(x)\big) = 0. \tag{4.37}$$

The relation (4.37) is written in the following matrix form:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \Phi''_{1m}(x;t) \\ \Phi''_{2m}(x;t) \\ \vdots \\ \Phi''_{mm}(x;t) \end{pmatrix} - \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \Phi'_{1m}(x;t) \\ \Phi'_{2m}(x;t) \\ \vdots \\ \Phi'_{mm}(x;t) \end{pmatrix}$$
$$+ \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{pmatrix} \times \begin{pmatrix} \Phi_{1m}(x;t) \\ \Phi_{2m}(x;t) \\ \vdots \\ \Phi_{mm}(x;t) \end{pmatrix}$$
$$+ \begin{pmatrix} (\rho+1)u^0(x;t)\Phi_{1m}(x;t) - D_u^{1m}\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big) \\ (\rho+1)u^0(x;t)\Phi_{2m}(x;t)) - D_u^{2m}\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big) \\ \vdots \\ (\rho+1)u^0(x;t)\Phi_{mm}(x;t)) - D_u^{mm}\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\big) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus, the system of equations (4.37) becomes $I_m X''(x;t) - I_m X'(x;t) + A_m X(x;t) - B_m = 0$, with $I_m = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$,

$$A_m = \begin{pmatrix} \lambda_1 + (1+\rho)[u^0(x;t)]^\rho & 0 & \cdots & 0 \\ 0 & \lambda_2 + (1+\rho)[u^0(x;t)]^\rho & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m + (1+\rho)[u^0(x;t)]^\rho \end{pmatrix},$$

$$X''(x;t) = \begin{pmatrix} \Phi''_{1m}(x;t) \\ \Phi_{2m}(x;t) \\ \vdots \\ \Phi''_{mm}(x;t) \end{pmatrix}, \ X'(x;t) = \begin{pmatrix} \Phi'_{1m}(x;t) \\ \Phi'_{2m}(x;t) \\ \vdots \\ \Phi'_{mm}(x;t) \end{pmatrix}, \ X(x;t) = \begin{pmatrix} \Phi_{1m}(x;t) \\ \Phi_{2m}(x;t) \\ \vdots \\ \Phi_{mm}(x;t) \end{pmatrix} \text{ and}$$

$$B_m = \begin{pmatrix} -D_u^{1m}\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\big) \\ \vdots \\ -D_u^{mm}\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\big) \end{pmatrix}.$$

Since $\det I_m \neq 0$, then the matrix I_m is invertible. So the system admits a solution defined in $]0;t_m[$.

Let us show that $t_m = T$ and look for the spaces in which the solution and the initial conditions of the problem (4.36)– (4.36) belong.

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \Phi''_{1m}(x;t) \\ \Phi''_{2m}(x;t) \\ \vdots \\ \Phi''_{mm}(x;t) \end{pmatrix} - \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \Phi'_{1m}(x;t) \\ \Phi'_{2m}(x;t) \\ \vdots \\ \Phi'_{mm}(x;t) \end{pmatrix}$$

$$+ \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m \end{pmatrix} \times \begin{pmatrix} \Phi_{1m}(x;t) \\ \Phi_{2m}(x;t) \\ \vdots \\ \Phi_{mm}(x;t) \end{pmatrix}$$

$$+ \begin{pmatrix} (\rho+1)u^0(x;t)\Phi_{1m}(x;t) - D_u^{1m}\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big) \\ (\rho+1)u^0(x;t)\Phi_{2m}(x;t)) - D_u^{2m}\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big) \\ \vdots \\ (\rho+1)u^0(x;t)\Phi_{mm}(x;t)) - D_u^{mm}\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\big) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus, the system of equations (4.37) becomes $I_m X''(x;t) - I_m X'(x;t) + A_m X(x;t) - B_m = 0$, with $I_m = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$,

$$A_m = \begin{pmatrix} \lambda_1 + (1+\rho)[u^0(x;t)]^\rho & 0 & \cdots & 0 \\ 0 & \lambda_2 + (1+\rho)[u^0(x;t)]^\rho & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_m + (1+\rho)[u^0(x;t)]^\rho \end{pmatrix},$$

$$X''(x;t) = \begin{pmatrix} \Phi''_{1m}(x;t) \\ \Phi_{2m}(x;t) \\ \vdots \\ \Phi''_{mm}(x;t) \end{pmatrix}, \; X'(x;t) = \begin{pmatrix} \Phi'_{1m}(x;t) \\ \Phi'_{2m}(x;t) \\ \vdots \\ \Phi'_{mm}(x;t) \end{pmatrix}, \; X(x;t) = \begin{pmatrix} \Phi_{1m}(x;t) \\ \Phi_{2m}(x;t) \\ \vdots \\ \Phi_{mm}(x;t) \end{pmatrix} \text{ and}$$

$$B_m = \begin{pmatrix} -D_u^{1m}\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\big) \\ \vdots \\ -D_u^{mm}\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\big) \end{pmatrix}.$$

Since $\det I_m \neq 0$, then the matrix I_m is invertible. So the system admits a solution defined in $]0; t_m[$.

Let us show that $t_m = T$ and look for the spaces in which the solution and the initial conditions of the problem (4.36)– (4.36) belong. □

Step 2: A priori estimate. Multiply the equation (4.36) by a function $\dfrac{\partial \Phi_m(x;t)}{\partial t}$ and integrate over Ω:

$$\begin{aligned} &\int_\Omega \frac{\partial^2 \Phi_m(x;t)}{\partial t^2}\frac{\partial \Phi_m(x;t)}{\partial t}dx - \int_\Omega \Delta\Phi_m(x;t)\frac{\partial \Phi_m(x;t)}{\partial t}dx \\ &- \int_\Omega \frac{\partial \Phi_m(x;t)}{\partial t}\frac{\partial \Phi_m(x;t)}{\partial t}dx + (\rho+1)\int_\Omega \big[u^0(x,t)\big]^\rho \Phi_m(x;t)\frac{\partial \Phi_m(x;t)}{\partial t}dx \\ &= \int_\Omega D_u^m\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\frac{\partial \Phi_m(x;t)}{\partial t}dx. \end{aligned} \tag{4.38}$$

By applying Green's formula

$$\begin{aligned} \int_\Omega \Delta\Phi_m(x;t)\frac{\partial \Phi_m(x;t)}{\partial t}dx &= -\frac{1}{2}\frac{d}{dt}\int_\Omega \big(\nabla\Phi_m(x;t); \nabla\Phi_m(x;t)\big)dx \\ &= -\frac{1}{2}\frac{d}{dt}\|\nabla\Phi_m(x;t)\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.39}$$

