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EXTENDING CONVERGENCE ANALYSIS OF A LANDWEBER METHOD FOR SOLVING NONLINEAR ILL-POSED PROBLEMS

Ly Van An

ABSTRACT. In this paper, I use the Landweber iterative method to solve nonlinear problems. I analyze convergence and estimate error using assumptions such as the Landweber method is considered a regularization scheme when the iteration is stopped at the appropriate stage using the bias principle. I use the same difference principle is used in standard diagrams to stop proposed repeating diagrams.

1. INTRODUCTION

The Landweber iteration or Landweber algorithm is an algorithm to solve ill-posed linear inverse problems, and it has been extended to solve non-linear problems that involve constraints. The method was first proposed in the 1950 by Louis Landweber, and it can be now viewed as a special case of many other more general methods [1] . This paper is concerned with operator equations of the form

$$F(x) := y. \tag{1.1}$$

In there:

(1) Here we assume that $\mathbf{X}, \mathbf{Y}$ are real Hilbert spaces.
(2) $F : G(F) \subseteq \mathbf{X} \to \mathbf{Y}$ is a nonlinear operator on domain $G(F) \subset \mathbf{X}$.

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Note: For convenience in this article, the indices of inner products $< \cdot, \cdot >$ and norms $\| \cdot \|$ are neglected but they can always be identified from the context in which they appear. In this situation, actual data may not be available. Instead, I may have to deal with approximate data $\hat{y}$ satisfying the condition,

$$\Big\| y - \hat{y} \Big\| \leq \delta, \tag{1.2}$$

where $\delta > 0$ 0 is the given noise level. In such circumstances, I consider the operator equation

$$F(x) = \hat{y}. \tag{1.3}$$

For $\hat{y}$ satisfying (1.2). We assume that (1.3) has a solution $x^{\div}$. Which experience is not the only solution that satisfies $F(x^{\div}) = y$ and F has a non-invertible, locally uniformly bounded Fréchet derivative in a ball $B_r(x_0) := \Big\{ x \in X : \Big\| x - x_0 \Big\| < r \Big\}$. The regularization procedures are usually employed to obtain a stable approximate solution for problems (1.1)(see [2]). Tikhonov regularization is one of the widely used regularization schemes for obtaining such a solution (see [3] [4])in which the approximate solution, $\hat{x}_\zeta$, is obtained by minimizing the Tikhonov functional

$$\Big\| F(x) - \hat{y} \Big\|^2 + \zeta \Big\| x - x_0 \Big\|^2. \tag{1.4}$$

Herein x_0 is an initial guess and $\zeta > 0$ is the regularization parameter. An alternative approach is to consider the iterative schemes. Gauss–Newton type and Landweber iterative schemes are widely used in literature [5, 6]. The convergence and convergence rate analysis of nonlinear ill-posed problems can be carried out only with stronger assumptions compared to its linear counterpart. However, one can expect only local convergence rather than a global convergence. The Gauss-Newton and Landweber iterative scheme studied in literature [7] also belong to this category. A simplified approach for Gauss-Newton scheme has been studied in [8]. Our aim is to consider a simplified form of Landweber iterative scheme that gives the same convergence result as that of the standard scheme but with weaker assumptions. In this paper, we prove that the method converges to the x_0-minimum norm solution $x^{\div}$ with the rate of convergence $O\big(\delta^{m\beta/m\beta+1}\big), 0 < \beta \leq \frac{1}{m}, m \geq 2$ with specific smoothness assumption. For simplicity of analysis, I set $H := F(x_0)$. The standard Landweber iterative method for solving (1.3) is

$$\hat{x}_{k+1} = \hat{x}_k + F'(\hat{x}_k)^*(\hat{y} - F'(\hat{x}_k)), k \in \mathbb{N}, \hat{x}_0 = x_0, \tag{1.5}$$

where $F'(\hat{x}_k)^*, F'(\hat{x}_k)$ denote the Fréchet derivative and its adjoint respectively. In order to establish convergence and convergence rate analysis for (1.5), many assumptions have been used in literature. For the sake of completeness, we recall them here as follows:

$$\left\| y - \hat{y} \right\| \leq \delta, \tag{1.6}$$

and that F fulfils

(1) $F'(x) \leq 1$ for $x \in B_r(x_0)$
(2) $\left\| F(x) - F(\hat{x}) - F'(x)(x - \hat{x}) \leq \eta \right\| F(x) - F(\hat{x}) \Big\|, \eta < \frac{1}{2}, x, \hat{x} \in B_r(x_0)$
(3) $F'(x) = R_x F'(x_*)$, $x_*, x \in B_r(x_0)$, Where $\{R_x : x \in B_r(x_0)\}$ is a family of bounded linear operators $R_x : Y \to Y$ with $\left\| R_x - I \right\| \leq C \left\| x - x_* \right\|, x_*, x \in B_r(x_0), C > 0$,
(4) $x_* - x_0 = (F'(x_*)^* F'(x_*))^v f, 0 < v < \frac{1}{2}, f \in X$.

It is proved that the method achieves the convergence rate of $O\big(\delta^{m\beta/m\beta+1}\big), 0 < \beta \leq \frac{1}{m}, m \geq 2$ and the best possible rate attained is $O(\delta^{1/m})$ at $\beta = 1/m$ [9]. In practice, it is very difficult to verify all these assumptions. In many cases, Assumptions (1) and (2) can be verified. However, Assumptions (3) and (4) are quite difficult to deal with although it has been verified for a couple of examples [10]. Keeping this in mind, in this paper, we consider the following simplified form of Landweber iterative scheme

$$\hat{x}_{k+1} = \hat{x}_k + H(\hat{y} - F'(\hat{x}_k)), k \in \mathbb{N}, \tag{1.7}$$

where $H = F'(x_0)$. I make use of the following assumptions to establish the convergence of the method and derive the convergence rate.

