ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **13** (2024), no.3, 311–321
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.13.3.4

ORTHOGONAL GENERALIZED (σ, τ)-DERIVATIONS IN SEMIPRIME Γ-NEAR RINGS

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ABSTRACT. Consider a 2-torsion-free semiprime Γ-near ring N. Assume that σ and τ are automorphisms on N. An additive map $d_1 : N \to N$ is called a (σ, τ)-derivation if it satisfies

$$d_1(u\alpha v) = d_1(u)\alpha\sigma(v) + \tau(u)\alpha d_1(v)$$

for all $u, v \in N$ and $\alpha \in \Gamma$. An additive map $D_1 : N \to N$ is termed a generalized (σ, τ)-derivation associated with the (σ, τ)-derivation d_1 if

$$D_1(u\alpha v) = D_1(u)\alpha\sigma(v) + \tau(u)\alpha d_1(v)$$

for all $u, v \in N$ and $\alpha \in \Gamma$. Consider two generalized (σ, τ)-derivations D_1 and D_2 on N. This paper introduces the concept of the orthogonality of two generalized (σ, τ)-derivations D_1 and D_2 and presents several results regarding the orthogonality of generalized (σ, τ)-derivations and (σ, τ)-derivations in a Γ-near ring.

1. INTRODUCTION

Bresar and Vukman [7] explored the concept of orthogonal derivations in rings. The concept of generalized derivation was introduced by Bresar [6]. Bell and

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2020 *Mathematics Subject Classification.* 16N60, 16Y30.
Key words and phrases. Semiprime Γ-near ring, (σ,τ)-derivation, Generalized (σ,τ)-derivation, Orthogonal generalized (σ,τ)-derivations, Left centralizer.
Submitted: 18.06.2024; *Accepted:* 02.07.2024; *Published:* 18.07.2024.

Mason [2] introduced derivations in near-rings. Park and Jung [3] investigated orthogonal generalized derivations in semiprime near-rings and obtained results on orthogonal generalized derivations in the form of a product. Ashraf et al.[8] examined orthogonal generalize derivations in Γ-rings. Sulaiman and Majeed [9] established some results related to the orthogonal properties of derivations on nonzero ideals of a 2-torsion-free semiprime Γ-ring. K.K. Dey et al.[4,5] focussed on the study of generalized derivations in semiprime Γ-rings and near-rings respectively. More recently, C.Jaya Subba Reddy et al.[12,1,11,10] studied the orthogonality of generalized symmetric reverse biderivations,symmetric reverse bi-(σ, τ)-derivations, generalized symmetric reverse bi-(σ, τ)-derivations in semi prime rings, and orthogonality of reverse (σ, τ)-derivations in semiprime Γ-rings.

2. Preliminaries

Near-Ring (Left Near-Ring):

- A near-ring (or left near-ring) on a set N with operations $+$ (addition) and $\cdot$ (multiplication) satisfies the following:
  - $(N, +)$ forms a group (not necessarily abelian).
  - $(N, \cdot)$ is a semigroup.
  - Distributive law holds: $u \cdot (v + w) = u \cdot v + u \cdot w$ for all $u, v, w \in N$.

Γ-Near-Ring:

- A Γ-near-ring is a triplet $(N, +, \Gamma)$ where:
  - $(N, +)$ is a group (not necessarily abelian).
  - Γ is a non-empty set of binary operations on N.
  - For each $\alpha \in \Gamma$, $(N, +, \alpha)$ forms a left near-ring.
  - $u\alpha(v\beta w) = (u\alpha v)\beta w$ for all $u, v, w \in N$ and $\alpha, \beta \in \Gamma$.

In a Γ-near-ring N, the subset $N_0 = \{u \in N \mid 0\alpha u = 0, \alpha \in \Gamma\}$ is known as the zero symmetric part of N. A Γ-near-ring N is considered zero-symmetric if $N = N_0$. Notably, it qualifies as a left Γ-near-ring due to its adherence to the left distributive law. N will denote a zero-symmetric left Γ-near-ring.

