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SOME CHARACTERIZATIONS OF TIMELIKE HELICES WITH THE F-CONSTANT VECTOR FIELD IN MINKOWSKI SPACE E_1^3

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ABSTRACT. In this paper we give characterizations of timelike normal, rectifying and osculating helices in Minkowski space E_1^3. Moreover, we examine the characterization of timelike helices whose axis U perpendicular to the F-constant vector field X, which is a generalization of these helices.

1. INTRODUCTION

We consider Minkowski space E_1^3 endowed with the Lorentzian metric

$$\langle , \rangle_L = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a coordinate system of R^3.Let $x \in E_1^3$ be given. The vector x is called spacelike, timelike and lightlike, if $\langle x, x \rangle > 0$ or $x = 0$, $\langle x, x \rangle < 0$ and $\langle x, x \rangle = 0$ respectively. The magnitude of a vector x is defined by $\|x\| = \sqrt{|\langle x, x \rangle|}$ [19].

Let $\alpha : I \subset \mathbb{R} \to E_1^3$ be a regular curve, i.e. $\alpha'(s) \neq 0$. The curve α is called spacelike, timelike and null, if $\alpha'(s)$ is spacelike, timelike and null(lightlike) for all $s \in I$, respectively [19].

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In Minkowski space, $\alpha : I \subset \mathbb{R} \to E_1^3$ is a circle if and only if the curvature is a non-zero constant and the torsion is zero [2].

Let $\alpha : I \to E_1^3$ be a non-null unit speed curve, $F(s) = \{F_1(s), F_2(s), F_3(s)\}$ be a moving orthonormal frame at $\alpha(s)$ and $\{e_1, e_2, e_3\}$ be the natural basis of E_1^3 Given a vector field X along the curve α, then

$$X' = \frac{d_r}{ds}(X) + D_F \times X,$$

where $\frac{d_r}{ds}(X)$ is the rate of change of X in the moving orthonormal frame F and the vector field D_F is the Darboux vector for the frame F. If $X' = D_F \times X$, then the vector field X is called constant with respect to the frame F (or F-constant vector field) [3].

The Frenet frame of the timelike curve α denoted by $F = \{T, N, B\}$ is moving frame and the Darboux vector D_F for the frame F is given by

$$D_F = \tau T + \kappa B,$$

where τ and κ are the torsion and the curvature functions of α, respectively. The Frenet vector fields are F-constant vector field. The Frenet equations are given as

$$\begin{aligned} T' &= \kappa N \\ N' &= \kappa T + \tau B \\ B' &= -\tau N \end{aligned} \tag{1.1}$$

with

$$\kappa = \left\langle T', N \right\rangle, \; \tau = \left\langle N', B \right\rangle.$$

Given a vector field $X \neq 0$ along the curve $\alpha : I \to E_1^3$, X is called a normal, rectifying and osculating vector field, if the vector $X(s)$ lies in the normal, rectifying and osculating plane for all $s \in I$, respectively.

A curve α is called cylindrical helix if its tangent vector field T makes a constant angle with a fixed direction and is called slant helix if its principal normal vector field N makes a constant angle with a fixed direction. Now we define the helix by taking a F-constant vector field X instead of the vector fields T or N. Also a curve α is called helix if a F-constant vector field X makes a constant angle with a fixed direction U. The vector field U is an axis of the helix. α is called normal, rectifying and osculating helix, if X is a normal, rectifying and osculating vector field. Without loss of generality, we will take the vector field X as a unit vector field and $\tau \neq 0$ throughout the article.

In [3] authors generalized a helix in the three dimensional Euclidean space, which the curve whose axis perpendicular to the F-constant vector field. In this paper we give characterizations of the timelike normal, rectifying and osculating helices in Minkowski space. Moreover, we examine the characterization of timelike helices whose axis U perpendicular to the F-constant vector field X, which is a generalization of these helices in E_1^3.

