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SINGULAR INTEGRAL EQUATIONS FOR A CRACK SUBJECTED NORMAL STRESS IN A HEATED PLATE

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ABSTRACT. In this paper, a crack in a heated plate is investigated, subjected to normal stress. Employing the relationship between the uniform and perturbation fields, as well as complex potential functions and stresses, the problems of heat conduction and heat stress are modeled as singular integral equations. The derivatives of the crack opening displacement function and the temperature jump function serve as the unknown functions. Gauss integration rules are applied to solve the obtained equations numerically. Analysis of the stress intensity factors(SIFs) for some particular crack configurations is presented.

1. INTRODUCTION

The heated crack phenomenon has a substantial effect on the stability of materials. This has aroused the interest of academic community in studying the characteristics and behavior of heat cracks. Among the crack problems of interest are penny-shaped cracks [1, 2], cusp-type cracks [3–5], straight cracks [6–9], and curved cracks [10–14].

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Chen [6] applied the complex potential function method to establish two sets of Fredholm integral equations for multiple thermally insulated cracks without traction in an infinite plate. Zhang et al. [7] used the Gauss-Chebyshev numerical integration method to solve Cauchy singular integral equations for elliptical inclusion and straight cracks in a finite plate. Gross [10] proposed the boundary element method for solving the thermal curve crack problem. Chen and Hasebe [11] constructed the heat conduction problem of a heat curve crack as a new logarithmic integral equation, while the heat stress problem with traction-free was established using the complex potential method. Hamzah et al. [12, 13] constructed a new set of hypersingular integral equations to solve the thermally insulated crack problems in bonded dissimilar materials. Nourazar et al. [14] applied the Fourier transform method to determine the thermo-mechanical dislocation with unknown density and solved the piezoelectric plane crack problem by establishing a singular integral equation. Chen et al. [16] and Savruk et al. [20] have summarized the establishment and solving methods of equations for many types of cracks and heated cracks.

In this study, a heated crack problem with stress is formulated by the complex potential method, which is decomposed into two integral equations involving the heat conduction problem and the heat stress problem. The obtained equations are appropriate for addressing heat crack problems subjected to various stresses, such as shear stress, tearing stress, or other stresses. In these situations, it is often difficult to get a closed-form solution. However, the numerical solution method employed in this paper is able to obtain the required results directly with high precision.

2. Heat conduction problem

The physical field is described as:

$$\varphi(z) = \varphi_u(z) + \varphi_p(z),$$

where the subscript u is for the uniform field and p is for the perturbation field.

The solution of the uniform field has the following expression:

$$\varphi_u(z) = \phi_u(z) + \mathrm{i}\psi_u(z) = -\frac{q_o}{\kappa}\left(e^{-i\beta} z\right), \tag{2.1}$$

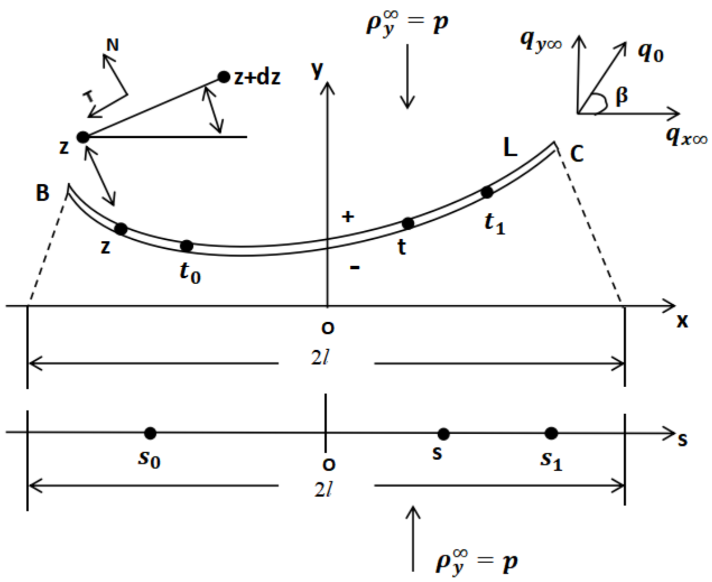


FIGURE 1. The curve crack subjected to normal stress in a heated plate and the length coordinate for curves.

where $\phi_u(z)$ denotes the temperature distribution, $\psi_u(z)$ is the heat stream function, κ is the heat conductivity, q_0 is the remote heat flux, and β is the angle between q_0 and the horizontal direction(see Fig.1).

