ORTHOGONAL GENERALIZED $(\sigma, \tau)$-DERIVATIONS IN SEMIPRIME $\Gamma$-NEAR RINGS

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ABSTRACT. Consider a 2-torsion-free semiprime $\Gamma$-near ring $N$. Assume that $\sigma$ and $\tau$ are automorphisms on $N$. An additive map $d_1 : N \to N$ is called a $(\sigma, \tau)$-derivation if it satisfies

$$d_1(uev) = d_1(u)\sigma(v) + \tau(u)\alpha d_1(v)$$

for all $u, v \in N$ and $\alpha \in \Gamma$. An additive map $D_1 : N \to N$ is termed a generalized $(\sigma, \tau)$-derivation associated with the $(\sigma, \tau)$-derivation $d_1$ if

$$D_1(uev) = D_1(u)\sigma(v) + \tau(u)\alpha d_1(v)$$

for all $u, v \in N$ and $\alpha \in \Gamma$. Consider two generalized $(\sigma, \tau)$-derivations $D_1$ and $D_2$ on $N$. This paper introduces the concept of the orthogonality of two generalized $(\sigma, \tau)$-derivations $D_1$ and $D_2$ and presents several results regarding the orthogonality of generalized $(\sigma, \tau)$-derivations and $(\sigma, \tau)$-derivations in a $\Gamma$-near ring.

1. INTRODUCTION

Bresar and Vukman [7] explored the concept of orthogonal derivations in rings. The concept of generalized derivation was introduced by Bresar [6]. Bell and

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Mason [2] introduced derivations in near-rings. Park and Jung [3] investigated orthogonal generalized derivations in semiprime near-rings and obtained results on orthogonal generalized derivations in the form of a product. Ashraf et al.[8] examined orthogonal generalized derivations in \( \Gamma \)-rings. Sulaiman and Majeed [9] established some results related to the orthogonal properties of derivations on nonzero ideals of a 2-torsion-free semiprime \( \Gamma \)-ring. K.K. Dey et al.[4,5] focussed on the study of generalized derivations in semiprime \( \Gamma \)-rings and near-rings respectively. More recently, C.Jaya Subba Reddy et al.[12,11,10] studied the orthogonality of generalized symmetric reverse biderivations, symmetric reverse bi-(\( \sigma, \tau \))-derivations, generalized symmetric reverse bi-(\( \sigma, \tau \))-derivations in semiprime rings, and orthogonality of reverse (\( \sigma, \tau \))-derivations in semiprime \( \Gamma \)-rings.

2. Preliminaries

Near-Ring (Left Near-Ring):
- A near-ring (or left near-ring) on a set \( N \) with operations \(+\) (addition) and \( \cdot \) (multiplication) satisfies the following:
  - \((N, +)\) forms a group (not necessarily abelian).
  - \((N, \cdot)\) is a semigroup.
  - Distributive law holds: \( u \cdot (v + w) = u \cdot v + u \cdot w \) for all \( u, v, w \in N \).

\( \Gamma \)-Near-Ring:
- A \( \Gamma \)-near-ring is a triplet \((N, +, \Gamma)\) where:
  - \((N, +)\) is a group (not necessarily abelian).
  - \( \Gamma \) is a non-empty set of binary operations on \( N \).
  - For each \( \alpha \in \Gamma \), \((N, +, \alpha)\) forms a left near-ring.
  - \( u\alpha(v\beta w) = (u\alpha v)\beta w \) for all \( u, v, w \in N \) and \( \alpha, \beta \in \Gamma \).

In a \( \Gamma \)-near-ring \( N \), the subset \( N_0 = \{u \in N \mid 0\alpha u = 0, \alpha \in \Gamma \} \) is known as the zero symmetric part of \( N \). A \( \Gamma \)-near-ring \( N \) is considered zero-symmetric if \( N = N_0 \). Notably, it qualifies as a left \( \Gamma \)-near-ring due to its adherence to the left distributive law. \( N \) will denote a zero-symmetric left \( \Gamma \)-near-ring.