2. Normal Helices

We assume that $\alpha : I \to E_1^3$ is a unit timelike curve with its Frenet frame $\{T, N, B\}$, curvature κ and non-zero torsion τ. Since α is timelike curve, then T is timelike, N and B are spacelike vector fields, i.e.

$$\langle T, T\rangle = -1, \langle N, N\rangle = 1, \langle B, B\rangle = 1.$$

Let α be a normal helix with an axis U. U is a constant vector field. Since α is a normal helix, then a unit vector field X is in the plane spanned by N and B. Thus we can write

$$X = \cos\phi N + \sin\phi B \tag{2.1}$$

and X is perpendicular to

$$U = fT + g(\sin\phi N - \cos\phi B), \tag{2.2}$$

where f and g are differentiable functions and $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is a nonzero constant. By differentiation of (2.2) and using the Frenet formula (1.1) we have

$$f' + g\kappa\sin\phi = 0, \tag{2.3}$$

$$f\kappa + g'\sin\phi - g\tau\cos\phi = 0, \tag{2.4}$$

$$-g'\cos\phi + g\tau\sin\phi = 0. \tag{2.5}$$

According to (2.5), we get

$$g = e^{\tan\phi\int\tau}. \tag{2.6}$$

From (2.4) and (2.6), we have

$$f = \frac{\cos 2\phi}{\cos\phi}\rho e^{\tan\phi \int \tau}, \tag{2.7}$$

where $\rho = \dfrac{\tau}{\kappa}$. Substitutiting the equations (2.6) and (2.7) into (2.3), we get the equation

$$\frac{\cos 2\phi}{\cos\phi}\left(\rho' + \rho\tau\tan\phi\right) = -\kappa\sin\phi, \tag{2.8}$$

and then

$$-\rho' = \frac{\cos\phi\sin\phi}{\cos 2\phi}\kappa + \rho\tau\tan\phi. \tag{2.9}$$

Hence we obtain

$$\frac{\kappa\cos 2\phi}{\tau^2\cos 2\phi + \kappa^2\cos^2\phi}\rho' = -\tan\phi. \tag{2.10}$$

Conversely, we assume that $\alpha : I \to E_1^3$ is a unit timelike curve satisfying (2.10) for a nonzero constant $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and we take the vector field

$$U = fT + g(\sin\phi N - \cos\phi B),$$

where $f = \dfrac{\cos 2\phi}{\cos\phi}\rho e^{\tan\phi\int\tau}$ and $g = e^{\tan\phi\int\tau}$. Then we get (2.4) and (2.5). Moreover, from (2.10) we have (2.8) and then we obtain (2.3). From (2.3), (2.4) and (2.5) we get $U' = 0$. Also the vector field U is constant. Since the vector field $X = \cos\phi N + \sin\phi B$ is perpendicular to U, then α is a normal helix.

Therefore we get following theorem.

Theorem 2.1. *Let $\alpha : I \to E_1^3$ be a unit timelike curve with its non-zero curvature κ and non-zero torsion τ. Then α is a normal helix with an axis U perpendicular to the vector field $X = \cos\phi N + \sin\phi B$ if and only if*

$$\frac{\kappa\cos 2\phi}{\tau^2\cos 2\phi + \kappa^2\cos^2\phi}\rho' = -\tan\phi,$$

for a nonzero constant $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Now we give the geometric interpretation of normal helices.

We assume that α is a normal helix with an axis U and U is perpendicular to $X = \cos\phi N + \sin\phi B$ for a nonconstant $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. X is F-constant vector field. We consider a cylinder parametrized by

$$\psi(t, y) = \alpha(t) + yU$$

and denoted by $C_{\alpha,U}$. The normal vector field of $C_{\alpha,U}$ is given by

$$Z = \psi_t \times \psi_y = T \times U = \cos\phi N + \sin\phi B$$

and then N makes a constant angle ϕ with Z. Also that is a characterization of the normal helix.