2. Basis for convergence analysis

In this section, I discuss the convergence of the scheme (1.7) and make use of the assumption

$$\left\| F(x) - F(y) - H(x - y) \right\| \leq \eta \left\| F(x) - F(y) \right\|, \tag{2.1}$$

where $\eta < \frac{1}{2}, x, y \in B_r(x_0) \subset \Omega(F); H = F'(x_0)$ and $H \leq 1$, (2.1) for establishing the result. We note that one can deduce the following relation from the assumption (2.1):

$$\frac{1}{1+\Gamma}\Big\|H(x-y)\Big\| \leq \Big\|F(x)-F(y)\Big\| \leq \frac{1}{1-\Gamma}\Big\|H(x-y)\Big\|. \tag{2.2}$$

The Landweber iteration has the inherent property that the iteration converges first and then diverges. Therefore in the case of noisy data, for obtaining stable solution, the iteration has to be stopped after certain steps say, $k_* = k_*(\delta, \hat{y})$. We employ the same stopping rule used in standard Landweber method for this purpose. As a stopping criteria for our iterative scheme, we use the following discrepancy principle:

$$\Big\|\hat{y} - F(\hat{x}_{k_*})\Big\| \leq \tau\delta \leq \Big\|\hat{y} - F(\hat{x}_{k_*})\Big\|, \quad 0 \leq k \leq k_0. \tag{2.3}$$

Some theorems for convergence analysis.

Theorem 2.1. *Assume that the equation $F(x) = y$ has a solution x^+ in $B_{\frac{r}{2}}(x_0)$ and F satisfies the following conditions*

$$\Big\|F(x)-F(y)-H(x-y)\Big\| \leq \eta\Big\|F(x)-F(y)\Big\|, \tag{2.4}$$

where $\eta < \frac{1}{2}, x, y \in B_r(x_0) \subset \Omega(F); H = F'(x_0)$ and $H \leq 1$. If $\hat{x}_k \in B_{\frac{r}{2}}(x^+)$ a sufficient condition for $\hat{x}_{k+1}$ to be a better approximation of x^+ than $\hat{x}_k$ is that

$$\Big\|\hat{y} - F(\hat{x}_k)\Big\| \geq \frac{2\delta(1+\gamma)}{1-2\gamma}, \quad 0 \leq k < k_*, \tag{2.5}$$

So, $\hat{x}_{k+1} \in B_{\frac{r}{2}}(x^+) \subset B_{\frac{r}{2}}(x_0)$.

Theorem 2.2. *Assume that the equation $F(x) = y$ has a solution x^+ in $B_{\frac{r}{2}}(x_0)$ and F satisfies the following conditions*

$$\Big\|F(x)-F(y)-H(x-y)\Big\| \leq \eta\Big\|F(x)-F(y)\Big\|, \tag{2.6}$$

where $\eta < \frac{1}{2}, x, y \in B_r(x_0) \subset \Omega(F); H = F'(x_0)$, $H \leq 1$, and $\hat{x}_k \in B_{\frac{r}{2}}(x^+)$ a sufficient condition for x_{k+1} to be a better approximation of x^+ than $\hat{x}_k$ is that

$$\Big\|\hat{y} - F(\hat{x}_k)\Big\| \geq \frac{2\delta(1+\gamma)}{1-2\gamma}, \quad 0 \leq k < k_*. \tag{2.7}$$

So, $\hat{x}_{k+1} \in B_{\frac{r}{2}}(x^+) \subset B_{\frac{r}{2}}(x_0)$. *If* $\left\|\hat{y} - F(\hat{x}_k)\right\| \geq \tau, 0 \leq k < k_*$. *For* $\tau \geq \frac{2\delta(1+\gamma)}{1-2\gamma}$, *then*

$$k_*(\tau\delta) \leq \sum_{k=0}^{k_*} \left\|\hat{y} - F(\hat{x}_k)\right\|^2 \leq \frac{\tau}{(1-2\gamma) - 2(1+\gamma)} \left\|x_0 - x^+\right\|^2. \tag{2.8}$$

Theorem 2.3. *Assume that the assumption*

$$\left\|F(x) - F(y) - H(x-y)\right\| \leq \eta \left\|F(x) - F(y)\right\|, \tag{2.9}$$

Where $\eta < \frac{1}{2}, x, y \in B_r(x_0) \subset \Omega(F); H = F'(x_0)$ *and* $H \leq 1$ *holds and that the equation* $F(x) = y$ *is solvable in* $B_r(x_0)$. *Then, the iterative scheme* (1.7) *with* $\hat{y}$ *replaced by y converges to the solution of* $F(x) = y$.

Theorem 2.4. *Let the assumptions of Theorem 2.3 hold and let the iterative scheme* (1.7) *is stopped according to the stopping criteria* (2.4). *Then the iterates* x_{k_*} *converges to the solution of* $F(x) = y$ *as* $\delta \to 0$.

Article layout:

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subsection.**3**.**1** :Describe the Landweber iterative method for the inverse problem

subsection.**3**.**2** :Landweber's Algorithm for the inverse problem

section.**4** : convergence analysis In this section, I give two lemmas, Lemma 4.1 and Lemma 4.2, as a basis for convergence analysis.