By introducing the relation (4.39) in (4.38), we obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\left\|\frac{\partial\Phi_m(x;t)}{\partial t}\right\|^2_{L^2(\Omega)}+\|\nabla\Phi_m(x;t)\|^2_{L^2(\Omega)}\right]\\ &+(\rho+1)\int_\Omega\big[u^0(x,t)\big]^\rho\Phi_m(x;t)\frac{\partial\Phi_m(x;t)}{\partial t}dx-\int_\Omega\left|\frac{\partial\Phi_m(x;t)}{\partial t}\right|^2dx\\ &=\int_\Omega D^m_u\big(u^0(x;t);f^0(x;t);u^0_0(x);u^0_1(x)\big)\frac{\partial\Phi_m(x;t)}{\partial t}dx.\end{aligned}\tag{4.40}$$

By integrating the equation (4.40) from 0 to t, we obtain:

$$\begin{aligned}&\left\|\frac{\partial\Phi_m(x;t)}{\partial t}\right\|^2_{L^2(\Omega)}+\|\nabla\Phi_m(x;t)\|^2_{L^2(\Omega)}+2(\rho+1)\\ &\cdot\int_\Omega\int_0^t\big[u^0(x,s)\big]^\rho\Phi_m(x;s)\frac{\partial\Phi_m(x;s)}{\partial s}dxds-2\int_0^t\int_\Omega\left|\frac{\partial\Phi_m(x;s)}{\partial s}\right|^2dxds\\ &=2\int_0^t\int_\Omega\frac{\partial\Phi_m(x;s)}{\partial s}D^m_u\big(u^0(x;s);f^0(x;s);u^0_0(x);u^0_1(x)\big)dxds\\ &+\|\Phi_{1m}(x)\|^2_{L^2(\Omega)}+\|\nabla\Phi_{0m}(x)\|^2_{L^2(\Omega)}.\end{aligned}\tag{4.41}$$

Thus, for all $t\in]0;T[$ we obtain:

$$\begin{aligned}&\left\|\frac{\partial\Phi_m(x;t)}{\partial t}\right\|^2_{L^2(\Omega)}+\|\nabla\Phi_m(x;t)\|^2_{L^2(\Omega)}\le 2(\rho+1)\\ &\cdot\int_\Omega\int_0^T\big|\big[(u^0(x,t)\big]^\rho\big||\Phi_m(x;t)|\left|\frac{\partial\Phi_m(x;t)}{\partial t}\right|dxdt+2\int_0^T\int_\Omega\left|\frac{\partial\Phi_m(x;t)}{\partial t}\right|^2dxdt\\ &+2\int_0^T\int_\Omega\left|\frac{\partial\Phi_m(x;t)}{\partial t}\right|\big|D^m_u\big(u^0(x;t);f^0(x;t);u^0_0(x);u^0_1(x)\big)\big|dxdt\\ &+\|\Phi_{1m}(x)\|^2_{L^2(\Omega)}+\|\nabla\Phi_{0m}(x)\|^2_{L^2(\Omega)}.\end{aligned}\tag{4.42}$$

Considering $\frac{1}{2}+\frac{1}{n}+\frac{1}{q}=1$ and applying Hölder's inequality to the term $2\int_\Omega\big|\big[(u^0(x,t)\big]^\rho\big||\Phi_m(x;t)|\left|\frac{\partial\Phi_m(x;t)}{\partial t}\right|dx$. We get:

$$\begin{aligned}&2\int_\Omega\big|\big[(u^0(x,t)\big]^\rho\big|\left|\frac{\partial\Phi_m(x;t)}{\partial t}\right|\big|\Phi_m(x;t)\big|dx\\ &\le 2\|\big[(u^0(x,t)\big]^\rho\|_{L^n(\Omega)}\left\|\frac{\partial\Phi_m(x;t)}{\partial t}\right\|_{L^2(\Omega)}\|\Phi_m(x;t)\|_{L^q(\Omega)}.\end{aligned}$$

Since $\big[(u^0(x,t)\big]^\rho$ is bounded in $L^n(\Omega)$ and $L^2(\Omega)\subset L^q(\Omega)$ we obtain:

$$\begin{aligned} & 2\int_\Omega \big|\big[(u^0(x,t)\big]^\rho\big|\Big|\frac{\partial\Phi_m(x;t)}{\partial t}\Big||\Phi_m(x;t)|dx \\ & \le 2c_{25}\Big\|\frac{\partial\Phi_m(x;t)}{\partial t}\Big\|_{L^2(\Omega)}\|\Phi_m(x;t)\|_{L^2(\Omega)}; \end{aligned} \tag{4.43}$$

$q = \dfrac{\rho+2}{\rho+1}$, $\rho > 0$. By applying Young's inequality to the second member of the relation (4.43), we obtain:

$$\begin{aligned} & 2\int_\Omega \big|\big[(u^0(x,t)\big]^\rho\big|\Big|\frac{\partial\Phi_m(x;t)}{\partial t}\Big||\Phi_m(x;t)|dx \\ & \le c_{25}\bigg(\Big\|\frac{\partial\Phi_m(x;t)}{\partial t}\Big\|_{L^2(\Omega)} + \|\Phi_m(x;t)\|_{L^2(\Omega)}\bigg). \end{aligned} \tag{4.44}$$

Applying Hölder's and Young's inequality to the term

$$\int_D |D_u^m\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)|\Big|\frac{\partial u(x;t)}{\partial t}\Big|dxdt.$$

We will have:

$$\begin{aligned} & \int_D \big|D_u^m\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\big|\Big|\frac{\partial\Phi_m(x;t)}{\partial t}\Big|dxdt \\ & \le \frac{1}{2}\bigg(\|D_u^m\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\|^2_{L^2(D)} + \Big\|\frac{\partial\Phi_m(x;t)}{\partial t}\Big\|^2_{L^2(D)}\bigg). \end{aligned} \tag{4.45}$$

By introducing the relations (4.45) and (4.44) in (4.42) we obtain:

$$\begin{aligned} & \Big\|\frac{\partial\Phi_m(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)} + \|\nabla\Phi_m(x;t)\|^2_{L^2(\Omega)} \\ & \le (\rho+1)c_{25}\int_0^T \bigg(\|\Phi_m(x;t)\|^2_{L^2(\Omega)} + \Big\|\frac{\partial\Phi_m(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)}\bigg)dt \\ & + 3\int_0^T \Big\|\frac{\partial\Phi_m(x;t)}{\partial t}\Big\|^2_{L^2(D)}dt + \|D_u^m\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\|^2_{L^2(D)} \\ & + \|\Phi_{1m}(x)\|^2_{L^2(\Omega)} + \|\nabla\Phi_{0m}(x)\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.46}$$

By adding member to member the term $\|\Phi_m(x;t)\|^2_{L^2(\Omega)}$ to the relation (4.46), we obtain:

$$\begin{aligned}
&\left\|\frac{\partial\Phi_m(x,t)}{\partial t}\right\|^2_{L^2(\Omega)}+\|\Phi_m(x;t)\|^2_{H^1(\Omega)}\\
&\leq(\rho+1)c_{25}\int_O^T\left(\|\Phi_m(x;t)\|^2_{L^2(\Omega)}+\left\|\frac{\partial\Phi_m(x;t)}{\partial t}\right\|^2_{L^2(\Omega)}\right)dt\\
&+3\int_0^T\left\|\frac{\partial\Phi_m(x;t)}{\partial t}\right\|^2_{L^2(D)}dt+\|D_u^m\big(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x)\big)\|^2_{L^2(D)}\\
&+\|\Phi_{1m}(x)\|^2_{L^2(\Omega)}+\|\nabla\Phi_{0m}(x)\|^2_{L^2(\Omega)}+\|\Phi_m(x;t)\|^2_{L^2(\Omega)}.
\end{aligned}\tag{4.47}$$