For the perturbation field, the following complex expression is used:

$$\varphi_p(z) = \phi_p(z) + \mathrm{i}\psi_p(z) = \frac{1}{\pi \mathrm{i}} \int_L \frac{\gamma(t)}{t - z} dt, \tag{2.2}$$

where $\gamma(t)$ is the temperature jump function.

Let z approaches t_0^+ and t_0^-, Eq.(2.2) gives [15]

$$\varphi_p^{\pm}(t_o) = \pm\gamma(t_o) + \frac{1}{\pi i} \int_L \frac{\gamma(t)dt}{t - t_o} \quad (t_o \in L), \tag{2.3}$$

where " + " and " − ", respectively, represent the displacement at point t_0 on the upper and lower crack L(see Fig.1).

Hence, we have

$$\gamma(t_o) = \frac{1}{2}\left(\varphi_p^+(t_o) - \varphi_p^-(t_o)\right) \quad (t_o \in L).$$

Since the crack faces is assumed to have a thermal insulation condition, we have

$$\psi_p(t) + \psi_u(t) = 0 \quad or \quad \psi_u(t) = -\psi_p(t) \quad (t \in L). \tag{2.4}$$

From Eqs.(2.1), (2.3) and (2.4), we have

$$\frac{1}{\pi} Re \int_L \frac{\gamma(t)}{t-t_0} dt = \frac{q_o}{\kappa} Re\left(ie^{-i\beta}t_0\right) + c \quad (t_0 \in L), \tag{2.5}$$

where "Re" refers to the real part and c is a constant.

3. Heat stress problem

The complex potentials $X(z)$, $Y(z)$ and $\varphi(z)$ are described by the displacements (ω, v), the stresses $(\rho_x, \rho_y, \rho_{xy})$, and the resultant forces (K, J) as follows [16]:

$$\rho_x + \rho_y = 4ReX'(z) \tag{3.1}$$

$$\rho_y - \rho_x + 2i\rho_{xy} = 2\left[\bar{z}X''(z) + Y'(z)\right] \tag{3.2}$$

$$f = -J + iK = X(z) + z\overline{X'(z)} + \overline{Y(z)} \tag{3.3}$$

$$2G_1(\omega + iv) = \lambda X(z) - z\overline{X'(z)} - \overline{Y(z)} + (1+\lambda)\sigma_t \int \varphi(z)dz \tag{3.4}$$

where G_1 is the shear modulus of elasticity. $\lambda = 3-4\nu_1$ and $\sigma_t = G_1(1+\nu_1)\alpha_t/(2(1-\nu_1))$ are used in the plane strain problem, while $\lambda = (3-\nu_1)/(1+\nu_1)$ and $\sigma_t = G_1(1+\nu_1)\alpha_t/2$ are used in the plane stress problem, α_t represents the coefficient of thermal expansion and ν_1 is the Poisson's ratio.

The complex potential functions are expressed as follows:

$$\begin{aligned} X(z) &= -\frac{1}{2\pi}\int_{\mathrm{L}} \ln(z-t)H(t)dt \\ Y(z) &= \frac{1}{2\pi}\int_L (t-z)\overline{H(t)dt} - \frac{1}{2\pi}\int_L \frac{\bar{t}}{t-z}H(t)dt. \end{aligned} \tag{3.5}$$

After differentiating Eqs.(3.3) and (3.4) in a specified direction, we have, respectively,

$$\begin{aligned} F_1\left(z, \frac{d\bar{z}}{dz}\right) &= \frac{d}{dz}\{-J+iK\} \\ &= X'(z) + \overline{X'(z)} + \frac{d\bar{z}}{dz}\left(z\overline{X''(z)} + \overline{Y'(z)}\right) = N + iT \end{aligned} \tag{3.6}$$

$$F_2\left(z, \frac{d\bar{z}}{dz}\right) = 2G_1 \frac{d}{dz}\{\omega + iv\}$$

$$= \lambda X'(z) - \overline{X'(z)} - \frac{d\bar{z}}{dz}\left(z\overline{X''(z)} + \overline{Y'(z)}\right) + (1+\lambda)\sigma_{\mathrm{t}}\varphi(z)$$

$$= (\lambda+1)X'(z) - F_1 + (\lambda+1)\sigma_{\mathrm{t}}\varphi(z) \tag{3.7}$$

where $F_1 = N + iT$ represents the normal and tangential tractions along the small crack segment $\overline{z, z+dz}$ (see Fig.1).