A Γ-near-ring N is termed semiprime if it possesses the property that $u\Gamma N\Gamma u = \{0\}$ implies $u = 0$ for all $u \in N$. A Γ-near-ring N is described as 2-torsion free if $2u = 0$ implies $u = 0$ for all $u \in N$. In our discussion, the term Γ-near-ring specifically refers to a left Γ-near-ring.

An additive map $d_1 : N \to N$ is referred to as a (σ,τ)-derivation if it satisfies the condition $d_1(u\alpha v) = d_1(u)\alpha\sigma(v) + \tau(u)\alpha d_1(v)$ for all $u, v \in N$ and $\alpha \in \Gamma$.

Let $D_1 : N \to N$ be an additive map, and $d_1 : N \to N$ be a (σ,τ)-derivation. If D_1 fulfills the condition $D_1(u\alpha v) = D_1(u)\alpha\sigma(v) + \tau(u)\alpha d_1(v)$ for all $u, v \in N$ and $\alpha \in \Gamma$, then D_1 is termed a generalized (σ,τ)-derivation connected with the (σ,τ)-derivation d_1.

Consider an additive mapping $d : N \to N$. It qualifies as a left centralizer if it obeys $d(u\alpha v) = d(u)\alpha v$ for all $u, v \in N$ and $\alpha \in \Gamma$. The broader notion of the generalized (σ,τ)-derivation encompasses both the specific cases of (σ,τ)-derivations and left centralizers.

Two generalized (σ,τ)-derivations D_1 and D_2 of a semiprime Γ-near-ring N are termed orthogonal if $D_1(u)\Gamma N\Gamma D_2(v) = D_2(v)\Gamma N\Gamma D_1(u) = 0$ holds true for all $u, v \in N$.

In this paper, we maintain the assumption that N is a 2-torsion-free semiprime Γ-near ring, with σ and τ being automorphisms on N. Additionally, we assume that d_1 and d_2 are (σ,τ)-derivations, while D_1 and D_2 are generalized (σ,τ)-derivations of N. It is further assumed that $d_1\tau = \tau d_1$, $d_2\tau = \tau d_2$, $\sigma d_1 = d_1\sigma$, $\sigma d_2 = d_2\sigma$, and $D_1\tau = \tau D_1$, $D_2\tau = \tau D_2$, $\sigma D_1 = D_1\sigma$, $\sigma D_2 = D_2\sigma$.

Lemma 2.1 ([5]. *,LEMMA 2.1] Suppose that N is a 2-torsion-free semiprime Γ-near ring and $a, b \in N$. Then the following conditions are identical:*

(1) $a\alpha u\beta b = 0$, $\forall u \in N$ *and* $\alpha, \beta \in \Gamma$.
(2) $b\alpha u\beta a = 0$, $\forall u \in N$ *and* $\alpha, \beta \in \Gamma$.
(3) $a\alpha u\beta b + b\alpha u\beta a = 0$, $\forall u \in N$ *and* $\alpha, \beta \in \Gamma$.

If one of the aforementioned conditions holds, then $a\Gamma b = b\Gamma a = 0$.

Lemma 2.2. *Suppose N is a Γ-near ring and D_1 a generalized (σ,τ)-derivation of N. Then the following statements are true:*

$$(i)\quad (D_1(u)\alpha\sigma(v) + \tau(u)\alpha d_1(v))\beta\sigma(w) = D_1(u)\alpha\sigma(v)\beta\sigma(w) + \tau(u)\alpha d_1(v)\beta\sigma(w),$$

$$(ii)\quad (d_1(u)\alpha\sigma(v) + \tau(u)\alpha d_1(v))\beta\sigma(w) = d_1(u)\alpha\sigma(v)\beta\sigma(w) + \tau(u)\alpha d_1(v)\beta\sigma(w).$$

for all $u, v, w \in N$ and $\alpha, \beta \in \Gamma$.