By setting

$$\Phi_m(x;t)=\int_0^t\frac{\partial\Phi_m(x;s)}{\partial s}ds+\Phi_m(x;0)$$

$$|\Phi_m(x;t)|^2\leq|\Phi_{0m}(x)|^2+2|\Phi_{0m}(x)|\int_0^t\left|\frac{\partial\Phi_m(x;s)}{\partial s}\right|ds+\left(\int_0^t\left|\frac{\partial\Phi_m(x;s)}{\partial s}\right|ds\right)^2.$$

Applying Young's inequality to the term $|\Phi_{0m}(x)|\int_0^t\left|\frac{\partial\Phi_m(x;s)}{\partial s}\right|ds$, We get:

$$|\Phi_{0m}(x)|\int_0^t\left|\frac{\partial\Phi_m(x;s)}{\partial s}\right|ds\leq\frac{1}{2}|\Phi_{0m}(x)|^2+\frac{1}{2}\left(\int_0^t\left|\frac{\partial\Phi_m(x;s)}{\partial s}\right|ds\right)^2,$$

thus

$$|\Phi_m(x;t)|^2\leq2\left[\left|\Phi_{0m}(x)\right|^2+\left(\int_0^t\left|\frac{\partial\Phi_m(x;s)}{\partial s}\right|ds\right)^2\right].$$

By applying Hölder's inequality to the term $\int_0^t\left|\frac{\partial\Phi_m(x;s)}{\partial s}\right|ds$, we obtain:

$$\left[\int_0^t\left|\frac{\partial\Phi_m(x;s)}{\partial s}\right|ds\right]^2\leq\left[\left(\int_0^t ds\right)^{\frac{1}{2}}\left(\int_0^t\left|\frac{\partial\Phi_m(x;s)}{\partial s}\right|^2ds\right)^{\frac{1}{2}}\right]^2,$$

$$|\Phi_m(x;t)|^2\leq2\left[|\Phi_{0m}(x)|^2+t\int_0^t\left|\frac{\partial\Phi_m(x;s)}{\partial s}\right|^2ds\right].$$

Taking $t\in]0;1[\subset]0;T[$, we have: $0<1-t<1$, and

$$|\Phi_m(x;t)|^2\leq2\left[|\Phi_{0m}(x)|^2+t\int_0^t\left|\frac{\partial\Phi_m(x;s)}{\partial s}\right|^2ds\right]+2(1-t)\int_0^t\left|\frac{\partial\Phi_m(x;s)}{\partial s}\right|^2ds,$$

$$|\Phi_m(x;t)|^2 \leq 2\bigg[|\Phi_{0m}(x)|^2 + \int_0^t \Big|\frac{\partial \Phi_m(x;s)}{\partial s}\Big|^2 ds\bigg]. \tag{4.48}$$

By integrating the equation (4.48) over Ω, we obtain:

$$\|\Phi_m(x;t)\|^2_{L^2(\Omega)} \leq 2\int_\Omega |\Phi_{0m}(x)|^2 dx + 2\int_0^t \bigg(\int_\Omega \Big|\frac{\partial \Phi_m(x;s)}{\partial s}\Big|^2 dx\bigg) ds,$$

$$\|\Phi_m(x;t)\|^2_{L^2(\Omega)} \leq 2\|\Phi_{0m}(x)\|^2_{L^2(\Omega)} + 2\int_0^t \Big\|\frac{\partial \Phi_m(x;s)}{\partial s}\Big\|^2_{L^2(\Omega)} ds.$$

Thus, for all $t \in]0;T[$ we get:

$$\|\Phi_m(x;t)\|^2_{L^2(\Omega)} \leq 2\Bigg(\|\Phi_{0m}(x)\|^2_{L^2(\Omega)} + \int_0^T \Big\|\frac{\partial \Phi_m(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)} dt\Bigg). \tag{4.49}$$

Introducing the relation (4.49) in (4.47), we obtain:

$$\begin{aligned} &\Big\|\frac{\partial \Phi_m(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)} + \|\Phi_m(x;t)\|^2_{H^1(\Omega)} \\ &\leq (\rho+1)c_{25}\int_0^T \bigg(\|\Phi_m(x;t)\|^2_{H^1(\Omega)} + \Big\|\frac{\partial \Phi_m(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)}\bigg) dt \\ &+ 5\int_0^T \Big\|\frac{\partial \Phi_m(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)} dt + \|D_u^m\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\|^2_{L^2(D)} \\ &+ 2\|\Phi_{0m}(x)\|^2_{L^2(\Omega)} + \|\Phi_{1m}(x)\|^2_{L^2(\Omega)} + \|\nabla\Phi_{0m}(x)\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.50}$$

By adding to the second member of the inequality (4.50) the following terms:

$$5\int_0^T \|\Phi_m(x;t)\|^2_{H^1(\Omega)} dt\,, \|\nabla\Phi_{0m}(x)\|^2_{L^2(\Omega)} \quad \text{and} \quad c_{26}\int_0^T \|\nabla\Phi_m(x;t)\|^2_{L^2(\Omega)},$$

where $c_{26} = (\rho+1)c_{25} + 5$.

By setting

$$c_{27} = \|D_u^m\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\|^2_{L^2(D)} + \|\Phi_{1m}(x)\|^2_{L^2(\Omega)} + 2\|\Phi_{0m}(x)\|_{H^1(\Omega)},$$

then we get:

$$\begin{aligned} &\Big\|\frac{\partial \Phi_m(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)} + \|\Phi_m(x;t)\|^2_{H^1(\Omega)} \\ &\leq c_{27} + c_{26}\int_0^T \bigg(\|\Phi_m(x;t)\|^2_{H^1(\Omega)} + \Big\|\frac{\partial \Phi_m(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)}\bigg) dt. \end{aligned} \tag{4.51}$$

By setting $F(t) = \left\|\dfrac{\partial\Phi_m(x;t)}{\partial t}\right\|^2_{L^2(\Omega)} + \|\Phi_m(x;t)\|^2_{H^1(\Omega)}$,

$$F(t) \leq c_{27} + c_{26}\int_0^T F(s)ds.$$

According to Gronwall's lemma $F(t) \leq c_{27}e^{c_{26}\int_0^T dt}$. $F(t) \leq M\ a.s$ for all $t \in]0;T[$ with $M = c_{27}e^{c_{26}T}$.