For the crack opening displacement function, the following relation is derived [11]:

$$G'(t) = -\frac{2G_1 i}{1+\lambda}\frac{d}{dt}\left[(\omega(t)+iv(t))^+ - (\omega(t)+iv(t))^-\right]$$

$$= H(t) - 2i\sigma_t\gamma(t) \quad (t \in L). \tag{3.8}$$

Single-valuedness condition of displacement reads

$$\int_L G'(t)dt = 0 \quad (t \in L). \tag{3.9}$$

Substituting Eq.(3.5) into Eq.(3.6), taking the limit as z approaches t_0^+ and t_0^-, and defining $t^* = t - t_0$, the equation for the heat stress problem is established as follows:

$$\frac{1}{\pi}\int_L \frac{H(t)\,\mathrm{d}t}{t-t_0} + \frac{1}{2\pi}\int_L P_1(t^*, t_0)\,H(t)\,\mathrm{d}t + \frac{1}{2\pi}\int_L P_2(t^*, t_0)\,\overline{H(t)}\mathrm{d}\bar{t}$$

$$= N(t_0) + \mathrm{i}T(t_0) \tag{3.10}$$

where

$$P_1(t^*, t_0) = -\frac{1}{t^*} + \frac{1}{\bar{t^*}}\frac{\mathrm{d}\bar{t_0}}{\mathrm{d}t_0}, \quad P_2(t^*, t_0) = \frac{1}{\bar{t^*}} - \frac{t^*}{(\bar{t^*})^2}\frac{\mathrm{d}\bar{t_0}}{\mathrm{d}t_0}. \tag{3.11}$$

Note that dt in Eqs.(3.9),(3.10) and (3.11) is a small increment, which can be expressed as follows:

$$dt = e^{i\theta}ds$$

where ds stands for a small arc length along the crack L, while θ is an angle formed between the tangent line and the horizontal direction at point t on the crack L.

Under the condition in Eq.(3.9), substituting Eq.(3.8) into Eq.(3.10), the new singular integral equation with $G'(t)$ and $\gamma(t)$ as unknowns is obtained as follows:

$$N(t_0) + \mathrm{i}T(t_0) = \frac{1}{\pi}\int_L \frac{G'(t)\,\mathrm{d}t}{t-t_0} + \frac{1}{2\pi}\int_L P_1(t^*,t_0)\,G'(t)\,\mathrm{d}t + \frac{1}{2\pi}\int_L P_2(t^*,t_0)\,\overline{G'(t)}\mathrm{d}\bar{t}$$

$$+\frac{i\sigma_t}{\pi}\int_L P_3(t^*,t_0)\,\gamma(t)\,\mathrm{d}t + \frac{i\sigma_t}{\pi}\int_L P_2(t^*,t_0)\,\gamma(t)\,\mathrm{d}\bar{t} \quad (t_0 \in L) \tag{3.12}$$

where

$$P_3(t^*,t_0) = \frac{1}{t^*} + \frac{1}{\bar{t^*}}\frac{\mathrm{d}\bar{t_0}}{\mathrm{d}t_0}.$$

The unknown temperature jump function $\gamma(t)$ in Eq.(3.12) can be found from Eq.(2.5), and combining with Eq.(3.9), the unknown $G'(t)$ can be found.

4. The length coordinate method for curves

The curve crack is mapped one-to-one to the real axis using the length coordinate method (see Fig.1). The functions $t_1(s)$ and $t_2(s)$, based on the dislocation's properties, the functions $\gamma(t)$ and $G'(t)$ can be written in the following forms(Chen et al. [16]), respectively:

$$\gamma(t)\,|_{t_1(s)} = \sqrt{l^2-s^2}R(s) \qquad (|s|<l)$$

$$G'(t)\,|_{t_2(s)} = \frac{Q(s)}{\sqrt{l^2-s^2}} \qquad (|s|<l) \tag{4.1}$$

where $Q(s) = Q_1(s) + \mathrm{i}Q_2(s)$.