3. Preliminaries

3.1. **Describe the Landweber iterative method for the inverse problem.** Let $F = (F_0, \ldots, F_{p-1})$ and $y = (y_0, \ldots, y_{p-1})$ then the Landweber iteration for solving

$$F_j(x) = y_j, j = 1, \ldots, p-1. \tag{3.1}$$

reads as follows

$$\begin{aligned} x_{k+1} &= x_k^\delta - F_j'(x_k^\delta)^*(F(x_k^\delta) - y^\delta) \\ &= x_k^\delta - \sum_{j=1}^{p-1} F_j'(x_k^\delta)^*(F(x_k^\delta) - y^\delta), k = 1, \ldots, . \end{aligned} \tag{3.2}$$

Let $B_r(x_0)$ be an open ball of radius r containing x_*.

I/ The conditions A_I

(1) F is Frèchet differentiable on $B_r(x_0)$
(2) $F'(x) \leq 1$ for $x \in B_r(x_0)$
(3) $\Big\|F(x) - F(\hat{x}) - F'(x)(x - \hat{x}) \leq \Big\|F(x) - F(\hat{x})\Big\|, \eta < \frac{1}{2}, x, \hat{x} \in B_r(x_0)$

are strong enough to ensure at least local convergence to a solution of

$$F_j(x) = y_j, j = 1, \ldots, p-1. \tag{3.3}$$

II/ The conditions A_{II}

If y^δ does not belong to the range of F , then the iterates x_k^δ of (3.2) cannot converge but still allow a stable approximation of x_* provided the iteration is stopped after $k_* = k_*^\delta$ steps acording to the generalized discrepancy principle

$$\Big\|y^\delta - F(x_{k_*}^\delta)\Big\| \leq \tau\delta \leq \Big\|y^\delta - F(x_k^\delta)\Big\|, 0 \leq k \leq k_0, \quad \text{for} \quad \tau > 2\frac{1+\eta}{1-2\eta} > 2. \tag{3.4}$$

When speaking of convergence rates to a solution of (3.1) of an iterative method $x_{k+1} = U(x_k)$ for solving an illposed problem we understand:

$$(a) \quad if \quad \delta = 0 \quad \text{the rate of} \quad \Big\|x_* - x_k\Big\| \to 0 \quad as \quad k \to \infty. \tag{3.5}$$

$$(b) \quad if \quad \delta > 0 \quad \text{the rate of} \quad \Big\|x_* - x_{k_*(\delta)}\Big\| \to 0 \quad \text{as} \quad \delta \to \infty. \tag{3.6}$$

Under the general assumptions A_I the rate of convergence of $x_k \to x_*$ as $k \to \infty$ (with precise data, i.e. $\delta = 0$) or $x_{k_*(\delta)}^\delta \to x_*$ as $\delta \to 0$ (with perturbed data) will, in general, be arbitrarily slow. This is known for linear ill-posed problems $Kx = y$ where the rate of convergence is almost ompletely determined by the tuple $(v; \|f\|)$ in the source-wise representation

$$x_* - x_0 = (K^*K)^v f, v > 0, f \in X \tag{3.7}$$

cf. Example 3.1 and Theorem 7.3 in (see [11]). The same parameters also determine the rate of convergence of Tikhonov regularization (see [12]); the corresponding numbers

$$x_* - x_0 = (F'(x_*)^* F'(x_*))^v f, v > 0, f \in X \tag{3.8}$$

play the same role in Tikhonov regularization for nonlinear problems (see [13]) and in many iterative regularization methods (see [14]). In contrast to Tikhonov regularization, assumption (3.8) (with $\|f\|$ being sufficiently small) is not enough to obtain convergence rates for the Landweber iteration; we need further properties of F: we require

$$F(x) = R_x F'(x_*), x \in B_r(x_0), \tag{3.9}$$

where $\{R_x : x \in B_r(x_0)\}$ is a family of bounded linear operators $R_x : Y \to Y$ with

$$\|R_x - I\| \leq C\|x - x_*\| x \in B_r(x_0), \tag{3.10}$$

and C is a positive constant. Note that in the linear case $R_x \equiv I$; therefore, (3.9) may be interpreted as a further restriction of the ”non-linearity" of F. In particular, (3.9) implies that

$$\mathcal{N}(F'(x_*)) \subset \mathcal{N}(F'(x)), x \in B_r(x_0).$$

It is not difficult to see that (3.9), (3.10) imply (3.9) with $\hat{x} = x_*$ for r being sufficiently small.

Theorem 3.1. *Assume that problem* (3.1)

$$F_j(x) = y_j, j = 1, \ldots, p-1, \tag{3.11}$$

has a solution in $B_r(x_0)$, that y^δsatisfes

$$\left\|y_j^\delta - y_j\right\| \leq \delta j \in \{0, 1, \ldots, p-1\}, \tag{3.12}$$

and that F fulfils

(1) $F'(x) \leq 1$ *for* $x \in B_r(x_0)$
(2) $\left\|F(x) - F(\hat{x}) - F'(x)(x - \hat{x}) \leq \eta\right\|F(x) - F(\hat{x})\Big\|, \eta < \frac{1}{2}, x, \hat{x} \in B_r(x_0)$
(3) $F(x) = R_x F'(x_*)$, $x \in B_r(x_0)$, *where* $\{R_x : x \in B_r(x_0)\}$ *is a family of bounded linear operators* $R_x : Y \to Y$ *with* $\|R_x - I\| \leq C\|x - x_*\|, x \in B_r(x_0)$.