$$\left\|\frac{\partial\Phi_m(x;t)}{\partial t}\right\|^2_{L^2(\Omega)} \leq c_{28} \Longrightarrow \sup_{t\in]0;T[} ess\left\|\frac{\partial\Phi_m(x;t)}{\partial t}\right\|_{L^2(\Omega)} \leq c_{29}$$

$$\frac{\partial\Phi_m(x;t)}{\partial t} \in L^\infty\Big(]0;T[;L^2(\Omega)\Big) \tag{4.52}$$

$$\|\Phi_m(x;t)\|^2_{H^1(\Omega)} \leq c_{30} \Longrightarrow \sup_{t\in]0;T[} ess\Big\|\Phi_m(x;t)\Big\|_{H^1(\Omega)} \leq c_{31}$$

$$\Phi(x;t) \in L^\infty\Big(]0;T[;H^1(\Omega)\Big). \tag{4.53}$$

The relation (4.51) shows that $\Phi_{0m}(x) \in H^1(\Omega)$, $\Phi_{1m}(x) \in L^2(\Omega)$, $D^m_u\big(u^0(x;t);f^0(x;t);u^0_0(x);u^0_1(x)\big) \in L^2(D)$; the relations (4.52) and (4.53) show that

$$\Phi_m(x;t) \in L^\infty\Big(]0;T[;H^1(\Omega);\Big),$$
$$\frac{\partial\Phi_m(x;t)}{\partial t} \in L^\infty\big(]0;T[;L^2(\Omega)\big).$$

Since

$$\|\Phi_m(x;t)\|_{H^1(\Omega)} \leq c_{31} \text{ (independent of } m), \tag{4.54}$$

and

$$\left\|\frac{\partial\Phi_m(x;t)}{\partial t}\right\|_{H^1(\Omega)} \leq c_{29} \text{ (independent of } m), \tag{4.55}$$

we deduce that $t_m = T$.

Lemma 4.1. *Let $\mathcal{O}$ be a bounded open set of $\mathbb{R}^n_x \times \mathbb{R}_t$, g_u and g of functions of $L^q(\mathcal{O})$ where $1 < q < \infty$ such that: $\|g_u\|_{L^q(\mathcal{O})} \leq C$, $g_u \to g$ a.s in $\mathcal{O}$ then $g_u \to g$ weakly in $L^q(\mathcal{O})$.*

Step 3: Passing to the limit. The relations (4.54) and (4.55) show that, when $m \to \infty$, $\Phi_m(x;t)$ remains in a bounded set of $L^\infty\big(]0;T[;H^1(\Omega)\big)$ and $\dfrac{\partial \Phi_m(x;t)}{\partial t}$ remains in a bounded set $L^\infty\big(]0;T[;L^2(\Omega)\big)$.

The sequence $(\Phi_m(x;t))_{m\geq 1}$ is bounded in $L^\infty\big(]0;T[;H^1(\Omega)\big)$ and that the Banach space being separable, consequently we can extract a subsequence $(\Phi_\mu(x;t))_{\mu\geq 1}$ from $(\Phi_m(x;t))_{\mu\geq 1}$ such as: $\|\Phi_\mu(x;t)\|_{H^1(\Omega)} \leq c_{31}$, $\Phi_\mu(x;t) \to \Phi(x;t)$ in $H^1(\Omega)$ then $(\Phi_m(x;t))_{\mu\geq 1} \to \Phi(x;t)$ weakly in $L^\infty\big(]0;T[;H^1(\Omega)\big)$.

Moreover the sequence $\left(\frac{\partial \Phi_m(x;t)}{\partial t}\right)_{m\geq 1}$ is bounded in $L^\infty\big(]0;T[;L^2(\Omega)\big)$ and the space being Banach separable, therefore we can extract a subsequence $\left(\frac{\partial \Phi_\mu(x;t)}{\partial t}\right)_{\mu\geq 1}$ of $\left(\frac{\partial \Phi_m(x;t)}{\partial t}\right)_{m\geq 1}$ such that $\left\|\frac{\partial \Phi_\mu(x;t)}{\partial t}\right\|_{L^2(\Omega)} \leq c_{29}$, $\frac{\partial \Phi\ _{mu}(x;t)}{\partial t} \to \frac{\partial \Phi(x;t)}{\partial t}$ in $L^2(\Omega)$ then $\frac{\partial \Phi_\mu(x;t)}{\partial t} \to \frac{\partial \Phi(x;t)}{\partial t}$ weakly in $L^\infty\big(]0;T[;L^2(\Omega)\big)$.

According to the equation (4.36), we consider

$$\Delta:\ H^1(\Omega) \to H^{-1}(\Omega) \Rightarrow \Delta \in \mathcal{L}\big(H^1(\Omega);H^{-1}(\Omega)\big),$$

$$\Phi_m(x;t) \in L^\infty\big(]0;T[;H^1(\Omega)\big) \Rightarrow \Delta\Phi_m(x;t) \in L^\infty\big(]0;T[;H^{-1}(\Omega)\big).$$

The space $L^\infty\big(]0;T[;H^{-1}(\Omega)\big)$ being a separable Banach space, we can extract a subsequence $\big(\Delta\Phi_\mu(x;t)\big)_{\mu\geq 1}$ from $\big(\Delta\Phi_m(x;t)\big)_{m\geq 1}$ such that: $\|\Delta\Phi_\mu(x;t)\|_{H^{-1}(\Omega)} \leq c_{32}$, $\Delta\Phi_\mu(x;t) \to \Delta\Phi(x;t)$ in $H^{-1}(\Omega)$ then $\Delta\Phi_\mu(x;t) \to \Delta\Phi(x;t)$ weakly in $L^\infty\big(]0;T[;H^{-1}(\Omega)\big)$, and

$$\begin{aligned}\frac{\partial^2 \Phi_m(x;t)}{\partial t^2} &\in L^\infty\big((]0;T[;H^{-1}(\Omega)\big) + L^\infty\big(]0;T[;L^2(\Omega)\big)\\ &+ L^\infty\big(]0;T[;L^n(\Omega)\cap H^1(\Omega)\big).\end{aligned}$$

In particular $\frac{\partial^2 \Phi_m(x;t)}{\partial t^2} \in L^\infty\big(]0;T[;L^2(\Omega)+H^{-1}(\Omega)\big)$.

The space $L^\infty\big(]0;T[;H^{-1}(\Omega)+L^2(\Omega)\big)$ being a separable Banach space, we can extract a subsequence $\left(\frac{\partial^2 \Phi_\mu(x;t)}{\partial t^2}\right)_{\mu\geq 1}$ of $\left(\frac{\partial^2 \Phi_m(x;t)}{\partial t^2}\right)_{m\geq 1}$ such that:

$$\left\|\frac{\partial^2 \Phi_m(x;t)}{\partial t^2}\right\|_{H^{-1}(\Omega)+L^2(\Omega)} \leq c_{33},\ \frac{\partial^2 \Phi_\mu(x;t)}{\partial t^2} \to \frac{\partial^2 \Phi(x;t)}{\partial t^2} \text{ dans } H^{-1}(\Omega)+L^2(\Omega),$$

then $\frac{\partial^2 \Phi_\mu(x;t)}{\partial t^2} \to \frac{\partial^2 \Phi(x;t)}{\partial t^2}$ weakly in $L^\infty\big(]0;T[;H^{-1}(\Omega)+L^2(\Omega)\big)$.