Proof. Let us consider

$$\begin{aligned} D_1((u\alpha v)\beta w) &= D_1(u\alpha v)\beta\sigma(w) + \tau(u\alpha v)\beta d_1(w) \\ &= (D_1(u)\alpha\sigma(v) + \tau(u)\alpha d_1(v))\beta\sigma(w) + \tau(u)\alpha\tau(v)\beta d_1(w), \end{aligned} \tag{2.1}$$

$$\begin{aligned} D_1(u\alpha(v\beta w)) &= D_1(u)\alpha\sigma(v\beta w) + \tau(u)\alpha d_1(v\beta w) \\ &= D_1(u)\alpha\sigma(v)\beta\sigma(w) + \tau(u)\alpha d_1(v)\beta\sigma(w) + \tau(u)\alpha\tau(v)\beta d_1(w). \end{aligned} \tag{2.2}$$

From the above two equations, we get

$$(D_1(u)\alpha\sigma(v) + \tau(u)\alpha d_1(v))\beta\sigma(w) = D_1(u)\alpha\sigma(v)\beta\sigma(w) + \tau(u)\alpha d_1(v)\beta\sigma(w).$$

Result (ii) can be proved in a similar way.

□

Lemma 2.3. *Suppose N is a semiprime Γ-near ring that is 2-torsion free. Let D_1 and D_2 be two generalized (σ, τ)-derivations of N. If D_1 and D_2 are orthogonal, then the following conditions are satisfied:*

(1) $D_1(u)\Gamma D_2(v) = D_2(u)\Gamma D_1(v) = 0$, $\forall u, v \in N$.
(2) d_1 *and* D_2 *are orthogonal and* $d_1(u)\sigma D_2(v) = 0 = D_2(v)\sigma d_1(u)$, $\forall u, v \in N$.
(3) d_2 *and* D_1 *are orthogonal and* $d_2(u)\sigma D_1(v) = 0 = D_1(v)\sigma d_2(u)$, $\forall u, v \in N$.
(4) d_1 *and* d_2 *are orthogonal and* $d_1 d_2 = 0$.
(5) $d_1 D_2 = D_2 d_1 = 0$ *and* $d_2 D_1 = D_1 d_2 = 0$.
(6) $D_1 D_2 = D_2 D_1 = 0$.

Proof.

To prove (i): Since D_1 and D_2 are orthogonal, we have $D_1(u)\alpha w\beta D_2(v) = 0$, $\forall u, v, w \in N$ and $\alpha, \beta \in \Gamma$. From Lemma 2.1, we know that $D_1(u)\alpha D_2(v) = D_2(v)\alpha D_1(u) = 0$, $\forall u, v \in N$ and $\alpha \in \Gamma$.

To prove (ii): By the condition (i), we have

$$D_1(u)\alpha D_2(v) = 0, \quad \forall u, v \in N \text{ and } \alpha \in \Gamma. \tag{2.3}$$

Replacing u by $w\beta u$, $\forall w \in N$, $\beta \in \Gamma$ in (2.3) and using the orthogonality of D_1 and D_2, we get $\tau(w)\beta d_1(u)\alpha D_2(v) = 0$.
Since τ is an automorphism of N, which is a semiprime Γ-near ring, we have

$$d_1(u)\alpha D_2(v) = 0, \quad \forall u, v \in N. \tag{2.4}$$

Replacing u by $u\beta w$, $\forall w \in N$, $\beta \in \Gamma$ in equation (2.4) and using the same, we get $d_1(u)\beta\sigma(w)\alpha D_2(v) = 0$. Keeping the fact that σ is an automorphism of N, we get $d_1(u)\beta N\alpha D_2(v) = \{0\}$ and we can write $D_2(v)\Gamma d_1(u)\Gamma N\Gamma D_2(v)\Gamma d_1(u) = 0$, $D_2(v)\Gamma d_1(u) = 0$ (Using the semiprimeness of N). Hence, proved.

To prove (iii): Similarly, by considering $D_2(u)\Gamma D_1(v) = 0$, for all $u, v \in N$, and proceeding in the same manner as in the previous case, we can prove $d_2(u)\Gamma D_1(v) = 0 = D_1(v)\Gamma d_2(u)$, for all $u, v \in N$.