Let β be a unit planar timelike curve with its unit tangent vector T_β, principal normal vector field N_β and curvature κ_β, U be a unit vector field perpendicular to the plane and $C_{\beta,U}$ be a cylinder parametrized by $\psi(t, y) = \beta(t) + yU$. Suppose that the unit normal vector field of C is given by

$$Z(t, y) = T_\beta(t) \times U = N_\beta(t)$$

and

$$\alpha(s) = \psi(t(s), y(s)), \quad s \in I$$

is unit timelike curve in $C_{\beta,U}$ such that N makes a constant angle ϕ with Z. Then the Frenet vector fields are

$$(2.11)\qquad \begin{aligned} T(s) &= \cosh\theta(s) T_\beta(t(s)) + \sinh\theta(s) U, && (1)\\ N(s) &= \sin\phi\left(\sinh\theta(s) T_\beta(t(s)) + \cosh\theta(s) U\right) + \cos\phi Z, && (2)\\ B(s) &= -\cos\phi\left(\sinh\theta(s) T_\beta(t(s)) + \cosh\theta(s) U\right) + \sin\phi Z, && (3) \end{aligned}$$

where $\theta : I \to R$ is a differentiable function with $t'(s) = \cosh\theta(s)$ and $y'(s) = \sinh\theta(s)$. Then we calculate easily

$$(2.12)\qquad U = \sinh\theta(s) T(s) + \cosh\theta(s)(\sin\phi N(s) - \cos\phi B(s)).$$

The F-constant vector field $X = \cos\phi N + \sin\phi B$ is perpendicular to U and then α is a normal helix. Also we get the following theorem.

Theorem 2.2. *A unit timelike curve $\alpha : I \to E_1^3$ is a normal helix with an axis U if and only if α lies on a cylinder and the normal vector field of the cylinder makes a constant angle ϕ with the principal normal vector field of α.*

If $\phi = 0$, then the curve $\alpha : I \to E_1^3$ is a geodesic of the cylinder. Thus α is a cylindrical helix. The well-known theorem of Lancret is shown that cylindrical helices as the geodesics of cylinders. Therefore, Theorem 2.2 is an extension of Lancret's theorem.

Let $\alpha : I \to E_1^3$ be a unit timelike curve in a cylinder C. By differentiation of equations (1) and (2) in (2.11) and using the Frenet formula (1.1), since $\tau_\beta = 0$, we have

$$\begin{aligned} \kappa(s)N(s) &= \theta'(s)(\sinh\theta(s)T_\beta(t(s)) + \cosh\theta(s)U) + \cosh^2\theta(s)\kappa_\beta(t(s))N_\beta(t(s)) \\ -\tau(s)N(s) &= -\theta'(s)\cos\phi\,(\cosh\theta(s)T_\beta(t(s)) + \sinh\theta(s)U) \\ &\quad + \cosh\theta(s)\kappa_\beta(t(s))(\sin\phi T_\beta(t(s)) - \cos\phi\sinh\theta(s)N_\beta(t(s))). \end{aligned}$$

From these equations, we get the following theorem.

Theorem 2.3. *Let $\alpha : I \to E_1^3$, $\alpha(s) = \psi(t(s), y(s))$ be a unit timelike curve in a cylinder $C_{\beta,U}$. The normal vector field of the cylinder $C_{\beta,U}$ makes a constant angle ϕ with the principal normal vector field of α if and only if there is a differentiable function θ such that the equations hold*

$$t'(s) = \cosh\theta(s), \tag{2.13}$$

$$y'(s) = \sinh\theta(s), \tag{2.14}$$

$$\theta'(s) = -\tan\phi\cosh^2\theta(s)\kappa_\beta(t(s)). \tag{2.15}$$

In addition, the curvature and torsion of α are the following equations

$$\kappa(s) = \frac{\cosh^2\theta(s)}{\cos\phi}\kappa_\beta(t(s)), \quad \tau(s) = \sinh\theta(s)\cosh\theta(s)\kappa_\beta(t(s)). \tag{2.16}$$

Moreover, from the equations (2.16), we get

$$\frac{\tau}{\kappa}(s) = \cos\phi\tanh\theta(s).$$

If the curve β is a timelike circle of E_1^3, then its curvature is a non-zero constant and torsion is zero. In addition, since

$$(\tanh\theta)'(s) = -\tan\phi\kappa_\beta(t(s))$$

is constant, the function $\frac{\tau}{\kappa}$ is a linear function, and then the curve α is rectifying curve in E_1^3. Also normal helices lies on circular cylinders are only rectifying curves in E_1^3.