By substituting Eq.(4.1) into Eqs.(2.5), (3.9) and (3.12), and defining $s^* = s - s_0$, the singular integral equations have the following form, respectively:

$$\frac{1}{\pi}Re\int_{-l}^{l} P_4(s,s_o)\,\frac{R(s)}{(s-s_o)(\sqrt{l^2-s^2})}\mathrm{d}s = \frac{q_o}{\kappa}Re\left(ie^{-i\beta}s_0\right) + c \quad (|s_o|<l) \tag{4.2}$$

and

$$\frac{1}{\pi}\int_{-l}^{l} P_5(s,s_o)\,\frac{Q(s)}{(s-s_o)(\sqrt{l^2-s^2})}\mathrm{d}s + \frac{1}{2\pi}\int_{-l}^{l} E_1(s,s_o)\,\frac{Q(s)}{\sqrt{l^2-s^2}}\mathrm{d}s$$

$$+\frac{1}{2\pi}\int_{-l}^{l} E_2(s,s_o)\,\frac{\overline{Q(s)}}{\sqrt{l^2-s^2}}\mathrm{d}s + \frac{i\sigma_t}{\pi}\int_{-l}^{l} E_3(s,s_o)\,\frac{R(s)}{\sqrt{l^2-s^2}}\mathrm{d}s$$

$$= N(s_o) + \mathrm{i}T(s_o) \quad (|s_o|<l) \tag{4.3}$$

where

$$P_4(s,s_o)=\sqrt{l^2-s^2}\frac{s^*}{t^*}\frac{\mathrm{d}t}{\mathrm{d}s},\quad P_5(s,s_o)=\frac{s^*}{t^*}\frac{\mathrm{d}t}{\mathrm{d}s}, E_1(s,s_o)=P_1(t^*,t_o)\frac{\mathrm{d}t}{\mathrm{d}s},$$
$$E_2(s,s_o)=P_2(t^*,t_o)\frac{\mathrm{d}\bar{t}}{\mathrm{d}s},\quad E_3(s,s_o)=\left[P_3(t^*,t_o)\frac{\mathrm{d}t}{\mathrm{d}s}+P_2(t^*,t_o)\frac{\mathrm{d}\bar{t}}{\mathrm{d}s}\right]\left(l^2-s^2\right).$$

From Eq.(3.9), we get

$$\int_{-l}^{l}\frac{F(s)Q(s)}{\sqrt{l^2-s^2}}ds=0\quad(\text{where } F(s)=\frac{dt}{ds}).\tag{4.4}$$

In solving Eqs.(4.2), (4.3), and (4.4), the following Gauss integration rules are applied [17,20]:

$$\frac{1}{\pi}\int_{-l}^{l}\frac{L_1(s)Q(s)\mathrm{d}s}{\sqrt{l^2-s^2}\,(s-s_{o,k})}=\frac{1}{M}\sum_{j=1}^{M}\frac{L_1(s_j)\,Q(s_j)}{s_j-s_{o,k}}$$
$$\frac{1}{\pi}\int_{-l}^{l}\frac{L_2(s)Q(s)\mathrm{d}s}{\sqrt{l^2-s^2}}=\frac{1}{M}\sum_{j=1}^{M}L_2(s_j)\,Q(s_j)\tag{4.5}$$

where

$$s_j=l\cos\frac{(j-0.5)\pi}{M}\quad(j=1,2,\dots M)$$
$$s_{o,k}=l\cos\frac{k\pi}{M}\quad(k=1,2,\dots M-1)$$

With the help of integration rules Eq.(4.5), we take the following steps:

- **(i)** convert Eq.(4.2) into a linear system with M unknowns $R(s_j)(j=1,2,\dots M)$ and M equations;
- **(ii)** simultaneously, Eqs.(4.3) and (4.4) are also converted into a linear system with M unknowns $Q(s_j)(j=1,2,\dots M)$, M unknowns $R(s_j)$ and M equations;
- **(iii)** obtain the values of $R(s_j)$ by solving step (i) of the linear system;
- **(iv)** substitute the values of $R(s_j)$ into the linear system in step (ii) to obtain the value of $Q(s_j)$ for M unknowns.