To prove (iv): Consider $D_1(u)\alpha D_2(v) = 0$, for all $u, v \in \mathbb{N}$ and $\alpha \in \Gamma$ (By (2.3)). Replacing u by $u\beta w$ and v by $v\delta t$, $\forall w, t \in \mathbb{N}, \beta, \delta \in \Gamma$ in the above equation, we get $D_1(u)\beta\sigma(w)\alpha D_2(v)\delta\sigma(t) + \tau(u)\beta d_1(w)\alpha D_2(v)\delta\sigma(t) + D_1(u)\beta\sigma(w)\alpha\tau(v)\delta d_2(t) + \tau(u)\beta d_1(w)\alpha\tau(v)\delta d_2(t) = 0$, for all $u, v, w, t \in \mathbb{N}$ and $\alpha, \beta, \delta \in \Gamma$. Using conditions (i), (ii) and (iii), the first, second and third terms vanish. Hence, we get $\tau(u)\beta d_1(w)\alpha\tau(v)\delta d_2(t) = 0$. Since τ is an automorphism on a semiprime Γ-Near ring, we get $d_1(w)\alpha\tau(v)\delta d_2(t) = 0$, for all $v, w, t \in N$ and $\alpha, \delta \in \Gamma$. Again using the automorphism property of τ and Lemma 2.1, we get $d_1(w)\alpha d_2(t) = 0$, for all $w, t \in \mathbb{N}$ and $\alpha \in \Gamma$. Therefore, we can conclude that d_1 and d_2 are orthogonal and so we can write that $d_1(u)\alpha v\beta d_2(w) = 0$, for all $u, v, w \in \mathbb{N}$ and $\alpha, \beta \in \Gamma$.

Hence, $d_1(d_1(u)\alpha v\beta d_2(w)) = 0$ and

$$d_1^2(u)\alpha\sigma(v)\beta\sigma(d_2(w)) + \tau(d_1(u))\alpha d_1(v)\beta\sigma(d_2(w)) + \tau(d_1(u))\alpha\tau(v)\beta d_1(d_2(w)) = 0.$$

Using $\sigma d_2 = d_2\sigma$, $\tau d_1 = d_1\tau$, and σ, τ are automorphisms of $\mathbb{N}$, we get

$$d_1^2(u)\alpha\sigma(v)\beta d_2(w) + d_1(u)\alpha d_1(v)\beta d_2(w) + d_1(u)\alpha\tau(v)\beta d_1 d_2(w) = 0.$$

Since d_1 and d_2 are orthogonal, the first two terms vanish and so

$$d_1(u)\alpha\tau(v)\beta d_1 d_2(w) = 0, \quad \forall u, v, w \in \mathbb{N} \text{ and } \alpha, \beta \in \Gamma. \tag{2.5}$$

Replacing u by $d_2(w)$ in equation (2.5), and using the semiprimeness of $\mathbb{N}$, we get

$$d_1 d_2(w) = 0 \quad \text{and so} \quad d_1 d_2 = 0.$$

To prove (v): Consider $d_1(u)\alpha w\beta D_2(v) = 0$. (Since d_1, D_2 are orthogonal by (ii)) Then, $D_2(d_1(u)\alpha w\beta D_2(v)) = 0$, for all $u, v, w \in \mathbb{N}$ and $\alpha, \beta \in \Gamma$,

$$D_2(d_1(u))\alpha\sigma(w)\beta\sigma(D_2(v)) + \tau(d_1(u))\alpha(d_2(w)\beta\sigma(D_2(v)) + \tau(w)\beta d_2(D_2(v)) = 0.$$

Using $\sigma D_2 = D_2\sigma$, $\tau d_1 = d_1\tau$, and σ, τ are automorphisms of $\mathbb{N}$, we get

$$D_2(d_1(u))\alpha\sigma(w)\beta D_2(v) + d_1(u)\alpha d_2(w)\beta D_2(v) + d_1(u)\alpha\tau(w)\beta d_2(D_2(v)) = 0.$$

By using (ii) and (iv), the second and third terms are zero, hence

$$D_2(d_1(u))\alpha\sigma(w)\beta D_2(v) = 0. \tag{2.6}$$

Writing v as $d_1(u)$ in equation (2.6) and using the semiprimeness of $\mathbb{N}$, we get $D_2 d_1 = 0$.