A \( \Gamma \)-near-ring \( N \) is termed semiprime if it possesses the property that \( u\Gamma N\Gamma u = \{0\} \) implies \( u = 0 \) for all \( u \in N \). A \( \Gamma \)-near-ring \( N \) is described as 2-torsion free if \( 2u = 0 \) implies \( u = 0 \) for all \( u \in N \). In our discussion, the term \( \Gamma \)-near-ring specifically refers to a left \( \Gamma \)-near-ring.
An additive map $d_1 : N \to N$ is referred to as a $(\sigma, \tau)$-derivation if it satisfies the condition $d_1(u\alpha v) = d_1(u)\alpha \sigma(v) + \tau(u)\alpha d_1(v)$ for all $u, v \in N$ and $\alpha \in \Gamma$.

Let $D_1 : N \to N$ be an additive map, and $d_1 : N \to N$ be a $(\sigma, \tau)$-derivation. If $D_1$ fulfills the condition $D_1(u\alpha v) = D_1(u)\alpha \sigma(v) + \tau(u)\alpha d_1(v)$ for all $u, v \in N$ and $\alpha \in \Gamma$, then $D_1$ is termed a generalized $(\sigma, \tau)$-derivation connected with the $(\sigma, \tau)$-derivation $d_1$.

Consider an additive mapping $d : N \to N$. It qualifies as a left centralizer if it obeys $d(u\alpha v) = d(u)\alpha v$ for all $u, v \in N$ and $\alpha \in \Gamma$. The broader notion of the generalized $(\sigma, \tau)$-derivation encompasses both the specific cases of $(\sigma, \tau)$-derivations and left centralizers.

Two generalized $(\sigma, \tau)$-derivations $D_1$ and $D_2$ of a semiprime $\Gamma$-near-ring $N$ are termed orthogonal if $D_1(u)\Gamma D_2(v) = D_2(v)\Gamma D_1(u) = 0$ holds true for all $u, v \in N$.

In this paper, we maintain the assumption that $N$ is a 2-torsion-free semiprime $\Gamma$-near ring, with $\sigma$ and $\tau$ being automorphisms on $N$. Additionally, we assume that $d_1$ and $d_2$ are $(\sigma, \tau)$-derivations, while $D_1$ and $D_2$ are generalized $(\sigma, \tau)$-derivations of $N$. It is further assumed that $d_1\tau = \tau d_1$, $d_2\tau = \tau d_2$, $\sigma d_1 = d_1 \sigma$, $\sigma d_2 = d_2 \sigma$, and $D_1\tau = \tau D_1$, $D_2\tau = \tau D_2$, $\sigma D_1 = D_1 \sigma$, $\sigma D_2 = D_2 \sigma$.

**Lemma 2.1** ([5], LEMMA 2.1) Suppose that $N$ is a 2-torsion-free semiprime $\Gamma$-near ring and $a, b \in N$. Then the following conditions are identical:

1. $a\alpha u\beta b = 0$, $\forall u \in N$ and $\alpha, \beta \in \Gamma$.
2. $b\alpha u\beta a = 0$, $\forall u \in N$ and $\alpha, \beta \in \Gamma$.
3. $a\alpha u\beta b + b\alpha u\beta a = 0$, $\forall u \in N$ and $\alpha, \beta \in \Gamma$.

If one of the aforementioned conditions holds, then $a\Gamma b = b\Gamma a = 0$.

**Lemma 2.2.** Suppose $N$ is a $\Gamma$-near ring and $D_1$ a generalized $(\sigma, \tau)$-derivation of $N$. Then the following statements are true:

1. $(D_1(u)\alpha \sigma(v) + \tau(u)\alpha d_1(v))\beta \sigma(w) = D_1(u)\alpha \sigma(v)\beta \sigma(w) + \tau(u)\alpha d_1(v)\beta \sigma(w)$.
2. $(d_1(u)\alpha \sigma(v) + \tau(u)\alpha d_1(v))\beta \sigma(w) = d_1(u)\alpha \sigma(v)\beta \sigma(w) + \tau(u)\alpha d_1(v)\beta \sigma(w)$.

for all $u, v, w \in N$ and $\alpha, \beta \in \Gamma$. 
Proof. Let us consider

\[ D_1((u\alpha v)\beta w) = D_1(u\alpha v)\beta \sigma(w) + \tau(u\alpha v)\beta d_1(w) \]

(2.1)

\[ = (D_1(u)\alpha \sigma(v) + \tau(u)d_1(v))\beta \sigma(w) + \tau(u)\alpha \tau(v)\beta d_1(w), \]

From the above two equations, we get

\[ D_1(u\alpha(v\beta w)) = D_1(u)\alpha \sigma(v)\beta w + \tau(u)\alpha d_1(v\beta w) \]