Example 1. $\beta(t) = (\sinh t, \cosh t, 0)$ *is a planar timelike circle with radius one and let* $C_{\beta,U}$ *be a cylinder parametrized by* $\psi(t,y) = \beta(t) + yU$*, where* $U = (0,0,1)$*.From Theorem 2.3 and* $\kappa_\beta = 1$*, we obtain*

$$\begin{aligned}\theta(s) &= -\arctan h(\tan(\phi)s),\\ t(s) &= \cot\phi \arcsin(\tan(\phi)s) + c_1,\\ y(s) &= -\cot\phi\sqrt{1-\tan^2(\phi)s^2} + c_2,\end{aligned}$$

where ϕ*,* c_1 *and* c_2 *are constants. Thus, normal helices in the cylinder* $C_{\beta,U}$ *is the following equation*

$$\begin{aligned}\alpha(s) &= \psi(t(s), y(s)) = (\sinh t(s), \cosh t(s), y(s))\\ &= (\sinh(\cot\phi \arcsin(\tan(\phi)s) + c_1), \cosh(\cot\phi \arcsin(\tan(\phi)s) + c_1),\\ &\quad -\cot\phi\sqrt{1-\tan^2(\phi)s^2} + c_2).\end{aligned}$$

From the equations (2.16), we get

$$\kappa(s) = \frac{\cos\phi}{\cos^2\phi - \sin^2(\phi)s^2}, \quad \tau(s) = \frac{\sin\phi\cos(\phi)s}{\cos^2\phi - \sin^2(\phi)s^2}, \tag{2.17}$$

and then the function

$$\frac{\tau}{\kappa}(s) = \sin(\phi)s$$

is linear function. Also the normal helices are rectifying curves. If we reparametrize the normal helices with the change of the parameter

$$\tan(\phi)s = \sin(\tan(\phi)t),$$

then

$$\alpha(t) = (\sinh(t + c_1), \cosh(t + c_1), -\cot\phi\cos(\tan(\phi)t) + c_2)$$

and

$$\kappa(t) = \frac{\sec\phi}{\cos^2(\tan(\phi)t)}, \quad \tau(t) = \frac{\sin(\tan(\phi)t)}{\cos^2(\tan(\phi)t)}.$$

3. Osculating Helices

We assume that $\alpha : I \to E_1^3$ is a unit timelike curve with its Frenet frame $\{T, N, B\}$, curvature κ and non-zero torsion τ. Let α be an osculating helix with an axis U. Since α is an osculating helix, then a unit vector field X is in the plane spanned by the timelike vector fieds T and the spacelike vector fieds N. The unit vector field X can be spacelike or timelike. Let's examine the two cases separately.

Case 1. Let X be spacelike vector field. Thus we can write

$$X = \sinh\phi T + \cosh\phi N, \tag{3.1}$$

and X is perpendicular to the vector field

$$U = f(\cosh\phi T + \sinh\phi N) + gB, \tag{3.2}$$

where f and g are differentiable functions and $\phi \in R$ is a nonzero constant. By differentiation of (3.2) and using the Frenet formula (1.1) we have

$$f'\cosh\phi + f\kappa\sinh\phi = 0, \tag{3.3}$$

$$f'\sinh\phi + f\kappa\cosh\phi - g\tau = 0, \tag{3.4}$$

$$f\tau\sinh\phi + g' = 0. \tag{3.5}$$

According to (3.3), we obtain

$$f = e^{-\tanh\phi\int\kappa}. \tag{3.6}$$

From (3.4) and (3.6), we have

$$g = \frac{1}{\rho\cosh\phi}e^{-\tanh\phi\int\kappa}, \tag{3.7}$$

where $\rho = \dfrac{\tau}{\kappa}$. Substitutiting the equations (3.6) and (3.7) into (3.5), we get the equation