Also $\big(D_u^m\big(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x)\big)_{m\geq 1}$ is bounded in $L^2(D)$, we can extract a subsequence $\big(D_u^\mu\big(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x)\big)$ of $\big(D_u^m\big(u^0(x;t);f^0(x;t);u_0^0(x);$

$u_1^0(x)\big)$ such that:

$$D_u^{\mu}\big(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x)\big) \to D_u\big(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x)\big) \text{ a.s.}$$

and strong in $L^2(D)$. Since m being fixed for $m=\mu$ and multiplying the equation (4.36) by $\omega_i \in V_m$. We get:

$$\Big(\frac{\partial^2\Phi_\mu(x;t)}{\partial t^2};\omega_i\Big) - \big(\Delta\Phi_\mu(x;t);\omega_i\big) - \Big(\frac{\partial\Phi_\mu(x;t)}{\partial t};\omega_i\Big) \\ + \big((1+\rho)[u^0(x;t)]^\rho\Phi_\mu(x;t);\omega_i\big) = \big(D_u^{\mu}\big(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x)\big);\omega_i\big),$$

for $i=1,\cdots,m$. Passing to the limit, we get:

$$\Big(\frac{\partial^2\Phi(x;t)}{\partial t^2};\omega_i\Big) - \big(\Delta\Phi(x;t);\omega_i\big) - \Big(\frac{\partial\Phi(x;t)}{\partial t};\omega_i\Big) + \big((1+\rho)[u^0(x;t)]^\rho\Phi(x;t);\omega_i\big) \\ = \Big(D_u\big(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x)\big);\omega_i\Big).$$

V_m being dense in $H^1(\Omega)$, so for all $v \in H^1(\Omega)$, $\omega_i \to v$ when $i \to \infty$,

$$\Big(\frac{\partial^2\Phi(x;t)}{\partial t^2};v\Big) - \big(\Delta\Phi(x;t);v\big) - \Big(\frac{\partial\Phi(x;t)}{\partial t};v\Big) \\ + (1+\rho)[u^0(x;t)]^\rho\Phi(x;t);v\Big) = \Big(D_u\big(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x)\big);v\Big),$$

$$\frac{\partial^2\Phi(x;t)}{\partial t^2} - \Delta\Phi(x;t) - \frac{\partial\Phi(x;t)}{\partial t} + (1+\rho)[u^0(x;t)]^\rho\Phi(x;t) \\ = D_u\big(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x)\big), \tag{4.56}$$

for all $v \in H^1(\Omega)$. Check that $\frac{\partial\Phi(x;t)}{\partial t}\Big|_{t=0} = \Phi_1(x)$ and $\Phi(x;t)\Big|_{t=0} = \Phi_0(x)$.

Definition 4.2. *We call distributional solution of the problem* (4.36)–(4.36)*, any function*

$$\Phi_m(x;t) \in L^\infty\big(]0;T[;H^1(\Omega)\big)$$

equal to $\Phi_{0m}(x)$ *for all* $t=0$ *and satisfies the following integral equality:*

$$\int_\Omega \Big(\Phi_{0m}(x) - \Phi_{1m}(x;t)\Big)\Psi(x;0)dx + \int_D \Big(\Big(\nabla\Phi_m(x;t);\nabla Psi(x;t)\Big) \\ + \Phi_m(x;t)\frac{\partial\Psi(x;t)}{\partial t} + (\rho+1)\big[u^0(x;t)\big]^\rho\Phi_m(x;t)\Psi(x;t) \\ - \frac{\partial\Phi_m(x,t)}{\partial t}\frac{\partial\Psi(x;t)}{\partial t}\Big)dxdt$$

$$\int_D D_u^m\Big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\Big)\Psi(x;t)dxdt$$

for all $\Psi(x;t) \in L^\infty\big(]0;T[;H^1(\Omega)\big)$*, whose trace for* $t = T$ *is equal to* 0*.*

Since $(\Phi_{0m}(x))_{m\geq 1}$ is bounded in $H^1(\Omega)$, we can extract a subsequence $(\Phi_{0\mu}(x))_{\mu\geq 1}$ of $(\Phi_{0m}(x))_{m\geq 1}$ such that: $\Phi_{0\mu}(x) \to \Phi_0(x)$ in $H^1(\Omega)$. Similarly $(\Phi_{1m}(x))_{m\geq 1}$ is bounded in $L^2(\Omega)$, we can extract a subsequence $(\Phi_{1\mu}(x))_{\mu\geq 1}$ of $(\Phi_{1m}(x))_{m\geq 1}$ such that:

$$\Phi_{1\mu}(x) \to \Phi_1(x) \text{ in } L^2(\Omega).$$

Multiply the equation (4.56) by $\Psi(x;t)$ and integrate over $]0;T[$:

$$\int_0^T \Big(\frac{\partial^2\Phi_m(x;t)}{\partial t^2}.\Psi(x;t) - \Delta\Phi_m(x;t)\Psi(x;t) - \frac{\partial\Phi_m(x;t)}{\partial t}.\Psi(x;t) + (1+\rho)$$

$$\cdot [u^0(x;t)]^\rho\Phi(x;t)\Psi(x;t)\Big)dt = \int_0^T D_u^m\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\Psi(x;t)dt.$$

With m being fixed for $m = \mu$, we get:

$$\begin{aligned}
&-\int_0^T \frac{\partial\Phi_\mu(x;t)}{\partial t}.\frac{\partial\Psi(x;t)}{\partial t}dt - \int_0^T \Delta\Phi_\mu(x;t)\Psi(x;t)dt \\
&+\int_0^T (1+\rho)[u^0(x;t)]^\rho\Phi(x;t)\Psi(x;t)dt - \int_0^T \frac{\partial\Phi_\mu(x;t)}{\partial t}.\Psi(x;t)dt \\
&= \int_0^T D_u^\mu\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\Psi(x;t)dt + \frac{\partial\Phi_\mu(x;0)}{\partial t}\Psi(x;0).
\end{aligned}$$

Passing to the limit, we get:

$$\begin{aligned}
&-\int_0^T \frac{\partial\Phi(x;t)}{\partial t}.\frac{\partial\Psi(x;t)}{\partial t}dt + \int_0^T (1+\rho)[u^0(x;t)]^\rho\Phi(x;t)\Psi(x;t) \\
&-\int_0^T \Delta\Phi(x;t)\Psi(x;t)dt + -\int_0^T \frac{\partial\Phi(x;t)}{\partial t}.\Psi(x;t)dt \\
&= \int_0^T D_u\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\Psi(x;t)dt + \frac{\partial\Phi(x;o)}{\partial t}.\Psi(x;0).
\end{aligned} \tag{4.57}$$