If $x_* - x_0$ satisfies

$$x_* - x_0 = (F'(x_*)^* F'(x_*))^v f, v > 0, f \in X$$

with some $0 < v \leq \frac{1}{2}$ *and* $\left\|f\right\|$ *being sufficiently small, then there exists a positive constant* c_*, *depending on* v

$$\left\|x_* - x_k^\delta\right\| \leq \left\|f\right\|(k+1)^{-v} \tag{3.13}$$

and

$$\left\|y^\delta - F(x_k^\delta)\right\| \leq \left\|f\right\|(k+1)^{-v-1/2} \tag{3.14}$$

for all $0 \leq k \leq k_*$. *For* $\delta = 0$ (3.1) *and* (3.13) *holds for all* $k \geq 0$. *Furthermore, for* $\delta > 0$

$$k_* \leq c_1(\left\|f\right\|/\delta)^{2/(2v+1)} \tag{3.15}$$

and

$$\left\|x_* - x_{k_*}^\delta\right\| \leq c_2\left\|f\right\|^{1/(2v+1)}\delta^{2v/(2v+1)} \tag{3.16}$$

for some constsnts $c_1, c_2 > 0$, *depending on* v *only.*

3.2. **Landweber's Algorithm for the inverse problem.** The Landweber algorithm for solving the system $Ax = b$ is

$$x_{k+1} = x_k + \gamma A^{\div}(b - Ax_k)$$

where γ is a selected parameter. We can write the Landweber iteration as

$$x_{k+1} = Tx_k$$

for

$$Tx = (I - \gamma A^{\div}A)x + A^{\div}b = Bx + h.$$

The Landweber algorithm actually solves the square linear system $A^{\div}A = A^{\div}b$ for a least-squares solution of $Ax = b$. When there is a unique solution or unique least-squares solution of $Ax = b$, say $\hat{x}$, then the error at the $k - th$ step is $e_k = \hat{x} - x_k$ and we see that

$$Be_k = e_{k+1}.$$

We want $e_k \to 0$, so we want $\left\|B\right\|_* < 1$, this means that both T and B are Euclidean strict contractions. Since B is Hermitian, B will be strict contractions if and only $\left\|B\right\|_* < 1$, where $\left\|B\right\|_* = \rho(B)$ is the matrix norm induced by the Euclidean vector norm. On the other hand, when there are multiple solutions of

$Ax = b$, the solution found by the Landweber algorithm will be the one closest to the starting vector. In this case, we cannot define e_k and we do not want $\left\|B\right\|_* < 1$; that is, we do not need that B be a strict contraction, but something weaker. As we shall see, since B is Hermitian, B will be av whenever γ lies in the interval $(0, 2/\rho(B))$.

4. CONVERGENCE ANALYSIS

In this section, I give two lemmas, Lemma 4.1 and Lemma 4.2, as a basis for convergence analysis.

Lemma 4.1. *Let $s \in [0,1]$ and suppose that $H : X \to Y$ be a bounded linear operator with the property $\left\|H\right\| \le 1$. Then*

$$\left\| \prod_{j_k=0}^{n_k-1} \prod_{j_{k-1}=0}^{n_{k-1}-1} \cdots \prod_{j_1=0}^{n_1-1} (I - B^*B)^{n_1+n_2+\ldots+n_k} (B^*B)^s \right\| \le (\prod_{j=1}^{k} n_j + 1)^{-s} \tag{4.1}$$

$$\left\| \prod_{j_k=0}^{n_k-1} \prod_{j_{k-1}=0}^{n_{k-1}-1} \cdots \prod_{j_1=0}^{n_1-1} (I - B^*B)^{n_1+n_2+\ldots+n_k} B^* \right\| \le (\prod_{j=1}^{k} n_j + 1)^{-1/2} \tag{4.2}$$

$$\begin{aligned} &\left\| \sum_{j_1=0}^{n_1-1} (I - B^*B)^{j_1} (BB^*)^s + \sum_{j_2=0}^{n_2-1} (I - B^*B)^{j_2} (BB^*)^s + \cdots \right. \\ &\left. + \sum_{j_k=0}^{n_k-1} (I - B^*B)^{j_k} (BB^*)^s \right\| \le (\prod_{j=1}^{k} n_j)^{1-s} \end{aligned} \tag{4.3}$$

Lemma 4.2. *Give $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_k$ are positive real numbers. Then, there is a positive constant $C(\max\{p_1, p_2, \ldots, p_k\}, \max\{q_1, q_2, \ldots, q_k\})$ independent of $n_1, n_2, \ldots, n_k$ so that*

$$\begin{aligned} &\sum_{j_1=0}^{n_1-1} (j_1+1)^{-p_1} (n_1 - j_1)^{-q_1} + \sum_{j_2=0}^{n_2-1} (j_2+1)^{-p_2} (n_2 - j_2)^{-q_2} + \cdots \\ &+ \sum_{j_k=0}^{n_k-1} (j_1+1)^{-p_k} (n_k - j_k)^{-q_k} \end{aligned}$$

$$\leq C(\max\{p_1,p_2,\dots,p_k\},\max\{q_1,q_2,\dots,q_k\}) \times (\sum_{j=1}^{k} n_j+1)^{1-\sum_{i=1}^k p_i-\sum_{i=1}^k q_i}$$

$$\times \begin{cases} 1 \\ \qquad \{\max\{p_1,\dots,p_k\},\max\{q_1,\dots,q_k\}\} < \frac{1}{m^2} \\ \ln(\prod_{j=1}^k n_j+1) \\ \qquad \{\max\{p_1,\dots,p_k\},\max\{q_1,\dots,q_k\}\} = \frac{1}{m^2} \\ (\prod_{j=1}^k n_j+1)^{\{\{\max\{p_1,\dots,p_k\},\max\{q_1,\dots,q_k\}\}-1} \\ \qquad \{\max\{p_1,\dots,p_k\},\max\{q_1,\dots,q_k\}\} > \frac{1}{m^2}, \\ m\geq 2. \end{cases} \tag{4.4}$$

Throughout this section, to simplify the notation we set $e_k := \hat{x}_k - x^+, a_k := \hat{x}_k - x_0$. First we prove the following result.