The values of $Q(-l)$ and $Q(l)$ can be obtained by the following formula, respectively [19]:

$$Q(-l)=\frac{1}{M}\sum_{j=1}^{M}(-1)^{j+M}Q(s_j)\tan((2j-1)\pi/4M)$$

and

$$Q(l) = \frac{1}{M}\sum_{j=1}^{M}(-1)^{j+1}Q\left(s_j\right)\cot((2j-1)\pi/4M).$$

Finally, the SIF at the left(B) and right(C) tips can be determined separately using the following derived formulas:

$$(K_1 - iK_2)_B = \sqrt{2\pi}\lim_{t\to t_B}\sqrt{\mid t - t_B\mid}G'(t) = \sqrt{2\pi}\lim_{s\to -l}\sqrt{l+s}\frac{Q(s)}{\sqrt{l^2-s^2}}$$
$$= \sqrt{\frac{\pi}{l}}Q(-l),$$

and

$$(K_1 - iK_2)_C = -\sqrt{2\pi}\lim_{t\to t_C}\sqrt{|t - t_C|}G'(t) = -\sqrt{2\pi}\lim_{s\to -l}\sqrt{l-s}\frac{Q(s)}{\sqrt{l^2-s^2}}$$
$$= -\sqrt{\frac{\pi}{l}}Q(l).$$

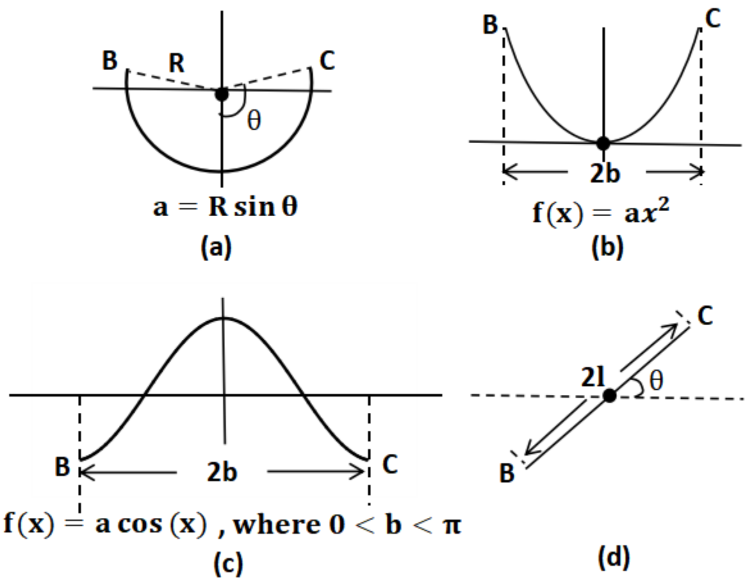


FIGURE 2. (a) A circular-arc-shaped crack, (b) a quadratic-shaped crack, (c) a cosine-shaped crack, and (d) a straight-shaped crack.

5. NUMERICAL EXAMPLES

Numerical examples for four different crack configurations (see Fig. 2) are presented. In the computation, we have used $M = 55$, $\beta = \pi/2$, $q_0 = 1$, $\sigma_t = 0.22$, $\alpha_t = 0.34$, $k = 0.34$ and the remote traction $\rho_y^\infty = p$.

5.1. **SIFs for a crack in Fig. 2(a).** The non-dimensional SIF for the crack in Fig. 2(a) is defined as follows:

$$\begin{aligned} K_{1C} &= K_{1B} = F_{1C}(\theta)p\sqrt{\pi a} \\ K_{2C} &= -K_{2B} = F_{2C}(\theta)p\sqrt{\pi a} \end{aligned} \quad (5.1)$$

where $a = Rsin\theta$.