On the other hand, by multiplying the equation (4.56) by $\Psi(x;t)$ and integrating over $]0;T[$ we will have:

$$-\int_0^T \frac{\partial\Psi(x;t)}{\partial t}.\frac{\partial\Psi(x;t)}{\partial t}dt + \int_0^T (1+\rho)[u^0(x;t)]^\rho\Phi(x;t)\Psi(x;t)dt$$

(4.58)
$$-\int_0^T \Delta\Phi(x;t)\Psi(x;t)dt - \int_0^T \frac{\partial\Phi(x;t)}{\partial t}.\Psi(x;t)dt$$
$$= \int_0^T D_u\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\Psi(x;t)dt + \Phi_1(x)\Psi(x;0).$$

By subtracting (4.57) and (4.58), we get:

$$\Big(\frac{\partial\Phi(x;0)}{\partial t} - \Phi_1(x)\Big)\Psi(x;0) = 0 \Rightarrow \frac{\partial\Phi(x;0)}{\partial t} = \frac{\partial\Phi(x;t)}{\partial t}\Big|_{t=0} = \Phi_1(x)$$

because $\Psi(x;0) \neq 0$. Multiply the equation (4.56) by $\Psi(x;t)$ and integrate over $]0;T[$.

$$\int_0^T \frac{\partial^2\Phi_m(x;t)}{\partial t^2}.\Psi(x;t)dt - \int_0^T \Delta\Phi_m(x;t)\Psi(x;t)dt + \int_0^T \Phi_m(x;t)\frac{\partial\Psi(x;t)}{\partial t}dt$$
$$+ \int_0^T (1+\rho)[u^0(x;t)]^\rho\Phi_m(x;t)\Psi(x;t)dt = -\Phi_m(x;0)\Psi(x;0)$$
$$+ \int_0^T D_u^m\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\Psi(x;t)dt.$$

m being fixed, for $m = \mu$ we will have:

$$\int_0^T \frac{\partial^2\Phi_\mu(x;t)}{\partial t^2}.\Psi(x;t)dt - \int_0^T \Delta\Phi_\mu(x;t)\Psi(x;t)dt - \int_0^T \Phi_\mu(x;t)\frac{\partial\Psi(x;t)}{\partial t}dt$$
$$+ \int_0^T (1+\rho)[u^0(x;t)]^\rho\Phi_\mu(x;t)\Psi(x;t)dt = -\Phi_\mu(x;0)\Psi(x;0)$$
$$+ \int_0^T D_u^\mu\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\Psi(x;t)dt.$$

Passing to the limit, we get:

$$\int_0^T \frac{\partial^2\Phi(x;t)}{\partial t^2}.\Psi(x;t)dt - \int_0^T \Delta\Phi(x;t)\Psi(x;t)dt + \int_0^T \Phi(x;t)\frac{\partial\Psi(x;t)}{\partial t}dt$$

(4.59)
$$+ \int_0^T (1+\rho)[u^0(x;t)]^\rho\Phi(x;t)\Psi(x;t)dt$$
$$= -\Phi(x;0)\Psi(x;0) + \int_0^T D_u\big(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x)\big)\Psi(x;t)dt.$$

By multiplying the equation (4.55) by $\Psi(x;t)$ and integrating over $]0 : T[$; we obtain:

$$\begin{aligned} &\int_0^T \frac{\partial^2\Phi(x;t)}{\partial t^2}.\Psi(x;t)dt - \int_0^T \Delta\Phi(x;t)\Psi(x;t)dt + \int_0^T \Phi(x;t)\frac{\partial\Psi(x;t)}{\partial t}dt \\ &+ \int_0^T \Big((1+\rho)[u^0(x;t)]^\rho\Phi(x;t)\Psi(x;t)\Big)dt \\ &= \int_0^T \big(D_u\big(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x)\big)\Psi(x;t)dt - \Phi_0(x)\Psi(x;0). \end{aligned} \tag{4.60}$$

By subtracting (4.59) and (4.60), we get

$$\big(\Phi(x;0)-\Phi_0(x)\big)\Psi(x;0) = 0 \Rightarrow \Phi(x;t)\Big|_{t=0} = \Phi_0(x) \text{ car } \Psi(x;0) \neq 0.$$

Hence $\Phi(x;t)$ is the solution of the problem (4.15)–(4.15) with the initial conditions:

$$\Phi(x;t)\mid_{t=0} = \Phi_0(x)$$

and

$$\frac{\partial\Phi(x;t)}{\partial t}\mid_{t=0} = \Phi_1(x).$$

4.2. **Uniqueness solution of the conjugate problem.**

Theorem 4.3. *Let* $\big(D_u\big(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x)\big) \in L^2(D)$*,* $u_0(x) \in H^1(\Omega)$*,* $u_1(x) \in L^2(\Omega)$ *of the given functions. Then the solution* $\Phi(x;t)$ *obtained in theorem* (4.15)–(4.15) *is unique.*

Proof. Let $\Phi(x;t)$ and $\Psi(x;t)$ be two solutions to the problem (4.15)–(4.15). Then the function $\gamma(x;t) = \Phi(x;t) - \Psi(x;t)$ checks for the following problem:

$$\begin{cases} \dfrac{\partial^2\gamma(x;t)}{\partial t^2} - \Delta\gamma(x;t) + \frac{\partial\gamma(x;t)}{\partial t} + (\rho+1)[u^0(x;t)]^\rho\gamma(x;t) = 0 \\ \dfrac{\partial\gamma(x;t)}{\partial t}\Big|_{t=0} = 0; x\in\Omega \\ \dfrac{\partial\gamma(x;t)}{\partial t}\Big|_{t=0} = 0; x\in\Omega \\ \dfrac{\partial\gamma(x;t)}{\partial\vec{n}}\Big|_{\partial\Omega} = 0; t\in]0;T[. \end{cases} \tag{4.61}$$

By multiplying the equation (4.61) by $\frac{\partial\gamma(x;t)}{\partial t}$ and integrating over Ω, we obtain:

$$\int_\Omega \Big(\frac{\partial^2\gamma(x;t)}{\partial t^2} - \Delta\gamma(x;t)\frac{\partial\gamma(x;t)}{\partial t} - \Big|\frac{\partial\gamma(x;t)}{\partial t}\Big|^2 + \big((\rho+1)[u^0(x;t)]^\rho\gamma(x;t)\big)\frac{\partial\gamma(x;t)}{\partial t}\Big)dx. \tag{4.62}$$

$$\int_\Omega \frac{\partial^2\gamma(x;t)}{\partial t^2}\frac{\partial\gamma(x;t)}{\partial t}dx = \frac{1}{2}\frac{d}{dt}\Big\|\frac{\partial\gamma(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)}. \tag{4.63}$$