$$\tau\sinh\phi = \frac{1}{\cosh\phi}\left(\frac{\rho'}{\rho^2} + \frac{\kappa}{\rho}\tanh\phi\right), \tag{3.8}$$

and then

$$\rho' = -\tau\tanh\phi + \tau\rho^2\sinh\phi\cosh\phi.$$

Hence we obtain

$$\frac{\kappa^2}{\kappa^2-\tau^2\cosh^2\phi}\left(\frac{\tau}{\kappa}\right)' = -\tau\tanh\phi \tag{3.9}$$

or

$$\frac{\tau}{\kappa^2-\tau^2\cosh^2\phi}\left(\frac{\kappa}{\tau}\right)' = \tanh\phi. \tag{3.10}$$

Conversely, we assume that $\alpha : I \to E_1^3$ is a unit timelike curve satisfying (3.10) for a nonzero constant $\phi \in R$ and we take the vector field

$$U = f(\cosh\phi T + \sinh\phi N) + gB,$$

where $f = e^{-\tanh\phi\int\kappa}$ and $g = \frac{1}{\rho\cosh\phi}e^{-\tanh\phi\int\kappa}$. Then we get (3.3) and (3.4). Moreover, from (3.10) we have (3.8) and then we obtain (3.5). From (3.3), (3.4) and (3.5) we get $U' = 0$. Also the vector field U is constant. Since the vector field $X = \sinh\phi T + \cosh\phi N$ is perpendicular to U, then α is an osculating helix.

Case 2. Let X be timelike vector field. Thus we can write

$$X = \cosh\phi T + \sinh\phi N,$$

and X is perpendicular to the vector field

$$U = f(\sinh\phi T + \cosh\phi N) + gB,$$

where f and g are differentiable functions and $\phi \in R$ is a nonzero constant. Similar to Case 1, it can be easily shown that

$$f = e^{-\coth\phi\int\kappa}, \quad g = -\frac{1}{\rho\sinh\phi}e^{-\coth\phi\int\kappa}$$

and then

$$\frac{\tau}{\kappa^2+\tau^2\sinh^2\phi}\left(\frac{\kappa}{\tau}\right)' = \coth\phi. \tag{3.11}$$

Conversely, we assume that $\alpha : I \to E_1^3$ is a unit timelike curve satisfying (3.11) for a nonzero constant $\phi \in R$ and then similar to Case 1, it can be easily shown that α is an osculating helix.

Therefore we get following theorems.

Theorem 3.1. *Let $\alpha : I \to E_1^3$ be a unit timelike curve with its non-zero curvature κ and non-zero torsion τ. Then α is an osculating helix with an axis U perpendicular* *to the spacelike vector field* $X = \sinh\phi T + \cosh\phi N$ *if and only if*

$$\frac{\tau}{\kappa^2 - \tau^2\cosh^2\phi}\left(\frac{\kappa}{\tau}\right)' = \tanh\phi,$$

for a nonzero constant $\phi \in R$.

Theorem 3.2. *Let* $\alpha : I \to E_1^3$ *be a unit timelike curve with its non-zero curvature* κ *and non-zero torsion* τ. *Then* α *is an osculating helix with an axis* U *perpendicular to the timelike vector field* $X = \cosh\phi T + \sinh\phi N$ *if and only if*

$$\frac{\tau}{\kappa^2 + \tau^2\sinh^2\phi}\left(\frac{\kappa}{\tau}\right)' = \coth\phi,$$

for a nonzero constant $\phi \in R$.

4. Rectifying Helices

We assume that $\alpha : I \to E_1^3$ is a unit timelike curve with its Frenet frame $\{T, N, B\}$, curvature κ and non-zero torsion τ. Let α be a rectifying helix with an axis U. Since α is a rectifying helix, then a unit vector field X is in the plane spanned by the timelike vector fieds T and the spacelike vector fieds B. The unit vector field X can be spacelike or timelike. Let's examine the two cases separately.