Theorem 4.1. *Assume that the equation $F(x) = y$ has a solution in $B_{\frac{r}{2}}(x_0)$ and F satisfies the following conditions*

(1) $\Big\|F(x)-F(y)-H(x-y)\Big\| \leq \eta\Big\|F(x)-F(y)\Big\|$, *where* $\eta < \frac{1}{2}, x, y \in B_r(x_0) \subset \Omega(F); H = F'(x_0)$ *and* $H \leq 1$,
(2) $x^{\div} - x_0 = (H^*H)^{\beta}\gamma, 0 < \beta \leq 1/2$,
(3) $\Big\|F(x) - F(y) - H(x-y)\Big\| \leq M_1\Big(\Big\|x-y\Big\| + \Big\|x-x_0\Big\|\Big)\Big\|F(x)-F(y)\Big\|$, $\forall x, y \in B_r(x_0)$.

Then, there exists a positive constant η depending on β only, with

$$\Big\|e_k\Big\| \leq \Big\|\gamma\Big\|(\prod_{j=1}^k n_j+1)^{-\beta}$$

$$\Big\|a_k\Big\| \leq \Big\|\gamma\Big\|(\prod_{j=1}^k n_j+1)^{-\beta}$$

$$\Big\|Ke_k\Big\| \leq \Big\|\gamma\Big\|(\prod_{j=1}^k n_j+1)^{-\beta-1/2}, for \quad 0\leq k\leq k_*. \tag{4.5}$$

Proof. First, I try to find out an expression for e_k and a_k, $0 \leq k \leq k_*$:

$$\begin{aligned} e_{k+1} &= e_k + H^*(\hat{y} - F(\hat{x_k})) - H^*(F(\hat{x_k}) - F(x^{\div})) + H^*(F(\hat{x_k}) - F(x^{\div})) \\ &= e_k - H^*(F(\hat{x_k}) - F(x^{\div})) + H^*(\hat{y} - F(\hat{x_k})) \\ &= (I - H^*H)e_k - H^*\Big[F(\hat{x_k}) - F(x^{\div}) - He_k\Big] + H^*(\hat{y} - F(\hat{x_k})) \qquad (4.6) \end{aligned}$$

$$\begin{aligned} a_{k+1} &= a_k + H^*(\hat{y} - F(\hat{x_k})) \\ &= a_k - H^*(F(\hat{x_k}) - F(x^{\div})) + H^*(\hat{y} - F(\hat{x_k})) \\ &= (I - H^*H)a_k - H^*\Big[F(\hat{x_k}) - F(x^{\div}) - Ha_k\Big] + H^*(\hat{y} - y) \\ &= (I - H^*H)a_k + H^*He_0 - H^*\Big[F(\hat{x_k}) - F(x^{\div}) - Ha_k\Big] + H^*(\hat{y} - y) \qquad (4.7) \end{aligned}$$

I put $Z_k = -\Big[F(\hat{x_k}) - F(x^{\div}) - He_k\Big]$. For $0 \leq k \leq k_*$. So

$$e_{k+1} = (I - H^*H)e_k - H^*Z_k + H^*(\hat{y} - y) \qquad (4.8)$$

and

$$a_{k+1} = (I - H^*H)a_k + (H^*H)^{\beta+1}\gamma - H^*Z_k + H^*(\hat{y} - y). \qquad (4.9)$$

For $0 \leq n \leq n_*$. Therefore closed expression for error is. Without loss of generality I hypothesize that $n_1 = n_2 = \ldots = n_k = n$. So I have

$$e_n = (I - H^*H)^{kn}e_0 + k\sum_{j=0}^{n-1}(I - H^*H)^j H^* Z_{n-j-1} \qquad (4.10)$$

$$+ \left[k\sum_{j=0}^{n-1}(I - H^*H)^j H^*\right](\hat{y} - y) \qquad (4.11)$$

and consequently

$$He_n = (I - H^*H)^{kn}e_0 + k\sum_{j=0}^{n-1}(I - H^*H)^j H^* Z_{n-j-1} \qquad (4.12)$$

$$+ \left[I + (I - H^*H)^{kn}H^*\right](\hat{y} - y). \qquad (4.13)$$

For $0 \leq n \leq n_*$,

$$a_n = k\sum_{j=0}^{n-1}(I - H^*H)^j(H^*H)^{\beta+1}\gamma + k\sum_{j=0}^{n-1}(I - H^*H)^j H^* Z_{n-j-1}$$

$$+ \left[k\sum_{j=0}^{n-1}(I - H^*H)^j H^*\right](\hat{y} - y). \tag{4.14}$$

So

$$a_n = k\sum_{j=0}^{n-1}(I - H^*H)^j(H^*H)^{\beta+1}\gamma \pm (I - H^*H)^{kn}(H^*H)^{\beta+1}\gamma \tag{4.15}$$

$$+ k\sum_{j=0}^{n-1}(I - H^*H)^j H^* Z_{n-j-1} + \left[k\sum_{j=0}^{n-1}(I - H^*H)^j H^*\right](\hat{y} - y)$$

$$= k\sum_{j=0}^{n-1}(I - H^*H)^j(H^*H)^{\beta+1}\gamma - (I - H^*H)^{kn}(H^*H)^{\beta+1}\gamma \tag{4.16}$$

$$+ k\sum_{j=0}^{n-1}(I - H^*H)^j H^* Z_{n-j-1} + \left[k\sum_{j=0}^{n-1}(I - H^*H)^j H^*\right](\hat{y} - y).$$