According to Green's formula

$$\int_\Omega \Delta\gamma(x;t)\frac{\partial\gamma(x;t)}{\partial t} = -\frac{1}{2}\frac{d}{dt}\|\nabla\gamma(x;t)\|^2_{L^2(\Omega)}. \tag{4.64}$$

By introducing the relations (4.64) and (4.63) in (4.62), we obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big[\Big\|\frac{\partial\gamma(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)} + \|\nabla\gamma(x;t)\|^2_{L^2(\Omega)}\Big] - \Big\|\frac{\partial\gamma(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)} \\ \leq \int_\Omega(\rho+1)\big|[u^0(x;t)]^\rho\big|\gamma(x;t)| \times \frac{\partial\gamma(x;t)}{\partial t}dx.\end{aligned} \tag{4.65}$$

Considering $\frac{1}{2}+\frac{1}{n}+\frac{1}{q}=1$ where $q=\frac{\rho+2}{\rho+1}$, $q>0$ and applying Hölder's inequality to the second member of the relation (4.65), we obtain:

$$\begin{aligned}\int_\Omega\Big((\rho+1)\big|[u^0(x;t)]^\rho\big||\gamma(x;t)|\Big)\Big|\frac{\partial\gamma(x;t)}{\partial t}\Big|dx \\ \leq (\rho+1)\|\gamma(x;t)\|_{L^q(\Omega)}\|[u^0(x;t)]^\rho\|_{L^n(\Omega)} \times \Big\|\frac{\partial\gamma(x;t)}{\partial t}\Big\|_{L^2(\Omega)}.\end{aligned} \tag{4.66}$$

Moreover $[u^0(x;t)]^\rho$ is bounded in $L^n(\Omega)$ and that $L^2(\Omega)\subset L^q(\Omega)$.

The relation (4.66) becomes:

$$\begin{aligned}(\rho+1)\int_\Omega\big|[u^0(x;t)]^\rho\big||\gamma(x;t)|\Big|\frac{\partial\gamma(x;t)}{\partial t}\Big|dx \\ \leq c_{34}\|\gamma(x;t)\|_{L^2(\Omega)}\Big\|\frac{\partial\gamma(x;t)}{\partial t}\Big\|_{L^2(\Omega)}.\end{aligned} \tag{4.67}$$

By applying Young's inequality to the second member of the relation (4.67), we will have:

$$2(\rho+1)\int_\Omega\big|[u^0(x;t)]^\rho\big||\gamma(x;t)|\Big|\frac{\partial\gamma(x;t)}{\partial t}\Big|dx$$

$$\leq c_{34}\Big(\|\gamma(x;t)\|^2_{L^2(\Omega)} + \Big\|\frac{\partial\gamma(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)}\Big). \tag{4.68}$$

By introducing the relation (4.68) in (4.65), we obtain:

$$\begin{aligned}&\frac{d}{dt}\Big[\Big\|\frac{\partial\gamma(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)} + \|\nabla\gamma(x;t)\|^2_{L^2(\Omega)}\Big] - 2\Big\|\frac{\partial\gamma(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)}\\ &\leq c_{34}\Big(\|\gamma(x;t)\|^2_{L^2(\Omega)} + \Big\|\frac{\partial\gamma(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)}\Big).\end{aligned} \tag{4.69}$$

By integrating from 0 to t the relation (4.69), we obtain:

$$\begin{aligned}&\Big\|\frac{\partial\gamma(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)} + \|\nabla\gamma(x;t)\|^2_{L^2(\Omega)} - \|\nabla\gamma(x;0)\|^2_{L^2(\Omega)}\\ &-2\int_0^t\Big\|\frac{\partial\gamma(x;s)}{\partial s}\Big\|^2_{L^2(\Omega)}ds \leq c_{34}\times\int_0^t\Big(\|\gamma(x;s)\|^2_{L^2(\Omega)} + \Big\|\frac{\partial\gamma(x;s)}{\partial s}\Big\|^2_{L^2(\Omega)}\Big)ds.\end{aligned} \tag{4.70}$$

Since $\gamma(x;0) = 0$, in particular $\nabla\gamma(x;0) = 0$. Thus, the relation (4.70) becomes:

$$\begin{aligned}&\Big\|\frac{\partial\gamma(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)} + \|\nabla\gamma(x;t)\|^2_{L^2(\Omega)} - 2\int_0^t\Big\|\frac{\partial\gamma(x;s)}{\partial s}\Big\|^2_{L^2(\Omega)}ds\\ &\leq c_{34}\times\int_0^t\Big(\|\gamma(x;s)\|^2_{L^2(\Omega)} + \Big\|\frac{\partial\gamma(x;s)}{\partial s}\Big\|^2_{L^2(\Omega)}\Big)ds.\end{aligned} \tag{4.71}$$

By adding the term $\|\gamma(x;t)\|^2_{L^2(\Omega)}$ member to member of the relation (4.71), we obtain:

$$\begin{aligned}&\Big\|\frac{\partial\gamma(x;t)}{\partial t}\Big\|^2_{L^2(\Omega)} + \|\gamma(x;t)\|^2_{H^1(\Omega)} - 2\int_0^t\Big\|\frac{\partial\gamma(x;s)}{\partial s}\Big\|^2_{L^2(\Omega)}ds\\ &\leq c_{34}\int_0^t\Big(\Big\|\frac{\partial\gamma(x;s)}{\partial s}\Big\|^2_{L^2(\Omega)} + \|\gamma(x;s)\|^2_{L^2(\Omega)}\Big)ds + \|\gamma(x;t)\|^2_{L^2(\Omega)}.\end{aligned} \tag{4.72}$$

By setting $\gamma(x;t) = \int_0^t \frac{\partial\gamma(x;t)}{\partial s}ds$, by squaring both sides and by applying Hölder's inequality in its second member, we obtain:

$$|\gamma(x;t)|^2 \leq t\int_0^t\Big|\frac{\partial\gamma(x;s)}{\partial s}\Big|^2 ds. \tag{4.73}$$

By integrating the relation (4.73) over Ω, we obtain:

$$\|\gamma(x;t)\|^2_{L^2(\Omega)} \leq t\int_0^t\Big\|\frac{\partial\gamma(x;s)}{\partial s}\Big\|^2_{L^2(\Omega)}ds \tag{4.74}$$

Taking $t \in]0;1[\subset]0;T[$ we have $0 < 1-t < 1$,

$$\|\gamma(x;t)\|^2_{L^2(\Omega)} \leq t\int_0^t \left|\frac{\partial\gamma(x;s)}{\partial s}\right|^2_{L^2(\Omega)} ds + (1-t)\int_0^t \left\|\frac{\partial\gamma(x;s)}{\partial s}\right\|^2_{L^2(\Omega)} ds$$

$$\|\gamma(x;t)\|^2_{L^2(\Omega)} \leq \int_0^t \left\|\frac{\partial\gamma(x;s)}{\partial s}\right\|^2_{L^2(\Omega)} ds \tag{4.75}$$