Case 1. Let X be spacelike vector field. Thus we can write

$$X = \sinh\phi T + \cosh\phi B,$$

and X is perpendicular to the vector field

$$U = f(\cosh\phi T + \sinh\phi B) + gN, \tag{4.1}$$

where f and g are differentiable functions and $\phi \in R$ is a nonzero constant. By differentiation of (4.1) and using the Frenet formula (1.1) we have

$$\begin{aligned} f'\cosh\phi + g\kappa &= 0, \\ f\kappa\cosh\phi - f\tau\sinh\phi + g' &= 0, \\ f'\sinh\phi + g\tau &= 0. \end{aligned}$$

According to the equations, we obtain

$$\frac{\tau}{\kappa} = \tanh\phi = \text{constant},$$

and then the curve $\alpha : I \to E_1^3$ is a cylindrical helix. Cylindrical helices are rectifying helices.

Case 2. Let X be timelike vector field. Thus we can write

$$X = \cosh\phi T + \sinh\phi B,$$

and X is perpendicular to the vector field

$$U = f(\sinh\phi T + \cosh\phi B) + gN,$$

where f and g are differentiable functions and $\phi \in R$ is a nonzero constant. Similar to Case 1, it can be easily shown that

$$\frac{\tau}{\kappa} = \coth\phi = \text{constant},$$

and then the curve $\alpha : I \to E_1^3$ is a cylindrical helix. Cylindrical helices are rectifying helices.

Therefore we get the following theorem.

Theorem 4.1. *Let $\alpha : I \to E_1^3$ be a unit timelike curve with its non-zero curvature κ and non-zero torsion τ. Then α is a rectifying helix with an axis U perpendicular to the spacelike (or timelike) vector field X if and only if the curve α is a cylindrical helix.*

5. The General Case

We assume that $\alpha : I \to E_1^3$ is a unit timelike curve with its Frenet frame $\{T, N, B\}$, curvature κ and non-zero torsion τ. The unit F-constant vector field X along α is given by the following equation

$$X = aT + bN + cB \tag{5.1}$$

with

$$\varepsilon = \langle X, X\rangle = -a^2 + b^2 + c^2 = \begin{cases} 1 & \text{if } X \text{ is spacelike} \\ -1 & \text{if } X \text{ is timelike} \\ 0 & \text{if } X \text{ is lightlike} \end{cases}$$

and perpendicular to the vector field

$$U = f(cT + aB) + g(-cN + bB), \tag{5.2}$$

where f and g are differentiable functions. By differentiation of (5.2) and using the Frenet formula (1.1) we have

$$f' - g\kappa = 0, \tag{5.3}$$

$$cg' - f(c\kappa - a\tau) + bg\tau = 0, \tag{5.4}$$

$$f'a + g'b - gc\tau = 0. \tag{5.5}$$

From (5.3) and (5.5), we have

$$g' = \frac{g}{b}(c\tau - a\kappa), \tag{5.6}$$

and from (5.4), we find the following equation

$$cg(c\tau - a\kappa) - bf(c\kappa - a\tau) + b^2 g\tau = 0.$$

Therefore we get

$$f = \lambda g, \qquad \text{where } \lambda = \frac{(b^2 + c^2)\tau - ac\kappa}{b(c\kappa - a\tau)}. \tag{5.7}$$

From (5.3) and (5.6), we have the equation

$$b\kappa = b\lambda' + \lambda(c\tau - a\kappa). \tag{5.8}$$

Hence, we obtain

$$\lambda' = \frac{\varepsilon c\kappa^2}{b(c\kappa - a\tau)^2}\left(\frac{\tau}{\kappa}\right)'. \tag{5.9}$$

The number ε can be zero or non-zero. Let's examine the two cases separately.

Case 1. Let ε be zero. From (5.9), $\lambda' = 0$, and then λ is a constant function. Hence, from (5.7), we obtain

$$\frac{\tau}{\kappa} = \frac{ac + \lambda bc}{b^2 + c^2 + \lambda ab} = \text{constant},$$

and then the curve $\alpha : I \to E_1^3$ is a cylindrical helix. Since every cylindrical helix is a rectifying helix, then α is a rectifying helix.