In order to prove the result for $0 \leq n < n^*$, I use induction. For $n = 0$, the proof is trivial and we assume that the result is true for $\forall j$ such that $0 \leq j < n$, where $n < n^*$:

$$\left\|e_n\right\| \leq \left\|(I - H^*H)^{kn}(H^*H)^{\beta}e_0\right\| + k\sum_{j=0}^{n-1}\left\|(I - H^*H)^j H^*\right\|\left\|Z_{n-j-1}\right\|$$

$$+ \left\|k\sum_{j=0}^{n-1}(I - H^*H)^j H^*\right\|\delta$$

$$\leq (n^k + 1)^{-\beta}\left\|\gamma\right\| + k\sum_{j=0}^{n-1}(j+1)^{-1/2}\left\|Z_{n-j-1}\right\| + \sqrt{n}\delta \tag{4.17}$$

$$\begin{aligned}\left\|a_n\right\| &\leq \left\|k\sum_{j=0}^{n}(I-H^*H)^j(H^*H)^{\beta+1}\right\|\left\|\gamma\right\| + \left\|(I-H^*H)^{nk}(H^*H)^{\beta+1}\right\|\left\|\gamma\right\| \\ &\quad + k\sum_{j=0}^{n-1}\left\|(I-H^*H)^jH^*\right\|\left\|Z_{n-j-1}\right\| + \left\|k\sum_{j=0}^{n-1}(I-H^*H)^jH^*\right\|\delta \\ &\leq (n^k+1)^{-\beta}\left\|\gamma\right\| + (n^k+1)^{-\beta-1}\left\|\gamma\right\| + k\sum_{j=0}^{n-1}(j+1)^{-1/2}\left\|Z_{n-j-1}\right\| + \sqrt{n}\delta \end{aligned} \tag{4.18}$$

$$\begin{aligned}\left\|He_n\right\| &\leq \left\|(I-H^*H)^{kn}He_0\right\| + k\sum_{j=0}^{n-1}\left\|(I-H^*H)^jHH^*\right\|\left\|Z_{n-j-1}\right\| + \delta \\ &\leq (n^k+1)^{-\beta-1/2}\left\|\gamma\right\| + k\sum_{j=0}^{n-1}(j+1)^{-1}\left\|Z_{n-j-1}\right\| + \delta. \end{aligned} \tag{4.19}$$

Now by making use of (2.2) and assumption (4.1), we get

$$\begin{aligned}\left\|Z_n\right\| &\leq \frac{c}{1-\alpha}\left\|He_n\right\|\left\|a_n\right\| + \frac{c}{1-\alpha}\left\|He_n\right\|\left\|e_n\right\| \\ &\leq mc\left\|He_n\right\|\left\|a_n\right\| + mc\left\|He_n\right\|\left\|e_n\right\|, m \geq 2. \end{aligned} \tag{4.20}$$

By using induction assumption, I have

$$\left\|Z_{n-j-1}\right\| \leq m^2c\eta^2\left\|\gamma\right\|^2(n-j)^{-2\beta-1/2}. \tag{4.21}$$

So

$$\begin{aligned}k\sum_{j=0}^{n-1}(j+1)^{-1/2}\left\|Z_{n-j-1}\right\| &\leq k\sum_{j=0}^{n-1}(j+1)^{-1/2}m^2c\eta^2\left\|\gamma\right\|^2(n-j)^{-2\beta-1/2} \\ &= m^2c\eta^2\left\|\gamma\right\|^2 k\sum_{j=0}^{n-1}(j+1)^{-1/2}(n-j)^{-2\beta-1/2} \end{aligned} \tag{4.22}$$

or

$$\begin{aligned}\sum_{j=0}^{n-1}(j+1)^{-1/2}\left\|Z_{n-j-1}\right\| &\leq \sum_{j=0}^{n-1}(j+1)^{-1/2}m^2c\eta^2\left\|\gamma\right\|^2(n-j)^{-2\beta-1/2} \\ &= \frac{m^2c\eta^2}{k}\left\|\gamma\right\|^2 k\sum_{j=0}^{n-1}(j+1)^{-1/2}(n-j)^{-2\beta-1/2}. \end{aligned} \tag{4.23}$$

Hence, by using the Lemma 4.2,

$$\sum_{j=0}^{n-1}(j+1)^{-1/2}(n-j)^{-2\beta-1/2}$$

$$\leq C(1/2,2\beta+1/2)(n^k+1)^{-2\beta}\begin{cases}\frac{1}{k} & If\beta<1/m^2\\ \frac{ln(n^k+1)}{k} & If\beta=1/m^2\\ \frac{(n^k+1)^{2\beta-1/2}}{k} & If\beta>1/m^2\end{cases}. \tag{4.24}$$

So

$$\sum_{j=0}^{n-1}(j+1)^{-1/2}\Big\|Z_{n-j-1}\Big\| \leq \frac{M_\beta}{k}\Big\|\gamma\Big\|^2(n^k+1)^{-\beta}+\frac{\sqrt{n^k}}{k}\delta, \tag{4.25}$$

where M is a constant that depends on β. So, I have

$$\begin{aligned}\Big\|e_n\Big\| &\leq \frac{1}{k}\left((n^k+1)^{-\beta}\Big\|\gamma\Big\|+M_\beta\Big\|\gamma\Big\|^2(n^k+1)^{-\beta}+\sqrt{n^k}\delta\right)\\ &\leq \frac{1}{k}(1+M_\beta\Big\|\gamma\Big\|)\Big\|\gamma\Big\|(n^k+1)^{-\beta}+\frac{\sqrt{n^k}}{k}\delta\\ &\leq \frac{1}{k}(m+M_\beta\Big\|\gamma\Big\|)\Big\|\gamma\Big\|(n^k+1)^{-\beta}+\frac{\sqrt{n^k}}{k}\delta, m\geq 2\\ &\leq \frac{1}{k}(m+M_\beta\Big\|\gamma\Big\|)\Big\|\gamma\Big\|(n^k+1)^{-\beta}+\frac{(n^k+1)^{1/2}}{k}\delta, m\geq 2 \end{aligned} \tag{4.26}$$