Similarly, we can prove $d_1 D_2 = 0$, $D_1 d_2 = 0 = d_2 D_1$, $D_1 D_2 = 0 = D_2 D_1$. Hence the result is proved. □

3. Main results

Theorem 3.1. *In a 2-torsion-free semiprime Γ-near ring N, suppose D_1 and D_2 are two generalized (σ, τ)-derivations. Then, the following assertions hold true:*

1. *D_1 and D_2 are orthogonal.*
2. *$D_1(u)\Gamma D_2(v) = 0 = d_1(u)\Gamma D_2(v) = 0, \quad \forall u, v \in N$.*
3. *$D_1(u)\Gamma D_2(v) = 0 = d_1(u)\Gamma d_2(v) = 0, \quad \forall u, v \in N$ and $d_1 D_2 = d_1 d_2 = 0$.*
4. *$D_1 D_2$ is a generalized (σ, τ)-derivation of N connected with a (σ, τ)-derivation $d_1 d_2$ of N, and $D_1(u)\Gamma D_2(v) = 0, \quad \forall u, v \in N$.*

Proof.

(i) $\implies$ (ii) is evident by (i) and (ii) of Lemma 2.3.

(i) $\implies$ (iii) is evident by (i), (iv), and (v) of Lemma 2.3.

(i) $\implies$ (iv): Suppose that D_1 and D_2 are orthogonal. Using condition (iv) of Lemma 2.3, it is easy to prove $d_1 d_2$ is a (σ, τ)-derivation of N. Also,

$$\begin{aligned} D_1 D_2(u\alpha v) = D_1(D_2(u\alpha v)) &= D_1 D_2(u)\alpha\sigma^2(v) + \tau(D_2(u))\alpha d_1(\sigma(v)) \\ &+ D_1(\tau(u))\alpha\sigma(d_2(v)) + \tau^2(u)\alpha d_1 d_2(v). \end{aligned}$$

Since τ, σ are automorphisms on N and using conditions (ii) and (iii) of Lemma 2.3, we get

$$D_1 D_2(u\alpha v) = D_1 D_2(u)\alpha\sigma(v) + \tau(u)\alpha d_1 d_2(v).$$

Therefore, $D_1 D_2$ is a generalized (σ, τ)-derivation of N corresponding to a (σ, τ)-derivation $d_1 d_2$ of N.

Also, by condition (i), we already proved that $D_1(u)\Gamma D_2(v) = 0, \quad \forall u, v \in N$. Hence, condition (iv) is proved.

(ii) $\implies$ (i): Suppose that $D_1(u)\Gamma D_2(v) = 0 = d_1(u)\Gamma D_2(v)$, for all $u, v \in N$. Consider

$$D_1(u)\Gamma D_2(v) = 0. \tag{3.1}$$

Replacing u by $u\beta w$, $w \in N$, $\beta \in \Gamma$ in the above equation and using condition (ii), we get

$$D_1(u)\beta\sigma(w)\alpha D_2(v) = 0.$$

Since σ is an automorphism on N, and using Lemma 2.1, we get

$$D_1(u)\alpha D_2(v) = 0 \quad \text{and so} \quad D_1 \text{ and } D_2 \text{ are orthogonal.}$$

(iii) $\implies$ (i): Suppose that $D_1(u)\Gamma D_2(v) = 0 = d_1(u)\Gamma d_2(v) = 0, \quad \forall u, v \in N$ and $d_1D_2 = d_1d_2 = 0$. Consider $d_1D_2 = 0$,

$$\begin{aligned} d_1D_2(u\alpha v) = 0 &= d_1(D_2(u)\alpha\sigma(v) + \tau(u)\alpha d_2(v)) \\ &= d_1(D_2(u))\alpha\sigma^2(v) + \tau(D_2(u))\alpha d_1(\sigma(v)) \\ &+ d_1(\tau(u))\alpha\sigma(d_2(v)) + \tau^2(u)\alpha d_1d_2(v). \end{aligned}$$

Since $\tau D_2 = D_2\tau$, $\sigma d_1 = d_1\sigma$, $\tau d_1 = d_1\tau$, $\sigma d_2 = d_2\sigma$, and σ, τ are automorphisms on N, we get

$$d_1D_2(u\alpha v) = d_1D_2(u)\alpha\sigma(v) + D_2(u)\alpha d_1(v) + d_1(u)\alpha d_2(v) + \tau(u)\alpha d_1d_2(v).$$