(2.2)

\[ = D_1(u)\alpha \sigma(v)\beta \sigma(w) + \tau(u)\alpha d_1(v)\beta \sigma(w) + \tau(u)\alpha \tau(v)\beta d_1(w). \]

From the above two equations, we get

\[ (D_1(u)\alpha \sigma(v) + \tau(u)\alpha d_1(v))\beta \sigma(w) = D_1(u)\alpha \sigma(v)\beta \sigma(w) + \tau(u)\alpha d_1(v)\beta \sigma(w). \]

Result (ii) can be proved in a similar way.

Lemma 2.3. Suppose \( N \) is a semiprime \( \Gamma \)-near ring that is 2-torsion free. Let \( D_1 \) and \( D_2 \) be two generalized \((\sigma, \tau)\)-derivations of \( N \). If \( D_1 \) and \( D_2 \) are orthogonal, then the following conditions are satisfied:

1. \( D_1(u)\Gamma D_2(v) = D_2(u)\Gamma D_1(v) = 0, \forall u, v \in N. \)
2. \( d_1 \) and \( D_2 \) are orthogonal and \( d_1(u)\sigma D_2(v) = 0 = D_2(v)\sigma d_1(u), \forall u, v \in N. \)
3. \( d_2 \) and \( D_1 \) are orthogonal and \( d_2(u)\sigma D_1(v) = 0 = D_1(v)\sigma d_2(u), \forall u, v \in N. \)
4. \( d_1 \) and \( d_2 \) are orthogonal and \( d_1d_2 = 0. \)
5. \( d_1D_2 = D_2d_1 = 0 \) and \( d_2D_1 = D_1d_2 = 0. \)
6. \( D_1D_2 = D_2D_1 = 0. \)

Proof.

To prove (i): Since \( D_1 \) and \( D_2 \) are orthogonal, we have \( D_1(u)\alpha w\beta D_2(v) = 0, \forall u, v, w \in N \) and \( \alpha, \beta \in \Gamma. \) From Lemma 2.1, we know that \( D_1(u)\alpha D_2(v) = D_2(v)\alpha D_1(u) = 0, \forall u, v \in N \) and \( \alpha \in \Gamma. \)

To prove (ii): By the condition (i), we have

\[ D_1(u)\alpha D_2(v) = 0, \forall u, v \in N \text{ and } \alpha \in \Gamma. \]

Replacing \( u \) by \( w\beta u, \forall u \in N, \beta \in \Gamma \) in (2.3) and using the orthogonality of \( D_1 \) and \( D_2, \) we get \( \tau(w)\beta d_1(u)\alpha D_2(v) = 0. \)

Since \( \tau \) is an automorphism of \( N, \) which is a semiprime \( \Gamma \)-near ring, we have

\[ d_1(u)\alpha D_2(v) = 0, \forall u, v \in N. \]
Replacing \( u \) by \( u\beta w \), \( \forall w \in N, \beta \in \Gamma \) in equation (2.4) and using the same, we get \( d_1(u)\beta \sigma(w)\alpha D_2(v) = 0 \). Keeping the fact that \( \sigma \) is an automorphism of \( N \), we get \( d_1(u)\beta N\alpha D_2(v) = \{0\} \) and we can write \( D_2(v)\Gamma d_1(u)\Gamma N\Gamma D_2(v)\Gamma d_1(u) = 0, D_2(v)\Gamma d_1(u) = 0 \) (Using the semiprimeness of \( N \)). Hence, proved.

To prove (iii): Similarly, by considering \( D_2(u)\Gamma D_1(v) = 0 \), for all \( u, v \in N \), and proceeding in the same manner as in the previous case, we can prove \( d_2(u)\Gamma D_1(v) = 0 = D_1(v)\Gamma d_2(u) \), for all \( u, v \in N \).