$$\begin{aligned}\Big\|a_n\Big\| &\leq \frac{1}{k}\left(m(n^k+1)^{-\beta}\Big\|\gamma\Big\|+M_\beta(n^k+1)^{-\beta}\Big\|\gamma\Big\|^2+\sqrt{n^k}\delta\right)\\ &\leq \frac{(m+M_\beta}{k}\Big\|\gamma\Big\|)\Big\|\gamma\Big\|(n^k+1)^{-\beta}+\frac{\sqrt{n^k}}{k}\delta, m\geq 2\\ &\leq \frac{(m+M_\beta}{k}\Big\|\gamma\Big\|)\Big\|\gamma\Big\|(n^k+1)^{-\beta}+\frac{(n^k+1)^{1/2}}{k}\delta, m\geq 2 \end{aligned} \tag{4.27}$$

By using Lemma 4.2,

$$k\sum_{j=0}^{n-1}(j+1)^{-1}(n-j)^{-2\beta-1/2}$$

$$\leq C(1/2,2\beta+1/2)(n^k+1)^{-2\beta}\begin{cases}\ln(n^k+1) & If\beta\leq 1/m^2\\ (n^k+1)^{2\beta-1/2} & If\beta>1/m^2\end{cases}. \tag{4.28}$$

Hence,

$$\sum_{j=0}^{n-1}(j+1)^{-1}\left\|Z_{n-j-1}\right\| \leq \frac{M_\beta}{k}\left\|\gamma\right\|^2(n^k+1)^{-\beta}. \tag{4.29}$$

So,

$$\left\|He_n\right\| \leq \frac{(m+M_\beta}{k}\left\|\gamma\right\|)\left\|\gamma\right\|(n^k+1)^{-\beta-1/2}+\frac{\delta}{k}, m\geq 2. \tag{4.30}$$

For $0 \leq n \leq n_*$ discrepancy principle (8) gives,

$$\left[\frac{m(1+\alpha)}{1-m\alpha}\right]\delta \leq \delta \leq \tau\delta < \left\|\hat{y}-F(\hat{x})\right\|\delta + \frac{1}{1-\alpha}\left\|He_n\right\|. \tag{4.31}$$

Hence, by making use of the above result (4.30), I get

$$\begin{aligned}\left[\frac{m(1+\alpha)}{1-m\alpha}\right]\delta &\leq \delta + \frac{1}{1-\alpha}(m+M_\beta\left\|\gamma\right\|)\left\|\gamma\right\|(n^k+1)^{-\beta-1/2}+\frac{1}{1-\alpha}\delta \\ &\leq \delta + \frac{1}{k(1-\alpha)}(m+M_\beta\left\|\gamma\right\|)\left\|\gamma\right\|(n^k+1)^{-\beta-1/2}+\frac{1}{k\left|1-m\alpha\right|}\delta \\ &\leq \frac{m(1+\alpha)}{\left|1-m\alpha\right|}\delta + \frac{1}{k(1-\alpha)}(m+M_\beta\left\|\gamma\right\|)\left\|\gamma\right\|(n^k+1)^{-\beta-1/2}. \end{aligned} \tag{4.32}$$

This would give

$$\frac{m^2\alpha}{\left|1-m\alpha\right|}\delta \leq \frac{1}{k(1-\alpha)}(m+M_\beta\left\|\gamma\right\|)\left\|\gamma\right\|(n^k+1)^{-\beta-1/2}.$$

So, I have

$$\delta \leq \frac{\left|1-m\alpha\right|}{km^2(1-\alpha)}(m+M_\beta\left\|\gamma\right\|)\left\|\gamma\right\|(n^k+1)^{-\beta} \tag{4.33}$$

$$\left\|e_n\right\| \leq \left(1+\frac{\left|1-m\alpha\right|}{km^2(1-\alpha)}\right)(m+M_\beta\left\|\gamma\right\|)\left\|\gamma\right\|(n^k+1)^{-\beta} \tag{4.34}$$

$$\left\|a_n\right\| \leq \left(1+\frac{\left|1-m\alpha\right|}{km^2(1-\alpha)}\right)(m+M_\beta\left\|\gamma\right\|)\left\|\gamma\right\|(n^k+1)^{-\beta} \tag{4.35}$$

$$\Big\|He_n\Big\| \leq \left(1+\frac{|1-m\alpha|}{km^2(1-\alpha)}\right)(m+M_\beta\Big\|\gamma\Big\|)\Big\|\gamma\Big\|(n^k+1)^{-\beta-1/2} \tag{4.36}$$

Or in general, we have something to prove

$$\delta \leq \frac{|1-m\alpha|}{km^2(1-\alpha)}(m+M_\beta\Big\|\gamma\Big\|)\Big\|\gamma\Big\|(n^k+1)^{-\beta} \tag{4.37}$$

$$\Big\|e_n\Big\| \leq \left(1+\frac{|1-m\alpha|}{km^2(1-\alpha)}\right)(m+M_\beta\Big\|\gamma\Big\|)\Big\|\gamma\Big\|(\prod_{j=1}^{k} n_j+1)^{-\beta} \tag{4.38}$$

$$\Big\|a_n\Big\| \leq \left(1+\frac{|1-m\alpha|}{km^2(1-\alpha)}\right)(m+M_\beta\Big\|\gamma\Big\|)\Big\|\gamma\Big\|(\prod_{j=1}^{k} n_j+1)^{-\beta} \tag{4.39}$$

$$\Big\|He_n\Big\| \leq \left(1+\frac{|1-m\alpha|}{km^2(1-\alpha)}\right)(m+M_\beta\Big\|\gamma\Big\|)\Big\|\gamma\Big\|(\prod_{j=1}^{k} n_j+1)^{-\beta-1/2} \tag{4.40}$$