Table 1 showns the non-dimensional SIF for the crack in Fig. 2(a) with and without heat. It is evidence that our results for crack without heat agree with the exact solution by Cotterell and Rice [18]. We observe that the Mode I non-dimensional SIF for a crack in a heated plate is higher than the crack without heat, whereas the Mode II non-dimensional SIF at tip C is smaller.

TABLE 1. Non-dimensional SIF for crack in Fig. 2(a).

θ (degrees)	10	20	30	40	50	60	70	80
$F1C*$	0.97363	0.89698	0.77793	0.62722	0.45745	0.28149	0.11076	-0.04466
$F1C_{exact}$	0.97358	0.89702	0.77790	0.62719	0.45749	0.28146	0.11074	-0.04468
$F1C_h$	0.99299	0.97043	0.92926	0.86529	0.77553	0.65971	0.52083	0.36576
$F2C*$	0.17235	0.33179	0.46729	0.57033	0.63591	0.66252	0.65111	0.60529
$F2C_{exact}$	0.17233	0.33182	0.46726	0.57030	0.63595	0.66250	0.65112	0.60530
$F2C_h$	-0.05069	-0.09791	-0.13961	-0.17650	-0.21192	-0.25144	-0.30205	-0.37027

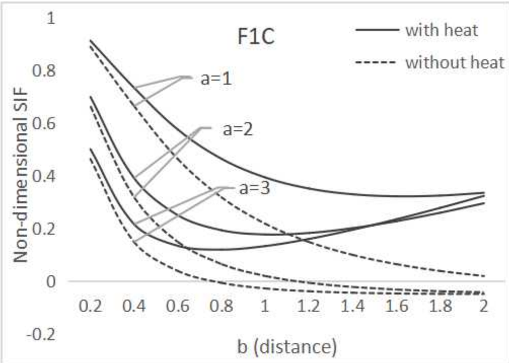
$*$ without heat; -exact solution without heat [18]; $-h$ with heat.

5.2. **SIFs for a crack in Fig. 2(b).** The non-dimensional SIF for the crack in Fig. 2(b) is defined as follows:

$$\begin{aligned} K_{1C} &= K_{1B} = F_{1C}(b)p\sqrt{\pi l} \\ K_{2C} &= -K_{2B} = F_{2C}(b)p\sqrt{\pi l} \end{aligned} \quad (5.2)$$

where $2l$ represents the crack length.

Figure 3 shows the non-dimensional SIF for the crack in Fig. 2(b) with and without heat at various parameter values, a. The results indicate that: (1) The Mode I non-dimensional SIF of a heated crack decreases with the continuous crack



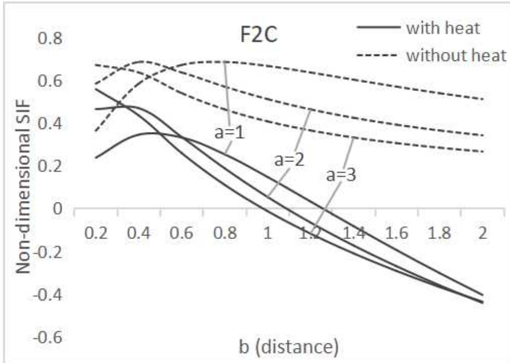


FIGURE 3. Non-dimensional SIF for crack in Fig. 2(b)

expansion, while Mode II non-dimensional SIF at tip C increases first and then decreases gradually. (2) Given the parameter a, the Mode I non-dimensional SIF for a crack in a heated plate is higher than the crack without heat, whereas the Mode II non-dimensional SIF at tip C is smaller.

5.3. **SIFs for a crack in Fig. 2(c).** The non-dimensional SIF of the crack in Fig. 2(c) is the same as Eq.(5.2).

Figure 4 depicts the relationship between the non-dimensional SIF of the crack in Fig. 2(c) under different parameter values of a and b with and without heat. It is found that: (1) When $a > 0$, the Mode I non-dimensional SIF for a crack in a heated plate is lower than the crack without heat. This behavior is due to the fact that the direction of the crack expansion in Fig. 2(c) is opposite to the remote heat flux, q_0. (2) Conversely, when $a < 0$, the Mode I non-dimensional SIF for a crack in a heated plate is higher.