To prove (iv): Consider \( D_1(u)\alpha D_2(v) = 0 \), for all \( u, v \in N \) and \( \alpha \in \Gamma \) (By (2.3)). Replacing \( u \) by \( u\beta w \) and \( v \) by \( v\delta t \), \( \forall w, t \in N, \beta, \delta \in \Gamma \) in the above equation, we get \( D_1(u)\beta \sigma(w)\alpha D_2(v)\delta \sigma(t) + \tau(u)\beta d_1(w)\alpha D_2(v)\delta \sigma(t) + D_1(u)\beta \sigma(w)\alpha \sigma(v)\delta d_2(t) + \tau(u)\beta d_1(w)\alpha \sigma(v)\delta d_2(t) = 0 \), for all \( u, v, w, t \in N \) and \( \alpha, \beta, \delta \in \Gamma \). Using conditions (i), (ii) and (iii), the first, second and third terms vanish. Hence, we get \( \tau(u)\beta d_1(w)\alpha \sigma(v)\delta d_2(t) = 0 \). Since \( \tau \) is an automorphism on a semiprime \( \Gamma \)-Near ring, we get \( d_1(v)\alpha \tau(v)\delta d_2(t) = 0 \), for all \( v, w, t \in N \) and \( \alpha, \beta, \delta \in \Gamma \). Again using the automorphism property of \( \tau \) and Lemma 2.1, we get \( d_1(w)\alpha d_2(t) = 0 \), for all \( w, t \in N \) and \( \alpha \in \Gamma \). Therefore, we can conclude that \( d_1 \) and \( d_2 \) are orthogonal and so we can write that \( d_1(u)\alpha \sigma(v)\beta d_2(w) = 0 \), for all \( u, v, w \in N \) and \( \alpha, \beta \in \Gamma \).

Hence, \( d_1(d_1(u)\alpha \sigma(v)\beta d_2(w)) = 0 \) and
\[
d_1^2(u)\alpha \sigma(v)\beta \sigma(d_2(w)) + \tau(d_1(u))\alpha \sigma(v)\beta d_2(w) + \tau(d_1(u))\alpha \sigma(v)\beta d_1 d_2(w) = 0.
\]
Using \( \sigma d_2 = d_2 \sigma, \tau d_1 = d_1 \tau \), and \( \sigma, \tau \) are automorphisms of \( N \), we get
\[
d_1^2(u)\alpha \sigma(v)\beta d_2(w) + d_1(u)\alpha d_1(v)\beta d_2(w) + d_1(u)\alpha \sigma(v)\beta d_1 d_2(w) = 0.
\]
Since \( d_1 \) and \( d_2 \) are orthogonal, the first two terms vanish and so
\[
(2.5) \quad d_1(u)\alpha \sigma(v)\beta d_1 d_2(w) = 0, \quad \forall u, v, w \in N \text{ and } \alpha, \beta \in \Gamma.
\]
Replacing \( u \) by \( d_2(w) \) in equation (2.5), and using the semiprimeness of \( N \), we get
\[
d_1 d_2(w) = 0 \quad \text{and so} \quad d_1 d_2 = 0.
\]

To prove (v): Consider \( d_1(u)\alpha \sigma(w)\beta D_2(v) = 0 \). (Since \( d_1, D_2 \) are orthogonal by (ii)) Then, \( D_2(d_1(u)\alpha \sigma(w)\beta D_2(v)) = 0 \), for all \( u, v, w \in N \) and \( \alpha, \beta \in \Gamma \),
\[
D_2(d_1(u)\alpha \sigma(w)\beta \sigma(D_2(v)) + \tau(d_1(u))\alpha(d_2(w)\beta \sigma(D_2(v)) + \tau(w)\beta d_2(D_2(v)) = 0.
\]
Using \( \sigma D_2 = D_2 \sigma, \tau d_1 = d_1 \tau \), and \( \sigma, \tau \) are automorphisms of \( N \), we get
By using (ii) and (iv), the second and third terms are zero, hence

\[ D_2(d_1(u))\alpha\sigma(w)\beta D_2(v) = 0. \]

Writing \( v \) as \( d_1(u) \) in equation (2.6) and using the semiprimeness of \( N \), we get

\[ D_2d_1 = 0. \]

Similarly, we can prove \( d_1D_2 = 0 = d_1D_2 = 0 = d_2D_1 \).