Since $d_1D_2 = d_1d_2 = 0$ and $d_1(u)\Gamma d_2(v) = 0$, by hypothesis,

$$0 = D_2(u)\alpha d_1(v). \tag{3.2}$$

Replacing u by $u\beta w$, $w \in N$, $\beta \in \Gamma$ in (3.2) and using the orthogonality of d_2 and d_1:

$$0 = D_2(u)\beta\sigma(w)\alpha d_1(v).$$

Since σ is an automorphism on N, using Lemma 2.1, we can have

$$D_2(u)\alpha d_1(v) = d_1(v)\alpha D_2(u) = 0 \quad \text{and so} \quad d_1(u)\Gamma D_2(v) = 0.$$

We have $D_1(u)\Gamma D_2(v) = 0, \quad \forall u, v \in N$ (by the hypothesis of condition (iii)). By condition (ii), we can write D_1 and D_2 are orthogonal. (iv) $\implies$ (i) Let D_1D_2 be a generalized (σ,τ)-derivation of N associated with a (σ,τ)-derivation d_1d_2 of N

and $D_1(u)\Gamma D_2(v) = 0, \quad \forall u, v \in N$.Then, we have

$$D_1D_2(u\alpha v) = D_1D_2(u)\alpha\sigma(v) + \tau(u)\alpha D_1D_2(v). \tag{3.3}$$

Also,

$$\begin{aligned} D_1D_2(u\alpha v) &= D_1(D_2(u\alpha v)) \\ &= D_1D_2(u)\alpha\sigma(v) + D_2(u)\alpha d_1(v) + D_1(u)\alpha d_2(v) \\ &+ \tau(u)\alpha d_1d_2(v). \end{aligned} \tag{3.4}$$

Comparing (3.3) and (3.4), we get

$$D_2(u)\alpha d_1(v) + D_1(u)\alpha d_2(v) = 0. \tag{3.5}$$

By the hypothesis of (iv), we have

$$D_1(u)\Gamma D_2(v) = 0. \tag{3.6}$$

Replacing v by $v\beta w$, $w \in N$, $\beta \in \Gamma$ in (3.6) and using the same, we get

$$D_1(u)\alpha\tau(v)\beta d_2(w) = 0$$

$$d_2(w)\gamma D_1(u)\alpha\tau(v)\beta d_2(w)\gamma D_1(u) = 0.$$

Since τ is an automorphism and using the semiprimeness of N, we get

$$d_2(w)\gamma D_1(u) = 0. \tag{3.7}$$

Replacing w by $v\beta w$, $w \in N$ in (3.7) and using the same,

$$d_2(v)\beta\sigma(w)\gamma D_1(u) = 0$$

$$D_1(u)\alpha d_2(v)\beta\sigma(w)\gamma D_1(u)\alpha d_2(v) = 0, \quad \forall u, v, w \in N, \gamma, \beta \in \Gamma$$

$$D_1(u)\alpha d_2(v)\Gamma\sigma(w)\Gamma D_1(u)\alpha d_2(v) = 0.$$

Since σ is an automorphism and using the semiprimeness of N, we get

$$D_1(u)\alpha d_2(v) = 0,$$

and hence we can write

$$D_2(u)\alpha d_1(v) = 0. \tag{3.8}$$

Replacing v by $v\beta w$, $w \in N$, $\beta \in \Gamma$ in (3.8) and using the same

$$D_2(u)\alpha\tau(v)\beta d_1(w) = 0$$

$$d_1(w)\gamma D_2(u)\Gamma\tau(v)\Gamma d_1(w)\gamma D_2(u) = 0$$
$$d_1(w)\gamma D_2(u) = 0 \quad \text{(By the semiprimeness of } N)$$

$$d_1(w)\Gamma D_2(u) = 0, \quad \forall u, w \in N. \tag{3.9}$$

By using the hypothesis of (iv), we have

$$D_1(u)\Gamma D_2(v) = 0. \tag{3.10}$$

Combining (3.9) and (3.10) and using condition (ii), We can conclude that D_1 and D_2 are orthogonal. ☐