Case 2. Let ε be non-zero. From the equation (5.8), we obtain

$$\frac{\varepsilon\kappa^2}{c\kappa\left[\left(\varepsilon - c^2\right)\kappa^2 - \left(3a^2 + \varepsilon\right)\tau^2\right] + a\tau\left[(\varepsilon - 3c^2)\kappa^2 + (\varepsilon + a^2)\tau^2\right]}\left(\frac{\tau}{\kappa}\right)' = \frac{1}{b} \tag{5.10}$$

and this equation can be rewritten as

$$\varepsilon b\kappa^2\left(\frac{\tau}{\kappa}\right)' = c\kappa\left[\left(\varepsilon - c^2\right)\kappa^2 - \left(3a^2+\varepsilon\right)\tau^2\right] + a\tau\left[(\varepsilon - 3c^2)\kappa^2 + (\varepsilon + a^2)\tau^2\right]. \tag{5.11}$$

Conversely, we assume that $\alpha : I \to E_1^3$ is a unit timelike curve satisfying (5.11) for some nonzero constants $a, b, c.$ and we take the non-zero vector field

$$U = f(cT + aB) + g(-cN + bB),$$

where f and g are given by the equations (5.7) and (5.6), respectively. And then It can be easily seen that the equations (5.3), (5.4) and (5.5) are satisfied. From the equations (5.3), (5.4) and (5.5) we get $U' = 0$. Also the vector field U is constant. Since the unit F-constant vector field $X = aT + bN + cB$ along α is perpendicular to U, then α is a helix.

Therefore we get following theorems.

Theorem 5.1. *Let $\alpha : I \to E_1^3$ be a unit timelike curve with its non-zero curvature κ and non-zero torsion τ. Then α is a helix with an axis U perpendicular to the spacelike or timelike vector field $X = aT + bN + cB$ along α if and only if*

$$\varepsilon b\kappa^2\left(\frac{\tau}{\kappa}\right)' = c\kappa\left[\left(\varepsilon - c^2\right)\kappa^2 - \left(3a^2+\varepsilon\right)\tau^2\right] + a\tau\left[(\varepsilon - 3c^2)\kappa^2 + (\varepsilon + a^2)\tau^2\right],$$

for non-zero number $\varepsilon = -a^2 + b^2 + c^2 = \pm 1$.

Theorem 5.2. *Let $\alpha : I \to E_1^3$ be a unit timelike curve with its non-zero curvature κ and non-zero torsion τ. Then α is a helix with an axis U perpendicular to the lightlike vector field $X = aT + bN + cB$ (i.e $\varepsilon = 0$) if and only if the curve α is a rectifying helix.*

Corollary 5.1. *Special cases of the equation (5.11) are as follows*

(1) *If $X = T$, then α is a plane curve and (5.11) reduces to $\rho = 0$.*
(2) *If $X = N$, then α is a cylindrical helix and (5.11) reduces to $\rho' = 0$.*
(3) *If $X = B$, then α is a plane curve and (5.11) reduces to $\rho = 0$.*
(4) *If $X = bN + cB$, then α is a normal helix and (5.11) reduces to (2.10).*
(5) *If $X = aT + bN$, then α is a osculating helix and (5.11) reduces to (3.10) or (3.11).*
(6) *If $X = aT + cB$, then α is a rectifying helix and (5.11) reduces to $\rho' = 0$.*
(7) *If $X = aT + bN + cB$, then α is a helix and (5.11) reduces to (5.10).*

Now we give the geometric interpretation of helices for the general case.