Hence the result is proved. \( \square \)

3. Main results

**Theorem 3.1.** In a 2-torsion-free semiprime \( \Gamma \)-near ring \( N \), suppose \( D_1 \) and \( D_2 \) are two generalized \((\sigma, \tau)\)-derivations. Then, the following assertions hold true:

1. \( D_1 \) and \( D_2 \) are orthogonal.
2. \( D_1(u)\Gamma D_2(v) = 0 = d_1(u)\Gamma D_2(v) = 0, \quad \forall u, v \in N. \)
3. \( D_1(u)\Gamma D_2(v) = 0 = d_1(u)\Gamma d_2(v) = 0, \quad \forall u, v \in N \text{ and } d_1D_2 = d_1d_2 = 0. \)
4. \( D_1D_2 \) is a generalized \((\sigma, \tau)\)-derivation of \( N \) connected with a \((\sigma, \tau)\)-derivation \( d_1d_2 \) of \( N \), and \( D_1(u)\Gamma D_2(v) = 0, \quad \forall u, v \in N. \)

**Proof.**

(i) \( \implies \) (ii) is evident by (i) and (ii) of Lemma 2.3.

(i) \( \implies \) (iii) is evident by (i), (iv), and (v) of Lemma 2.3.

(i) \( \implies \) (iv): Suppose that \( D_1 \) and \( D_2 \) are orthogonal. Using condition (iv) of Lemma 2.3, it is easy to prove \( d_1d_2 \) is a \((\sigma, \tau)\)-derivation of \( N \). Also,

\[
D_1D_2(u\alpha v) = D_1(D_2(u\alpha v)) = D_1D_2(u)\alpha\sigma^2(v) + \tau(D_2(u))\alpha d_1(\sigma(v)) + D_1(\tau(u))\alpha\sigma(d_2(v)) + \tau^2(u)\alpha d_1d_2(v).
\]

Since \( \tau, \sigma \) are automorphisms on \( N \) and using conditions (ii) and (iii) of Lemma 2.3, we get

\[
D_1D_2(u\alpha v) = D_1D_2(u)\alpha\sigma(v) + \tau(u)\alpha d_1d_2(v).
\]

Therefore, \( D_1D_2 \) is a generalized \((\sigma, \tau)\)-derivation of \( N \) corresponding to a \((\sigma, \tau)\)-derivation \( d_1d_2 \) of \( N \).
Also, by condition (i), we already proved that $D_1(u)\Gamma D_2(v) = 0, \ \forall u, v \in N$. Hence, condition (iv) is proved.

(ii) $\Rightarrow$ (i): Suppose that $D_1(u)\Gamma D_2(v) = 0 = d_1(u)\Gamma D_2(v)$, for all $u, v \in N$. Consider

$$(3.1) \quad D_1(u)\Gamma D_2(v) = 0.$$ 

Replacing $u$ by $u\beta w, w \in N, \ \beta \in \Gamma$ in the above equation and using condition (ii), we get

$$D_1(u)\beta \sigma(w)\alpha D_2(v) = 0.$$ 

Since $\sigma$ is an automorphism on $N$, and using Lemma 2.1, we get

$$D_1(u)\alpha\sigma D_2(v) = 0 \quad \text{and so} \quad D_1 \text{ and } D_2 \text{ are orthogonal.}$$

(iii) $\Rightarrow$ (i): Suppose that $D_1(u)\Gamma D_2(v) = 0 = d_1(u)\Gamma d_2(v) = 0, \ \forall u, v \in N$ and $d_1D_2 = d_1d_2 = 0$. Consider $d_1D_2 = 0$,

$$d_1D_2(u\sigma v) = 0 = d_1(D_2(u)\alpha\sigma(v) + \tau(u)\alpha d_2(v))$$

$$= d_1(D_2(u)\alpha\sigma^2(v) + \tau(D_2(u))\alpha d_1(\sigma(v))$$

$$+ d_1(\tau(u))\alpha\sigma(d_2(v)) + \tau^2(u)\alpha d_1d_2(v).$$

Since $\tau D_2 = D_2\tau, \ \sigma d_1 = d_1\sigma, \ \tau d_1 = d_1\tau, \ \sigma d_2 = d_2\sigma$, and $\sigma, \tau$ are automorphisms on $N$, we get

$$d_1D_2(u\sigma v) = d_1D_2(u)\alpha\sigma(v) + D_2(u)\alpha d_1(\sigma(v)) + d_1(u)\alpha d_2(v) + \tau(u)\alpha d_1d_2(v).$$

Since $d_1D_2 = d_1d_2 = 0$ and $d_1(u)\Gamma d_2(v) = 0$, by hypothesis,

$$(3.2) \quad 0 = D_2(u)\alpha d_1(v).$$

Replacing $u$ by $u\beta w, w \in N, \ \beta \in \Gamma$ in (3.2) and using the orthogonality of $d_2$ and $d_1$.