Theorem 3.2. *Suppose N is a Γ-near ring that is semiprime and 2-torsion free. Let D_1 and D_2 be two generalized (σ,τ)-derivations of N. Assume that D_1 is orthogonal to d_2 and D_2 is orthogonal to d_1. Then, we deduce the following:*

(i) *D_1D_2 is a left centralizer of N and $d_1d_2 = 0$.*
(ii) *D_2D_1 is a left centralizer of N and $d_2d_1 = 0$.*

Proof. Suppose that D_1 and D_2 are orthogonal to d_2 and d_1, then

$$D_1(u)\alpha w\beta d_2(v) = 0, \quad \forall u, v, w \in N \text{ and } \alpha, \beta \in \Gamma. \tag{3.11}$$

Replacing u by $w\delta u$, $w \in N$, $\delta \in \Gamma$ in (3.11) and using the same, we get

$$\tau(w)\delta d_1(u)\alpha w\beta d_2(v) = 0$$

and $d_1(u)\alpha w\beta d_2(v)\gamma\tau(w)\delta d_1(u)\alpha w\beta d_2(v) = 0$.

Since τ is an automorphism on a semiprime Γ-near ring N, we get $d_1(u)\alpha w \beta d_2(v) = 0$, for all $u, v, w \in N$ and $\alpha, \beta \in \Gamma$. Therefore, d_1 and d_2 are orthogonal. Hence,

$$d_1d_2 = 0. \tag{3.12}$$

Since D_1, D_2 are two generalized (σ,τ)-derivations of N, we can write

$$D_1D_2(u\alpha v) = D_1(D_2(u\alpha v)) = D_1D_2(u)\alpha\sigma(v) + D_1(u)\alpha d_2(v)$$
$$+ D_2(u)\alpha d_1(v) + \tau(u)\alpha d_1d_2(v).$$

Since D_1, D_2 are orthogonal to d_2, d_1, we have $D_1(u)\alpha d_2(v) = 0 = D_2(u)\alpha d_1(v)$. Also, by (3.12), we have $d_1d_2 = 0$. Then the above equation reduces to $D_1D_2(u\alpha v) = D_1D_2(u)\alpha\sigma(v)$.

Hence, D_1D_2 is a left centralizer of N. Similarly, we can prove result (ii) also. □

Theorem 3.3. *In a semiprime Γ-near ring N which is 2-torsion free, considering D_1 as a generalized (σ,τ)-derivation on N, the condition $D_1(u)\Gamma D_1(v) = 0$ for all $u, v \in N$ implies D_1 and d_1 are identically zero.*

Proof. By the hypothesis,

$$D_1(u)\Gamma D_1(v) = 0, \forall u, v \in N. \tag{3.13}$$

Replacing v by $v\beta w$, $w \in N$, $\beta \in \Gamma$ in (3.13) and using the same, we get

$$D_1(u)\alpha\tau(v)\beta d_1(w) = 0.$$

Since τ is an automorphism of N and using Lemma 2.1, we can have

$$D_1(u)\alpha d_1(w) = 0 = d_1(w)\alpha D_1(u).$$

Consider

$$d_1(w)\alpha D_1(u) = 0, \forall u, w \in N, \alpha \in \Gamma. \tag{3.14}$$

Replacing u by $u\beta w$, $w \in N$, $\beta \in \Gamma$ in (3.14) and using the same equation, we get

$$d_1(w)\alpha\tau(u)\beta d_1(w) = 0, \quad \forall u, w \in N, \alpha, \beta \in \Gamma.$$

Since τ is an automorphism on a semiprime Γ-near ring N, we get $d_1 = 0$.

Again, consider $D_1(u)\Gamma D_1(v) = 0$ (by the hypothesis). Replace u by $u\alpha v$, for all $v \in N$, $\alpha \in \Gamma$ in the above equation and using (3.14), we get

$$D_1(u)\alpha\sigma(v)\beta D_1(v) = 0.$$

Since σ is an automorphism and using N is semiprime, we get $D_1 = 0$. Hence proved. □

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