5.4. **SIFs for a crack in Fig. 2(d).** The non-dimensional SIF for the crack in Fig. 2(d) is defined as follows:

$$K_{1C} = K_{1B} = F_{1C}(\theta)p\sqrt{\pi l}$$
$$K_{2B} = F_{2B}(\theta)p\sqrt{\pi l}, \quad K_{2C} = F_{2C}(\theta)p\sqrt{\pi l} \tag{5.3}$$

Figure 5 shows the non-dimensional SIF for the crack in Fig. 2(d) with and without heat at various crack lengths, l. It is found that the temperature and crack length have a small effect on the values of Mode I non-dimensional SIFs; however, Mode

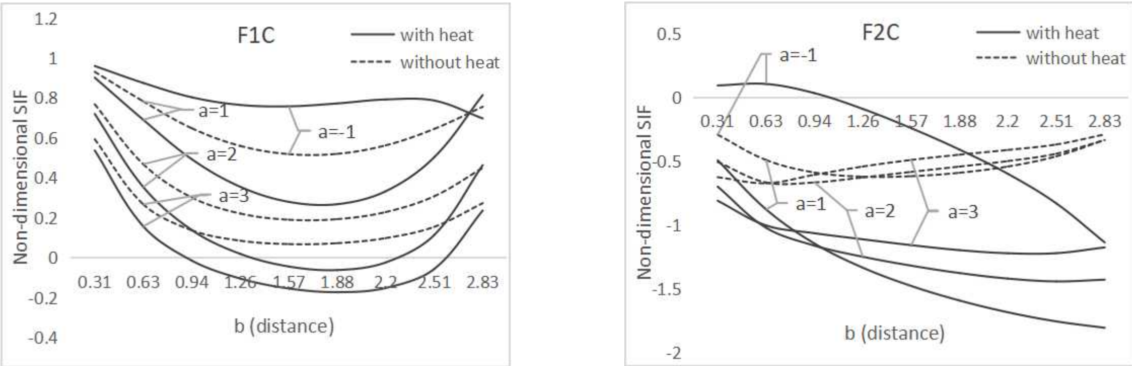
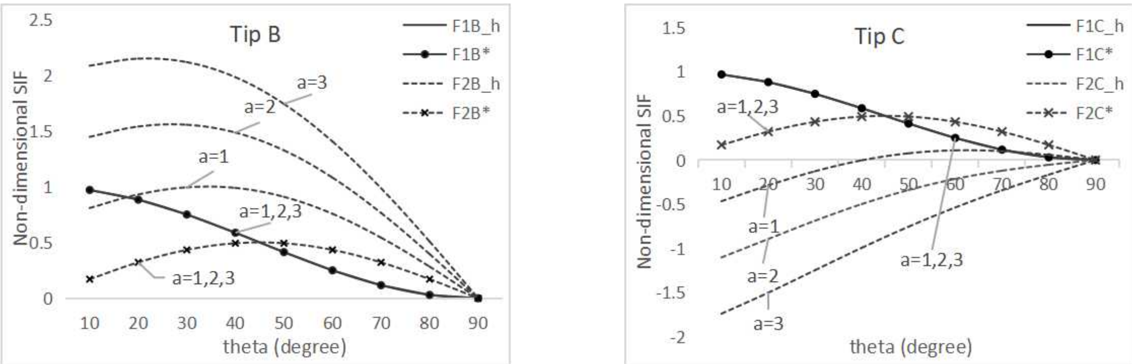


FIGURE 4. Non-dimensional SIF for crack in Fig. 2(c)



$*$ without heat; $-h$ with heat.

FIGURE 5. Non-dimensional SIF for a crack in Fig. 2(d)

II non-dimensional SIFs increases at tip B with crack length and decreases at tip C.

6. CONCLUSION

In the present work, the crack subjected normal stress in an infinite heated plate is formulated into two singular integral equations, and solved using Gauss integration rules. The numerical results exhibit that: (1) If the crack expansion is in the same direction as the remote heat flux, the Mode I non-dimensional SIF for a crack in a heated plate is higher than the crack without heat; however, if the direction is opposite, the Mode I non-dimensional SIF for a crack in a heated plate is smaller. (2) The non-dimensional SIF of a heated crack is influenced by the configuration and position of the crack.

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