$$0 = D_2(u)\beta \sigma(w)\alpha d_1(v).$$

Since $\sigma$ is an automorphism on $N$, using Lemma 2.1, we can have

$$D_2(u)\alpha d_1(v) = d_1(v)\alpha D_2(u) = 0 \quad \text{and so} \quad d_1(u)\Gamma D_2(v) = 0.$$ 

We have $D_1(u)\Gamma D_2(v) = 0, \ \forall u, v \in N$ (by the hypothesis of condition (iii)). By condition (ii), we can write $D_1$ and $D_2$ are orthogonal. (iv) $\Rightarrow$ (i) Let $D_1D_2$ be a generalized $(\sigma, \tau)$-derivation of $N$ associated with a $(\sigma, \tau)$-derivation $d_1d_2$ of $N$. 


and \( D_1(u) \Gamma D_2(v) = 0, \quad \forall u, v \in N. \)

Then, we have

\[ D_1 D_2(u \alpha v) = D_1 D_2(u) \alpha \sigma(v) + \tau(u) \alpha D_1 D_2(v). \quad (3.3) \]

Also,

\[ D_1 D_2(u \alpha v) = D_1 D_2(u \alpha v) \]

\[ = D_1 D_2(u) \alpha \sigma(v) + D_2(u) \alpha d_1(v) + D_1(u) \alpha d_2(v) \]

\[ + \tau(u) \alpha d_1 d_2(v). \quad (3.4) \]

Comparing (3.3) and (3.4), we get

\[ D_2(u) \alpha d_1(v) + D_1(u) \alpha d_2(v) = 0. \quad (3.5) \]

By the hypothesis of (iv), we have

\[ D_1(u) \Gamma D_2(v) = 0. \quad (3.6) \]

Replacing \( v \) by \( v \beta w, \quad w \in N, \quad \beta \in \Gamma \) in (3.6) and using the same,

\[ D_1(u) \alpha \tau(v) \beta d_2(w) = 0 \]

\[ d_2(w) \gamma D_1(u) \alpha \tau(v) \beta d_2(w) \gamma D_1(u) = 0. \]

Since \( \tau \) is an automorphism and using the semiprimeness of \( N \), we get

\[ d_2(w) \gamma D_1(u) = 0. \quad (3.7) \]

Replacing \( w \) by \( v \beta w, \quad w \in N \) in (3.7) and using the same,

\[ d_2(v) \beta \sigma(w) \gamma D_1(u) = 0 \]

\[ D_1(u) \alpha d_2(v) \beta \sigma(w) \gamma D_1(u) \alpha d_2(v) = 0, \quad \forall u, v, w \in N, \gamma, \beta \in \Gamma \]

\[ D_1(u) \alpha d_2(v) \Gamma \sigma(w) \Gamma D_1(u) \alpha d_2(v) = 0. \]

Since \( \sigma \) is an automorphism and using the semiprimeness of \( N \), we get

\[ D_1(u) \alpha d_2(v) = 0, \]

and hence we can write

\[ D_2(u) \alpha d_1(v) = 0. \quad (3.8) \]

Replacing \( v \) by \( v \beta w, \quad w \in N, \quad \beta \in \Gamma \) in (3.8) and using the same

\[ D_2(u) \alpha \tau(v) \beta d_1(w) = 0 \]
ORTHOGONAL GENERALIZED \((\sigma, \tau)\)-DERIVATIONS IN SEMIPRIME \(\Gamma\)-NEAR RINGS

\[ d_1(w)\gamma D_2(u)\Gamma(v)\Gamma d_1(w)\gamma D_2(u) = 0 \]
\[ d_1(w)\gamma D_2(u) = 0 \quad \text{(By the semiprimeness of} \ N) \]

(3.9) \[ d_1(w)\Gamma D_2(u) = 0, \quad \forall u, w \in N. \]

By using the hypothesis of (iv), we have

(3.10) \[ D_1(u)\Gamma D_2(v) = 0. \]

Combining (3.9) and (3.10) and using condition (ii), We can conclude that \(D_1\) and \(D_2\) are orthogonal. \(\square\)

**Theorem 3.2.** Suppose \(N\) is a \(\Gamma\)-near ring that is semiprime and 2-torsion free. Let \(D_1\) and \(D_2\) be two generalized \((\sigma, \tau)\)-derivations of \(N\). Assume that \(D_1\) is orthogonal to \(d_2\) and \(D_2\) is orthogonal to \(d_1\). Then, we deduce the following:

(i) \(D_1D_2\) is a left centralizer of \(N\) and \(d_1d_2 = 0\).