By introducing the relation (4.75) in (4.72), we get:

$$\begin{aligned}&\left\|\frac{\partial\gamma(x;t)}{\partial t}\right\|^2_{L^2(\Omega)} + \|\gamma(x;t)\|^2_{H^1(\Omega)}\\ &\leq c_{34}\int_0^t \left(\left\|\frac{\partial\gamma(x;s)}{\partial s}\right\|^2_{L^2(\Omega)} + \|\gamma(x;s)\|^2_{L^2(\Omega)}\right) ds + 3\int_0^t \left\|\frac{\partial\gamma(x;s)}{\partial s}\right\|^2_{L^2(\Omega)} ds.\end{aligned} \tag{4.76}$$

By adding the terms $c_{35}\int_0^t \|\nabla\gamma(x;s)\|^2_{L^2(\Omega)} ds$, $3\displaystyle\int_0^t \|\gamma(x;t)\|^2_{L^2(\Omega)}$ in the second member of the relation (4.76) and setting $c_{35} = c_{34} + 3$, we get:

$$\begin{aligned}&\left\|\frac{\partial\gamma(x;t)}{\partial t}\right\|^2_{L^2(\Omega)} + \|\gamma(x;t)\|^2_{H^1(\Omega)}\\ &\leq c_{35}\int_0^t \left(\left\|\frac{\partial\gamma(x;s)}{\partial s}\right\|^2_{L^2(\Omega)} + \|\gamma(x;s)\|^2_{H^1(\Omega)}\right) ds.\end{aligned} \tag{4.77}$$

For all $t \in]0;T[$ the relation (4.77) becomes

$$\left\|\frac{\partial\gamma(x;t)}{\partial t}\right\|^2_{L^2(\Omega)} + \|\gamma(x;t)\|^2_{H^1(\Omega)} \leq c_{35}\int_0^T \left(\left\|\frac{\partial\gamma(x;s)}{\partial s}\right\|^2_{L^2(\Omega)} + \|\gamma(x;s)\|^2_{H^1(\Omega)}\right) ds.$$

According to Gronwall's lemma $\left\|\frac{\partial\gamma(x;t)}{\partial t}\right\|^2_{L^2(\Omega)} + \|\gamma(x;t)\|^2_{H^1(\Omega)} \leq 0$ a.s for all $t \in]0;T[$ we have

$$\begin{cases}\frac{\partial\gamma(x;t)}{\partial t} = 0\\ \gamma(x;t) = 0\end{cases} \quad\Rightarrow\quad \gamma(x;t) = 0 \quad\Leftrightarrow\quad \Phi(x;t) = \Psi(x;t).$$

Hence the solution $\Phi(x;t)$ of the problem (4.15)–(4.15) is unique. □

The solution $\Phi(x;t)$ of the problem (4.15)–(4.15) being unique, therefore the final increment of the functional J at the point $\left(f^0(x;t), u_0^0(x), u_1^0(x)\right)$ becomes:

$$\begin{aligned}&\Delta J(f^0(x;t), u_0^0(x), u_1^0(x))\\ &= \int_D \left[D_f(f^0(x;t), u_0^0(x), u_1^0(x)) + \Phi(x;t)\right)\delta f\end{aligned}$$

$$+ D_{u_0}(f^0(x;t);u_0^0(x);u_1^0(x)) + \Phi(x;0))\delta u_0$$
$$+ (D_{u_1}(f^0(x;t),u_0^0(x);u_1^0(x)) + \Phi(x;0))\delta u_1\big]dxdt$$
$$+ 0(\|\delta f\|_{L^2(D)} + \|\delta u_0\|_{H^1(\Omega)} + \|\delta u_1\|_{L^2(\Omega)}).$$

Consider the following Hamiltonian functions:

$$H^{(1)}(u(x;t);f(x;t);u_0(x);u_1(x);\Phi(x;t))$$
$$= D(u(x;t);f(x;t):u_0(x);u_1(x)) + \Phi(x;t)f(x;t),$$
$$H^{(2)}(u(x;t);f(x;t);u_0(x);u_1(x);\Phi(x;0))$$
$$= D(u(x,t);f(x;t);u_0(x);u_1(x)) + \frac{1}{T}\int_0^T \Phi(x;0)u_0(x)dt,$$
$$H^{(3)}(u(x;t),f(x;t),u_0(x);u_1(x);\Phi(x;0)$$
$$= D(u(x;t);f(x;t);u_0(x);u_1(x)) + \frac{1}{T}\int_0^T \Phi(x;0)u_1(x)dt.$$

On the one hand the increment of the functional J at the point $(f^0(x;t)u_0^0(x);u_1^0(x))$ is written:

$$\Delta J(f^0(x;t);u_0^0(x);u_1^0(x))$$
$$= \int_D \left[\frac{\partial H^{(1)}}{\partial f}(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x);\Phi(x;t))\delta f\right.$$
$$+ \frac{\partial H^{(2)}}{\partial u_0}(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x);\Phi(x;0))\delta u_0$$
$$\left.+ \frac{\partial H^{(3)}}{\partial u_1}(u^0(x;t);f^0(x;t);u_0^0(x);u_1^0(x);\Phi(x,0)) \times \delta u_1\right]dxdt$$
$$+ 0(\|\delta f\|_{L^2(D)} + \|\delta u_0\|_{H^1(\Omega)} + \|\delta u_1\|_{L^2(\Omega)}).$$

On the other hand the increment of the functional J at the point $(f^0(x;t);u_0^0(x);u_1^0(x))$ is written as follows:

$$\Delta J(f^0(x;t);u_0^0(x);u_1^0(x))$$
$$= J_f(f^0(x;t);u_0^0(x);u_1^0(x))\delta f + J_{u_0}(f^0(x;t);u_0^0(x);u_1^0(x))\delta u_0$$
$$+ J_{u_1}(f^0(x;t);u_0^0(x);u_1^0(x))\delta u_1 + 0(\|\delta f\|_{L^2(D)} + \|\delta u_0\|_{H^1(\Omega)} + \|\delta u_1\|_{L^2(\Omega)}).$$

Hence the functional J is differentiable at the point $(f^0(x;t); u_0^0(x); u_1^0(x))$ in the sense of Fréchet relative the variables $f(x;t)$, $u_0(x)$, $u_1(x)$ and its partial derivatives at the point $(f^0(x;t); u_0^0(x); u_1^0(x))$ are written as follows:

$$J_f(f^0(x;t); u_0^0(x); u_1^0(x)) = \int_D \frac{\partial H^{(1)}}{\partial f}(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x); \Phi(x;t))dxdt$$

$$J_{u_0}(f^0(x;t); u_0^0(x); u_1^0(x)) = \int_D \frac{\partial H^{(2)}}{\partial u_0}(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x); \Phi(x;0))dxdt$$

$$J_{u_1}(f^0(x;t); u_0^0(x); u_1^0(x)) = \int_D \frac{\partial H^{(3)}}{\partial u_1}(u^0(x;t); f^0(x;t); u_0^0(x); u_1^0(x); \Phi(x;0))dxdt.$$

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