Let β be a unit planar timelike curve with its unit tangent vector T_β, principal normal vector field N_β and curvature κ_β, U be a unit vector field perpendicular to the plane and $C_{\beta,U}$ be a cylinder parametrized by $\psi(t,y) = \beta(t) + yU$. Suppose that the unit normal vector field of C is given by

$$Z(t,y) = T_\beta(t) \times U = N_\beta(t)$$

and

$$\alpha(s) = \psi(t(s), y(s)), \quad s \in I$$

is unit timelike curve in $C_{\beta,U}$ such that N makes an angle $\phi(s) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with Z. Then the Frenet vector fields are

$$\begin{aligned} T(s) &= \cosh\theta(s)T_\beta(t(s)) + \sinh\theta(s)U, \\ N(s) &= \sin\phi(s)\left(\sinh\theta(s)T_\beta(t(s)) + \cosh\theta(s)U\right) + \cos\phi(s)Z, \\ B(s) &= -\cos\phi(s)\left(\sinh\theta(s)T_\beta(t(s)) + \cosh\theta(s)U\right) + \sin\phi(s)Z, \end{aligned}$$

where $\theta : I \to R$ is a differentiable function with $t'(s) = \cosh\theta(s)$ and $y'(s) = \sinh\theta(s)$. Then we calculate easily

$$U = \sinh\theta(s)T(s) + \cosh\theta(s)(\sin\phi(s)N(s) - \cos\phi(s)B(s)). \tag{5.12}$$

Since α is a helix with the F-constant vector field $X = aT + bN + cB$ that is perpendicular to U and $\cosh\theta(s) \neq 0$ (otherwise, α would be a plane curve) then we get

$$-a\tanh\theta + b\sin\phi - c\cos\phi = 0 \tag{5.13}$$

Moreover, from (5.12), we obtain following equations

$$\langle T, U\rangle = \sinh\theta, \quad \sin\phi\,\langle B, U\rangle + \cos\phi\,\langle N, U\rangle = 0.$$

Conversely, suppose that the equation (5.13) is satisfed, then it can be easily seen that α is a helix.

Also we get the following theorem.

Theorem 5.3. *Let $\alpha : I \to E_1^3$, $\alpha(s) = \psi(t(s), y(s))$ be a unit timelike curve with its non-zero curvature κ and non-zero torsion τ. Then the curve α is a helix with its axis U perpendicular to the unit F-constant vector field $X = aT + bN + cB$ along α if and* *only if α lies in a cylinder $C_{\beta,U}$ and satisfies the following equation*

$$-a\tanh\theta + b\sin\phi - c\cos\phi = 0,$$

with a, b, c are real constants and the differentiable functions $\theta : I \to R$, $\phi : I \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that the following equations hold

$$\langle T, U\rangle = \sinh\theta, \quad \sin\phi\,\langle B, U\rangle + \cos\phi\,\langle N, U\rangle = 0.$$

Corollary 5.2. *Let $\alpha : I \to E_1^3$, $\alpha(s) = \psi(t(s), y(s))$ be a unit timelike curve in a cylinder $C_{\beta,U}$. Then the curve α satisfies (5.13), for a nonconstant differentiable function $\phi : I \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if and only if the equations hold*

$$\begin{aligned}
t'(s) &= \frac{a}{\sqrt{a^2 - (b\sin\phi - c\cos\phi)^2}},\\
y'(s) &= \frac{b\sin\phi - c\cos\phi}{\sqrt{a^2 - (b\sin\phi - c\cos\phi)^2}},\\
\kappa_\beta(t(s)) &= -\frac{\phi'(b\cos\phi + c\sin\phi)}{a\tan\phi}.
\end{aligned}$$

In addition, we have the curvature and torsion of α are the following equations

$$\kappa(s) = \frac{\cosh^2\theta(s)}{\cos\phi}\kappa_\beta(t(s)) = -\frac{a\phi'(b\cos\phi + c\sin\phi)}{\sin\phi(a^2 - (b\sin\phi - c\cos\phi)^2)},$$

and

$$\tau(s) = \sinh\theta(s)\cosh\theta(s)\kappa_\beta(t(s)) = -\frac{\phi'((b^2 - c^2)\sin 2\phi - 2bc\cos 2\phi)}{2(a^2 - (b\sin\phi - c\cos\phi)^2)\tan\phi}.$$

Moreover, from the equations, we get

$$\frac{\tau}{\kappa}(s) = \cos\phi\tanh\theta(s) = \cos\phi\frac{b\sin\phi - c\cos\phi}{a}.$$

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