(ii) \(D_2D_1\) is a left centralizer of \(N\) and \(d_2d_1 = 0\).

**Proof.** Suppose that \(D_1\) and \(D_2\) are orthogonal to \(d_2\) and \(d_1\), then

(3.11) \[ D_1(u)\alpha w\beta d_2(v) = 0, \quad \forall u, v, w \in N \text{ and } \alpha, \beta \in \Gamma. \]

Replacing \(u\) by \(w\delta u, w \in N, \delta \in \Gamma\) in (3.11) and using the same, we get

\[ \tau(w)\delta d_1(u)\alpha w\beta d_2(v) = 0 \]

and \(d_1(u)\alpha w\beta d_2(v)\gamma \tau(w)\delta d_1(u)\alpha w\beta d_2(v) = 0.\)

Since \(\tau\) is an automorphism on a semiprime \(\Gamma\)-near ring \(N\), we get \(d_1(u)\alpha w\beta d_2(v) = 0, \text{ for all } u, v, w \in N \text{ and } \alpha, \beta \in \Gamma. \) Therefore, \(d_1\) and \(d_2\) are orthogonal. Hence,

(3.12) \[ d_1d_2 = 0. \]

Since \(D_1, D_2\) are two generalized \((\sigma, \tau)\)-derivations of \(N\), we can write

\[ D_1D_2(u\alpha v) = D_1(D_2(u\alpha v)) = D_1D_2(u)\alpha \sigma(v) + D_1(u)\alpha d_2(v) + D_2(u)\alpha d_1(v) + \tau(u)\alpha d_1d_2(v). \]

Since \(D_1, D_2\) are orthogonal to \(d_2, d_1\), we have \(D_1(u)\alpha d_2(v) = 0 = D_2(u)\alpha d_1(v). \)

Also, by (3.12), we have \(d_1d_2 = 0. \) Then the above equation reduces to \(D_1D_2(u\alpha v) = D_1D_2(u)\alpha \sigma(v). \)
Hence, $D_1 D_2$ is a left centralizer of $N$. Similarly, we can prove result (ii) also.

Theorem 3.3. In a semiprime $\Gamma$-near ring $N$ which is 2-torsion free, considering $D_1$ as a generalized $(\sigma, \tau)$-derivation on $N$, the condition $D_1(u) \Gamma D_1(v) = 0$ for all $u, v \in N$ implies $D_1$ and $d_1$ are identically zero.

Proof. By the hypothesis,

(3.13) $D_1(u) \Gamma D_1(v) = 0, \forall u, v \in N.$

Replacing $v$ by $v \beta w, w \in N, \beta \in \Gamma$ in (3.13) and using the same, we get

$D_1(u) \alpha \tau(v) \beta d_1(w) = 0.$

Since $\tau$ is an automorphism of $N$ and using Lemma 2.1, we can have

$D_1(u) \alpha d_1(w) = 0 = d_1(w) \alpha D_1(u).$

Consider

(3.14) $d_1(w) \alpha D_1(u) = 0, \forall u, w \in N, \alpha \in \Gamma.$

Replacing $u$ by $u \beta w, w \in N, \beta \in \Gamma$ in (3.14) and using the same equation, we get

$d_1(w) \alpha \tau(u) \beta d_1(w) = 0, \forall u, w \in N, \alpha, \beta \in \Gamma.$

Since $\tau$ is an automorphism on a semiprime $\Gamma$-near ring $N$, we get $d_1 = 0.$

Again, consider $D_1(u) \Gamma D_1(v) = 0$ (by the hypothesis). Replace $u$ by $u \alpha v$, for all $v \in N, \alpha \in \Gamma$ in the above equation and using (3.14), we get

$D_1(u) \alpha \sigma(v) \beta D_1(v) = 0.$

Since $\sigma$ is an automorphism and using $N$ is semiprime, we get $D_1 = 0.$ Hence proved.

References


ORTHOGONAL GENERALIZED $($σ, τ$)$-DERIVATIONS IN SEMIPRIME Γ-NEAR RINGS


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