□

Theorem 4.2. *Assume that the equation $F(x) = y$ has a solution in $B_{\frac{r}{2}}(x_0)$ and F satisfies the following conditions*

(1) $\Big\|F(x)-F(y)-H(x-y)\Big\| \leq \eta\Big\|F(x)-F(y)\Big\|$, *where:* $\eta < \frac{1}{2}, x, y \in B_r(x_0) \subset \Omega(F); H = F'(x_0)$ *and* $H \leq 1$,
(2) $x^{\div} - x_0 = (H^*H)^\beta\gamma, 0 < \beta \leq 1/2$,
(3) $\Big\|F(x) - F(y) - H(x-y)\Big\| \leq M_1(\Big\|x-y\Big\| + \Big\|x-x_0\Big\|)\Big\|F(x) - F(y)\Big\|$. $\forall x, y \in B_r(x_0)$

Then, according to the assumption of the theorem, we always have

$$\Big\|n_*\Big\| \leq M_1\Big(\frac{\|\gamma\|}{\delta}\Big)^{\frac{m}{m\beta+1}}.$$

$$\Big\|x^{\div} - \hat{x}_{n_*}\Big\| \leq M_2\Big\|\gamma\Big\|^{\frac{1}{m\beta+1}}\delta^{\frac{m}{m\beta+1}} \tag{4.41}$$

a positive constant M_1, M_2 depending on β only, $m \geq 2$.

Proof. From (4.46) For $0 \leq n \leq n_*$. Therefore closed expression for error is. Without loss of generality I hypothesize that $n_1 = n_2 = \ldots = n_k = n$. So I have

$$\begin{aligned} e_{n_*} &= (I - H^*H)^{kn_*}\gamma + k\sum_{j=0}^{n_*-1}(I - H^*H)^j H^* Z_{n-j-1} \\ &+ \left[k\sum_{j=0}^{n_*-1}(I - H^*H)^j H^*\right](\hat{y} - y) \\ &= (H^*H)^\beta Q_{n_*} + \left[k\sum_{j=0}^{n_*-1}(I - H^*H)^j H^*\right](\hat{y} - y). \end{aligned} \tag{4.42}$$

In there

$$Q_{n_*} = (I - H^*H)^{kn_*}e_0 + k\sum_{j=0}^{n_*-1}(I - H^*H)^j (H^*H)^{1/2-\beta}\hat{Z}_{n_*-j-1}, \tag{4.43}$$

with

$$\left\|\hat{Z}_{n_*}\right\| = \left\|Z_j\right\|, j = 0, 1, \ldots, n_* - 1,$$

$$\begin{aligned} \left\|Q_{n_*}\right\| &\leq \left\|(I - H^*H)^{kn_*}\gamma\right\| + k\sum_{j=0}^{n_*-1}\left\|(I - H^*H)^j (H^*H)^{1/2-\beta}\right\|\left\|Z_{n_*-j-1}\right\| \\ &\leq (n_* + 1)^0\left\|\gamma\right\| + k\sum_{j=0}^{n_*-1}\left\|(j+1)^{\beta-1/2}\right\|\left\|Z_{n_*-j-1}\right\|. \end{aligned} \tag{4.44}$$

So,

$$k\sum_{j=0}^{n-1}(j+1)^{-1/2}\left\|Z_{n_*-j-1}\right\| \leq m^2 c\eta^2\left\|\gamma\right\|^2 k\sum_{j=0}^{n-1}(j+1)^{-1/2}(n_* - j)^{-2\beta-1/2}. \tag{4.45}$$

Hence, by using the Lemma 4.2,

$$\begin{aligned} &k\sum_{j=0}^{n-1}(j+1)^{\beta-1/2}(n-j)^{-2\beta-1/2} \\ &\leq C(1/2-\beta, 2\beta+1/2)(n^k+1)^{-2\beta}\begin{cases} 1 & If \beta < 1/m^2 \\ ln(n^k+1) & If \beta = 1/m^2 \\ (n^k+1)^{2\beta-1/2} & If \beta > 1/m^2 \end{cases} \end{aligned} \tag{4.46}$$

$$\sum_{j=0}^{n-1}(j+1)^{\beta-1/2}(n-j)^{-2\beta-1/2}$$

$$\leq C(1/2-\beta,2\beta+1/2)(n^k+1)^{-2\beta}\begin{cases}\frac{1}{k} & If \beta < 1/m^2\\ \frac{ln(n^k+1)}{k} & If \beta = 1/m^2\\ \frac{(n^k+1)^{2\beta-1/2}}{k} & If \beta > 1/m^2\end{cases}. \tag{4.47}$$

We know that γ has to be small, therefore I consider $\gamma \leq 1$. Hence I have,

$$Q_{k_*} \leq \|\gamma\| + \hat{M}_\beta\|\gamma\|^2 \leq (\hat{M}_\beta + 1)\|\gamma\|.$$

Therefore,

$$\begin{aligned}\Big\|H(H^*H)^\beta Q_{n_*}\Big\| &= \Big\|He_n - (I-(I-H^*H)^{kn}(\hat{y}-y)\Big\|\\ &\leq \Big\|He_n\Big\| + \frac{\delta}{k}\\ &\leq \Big\|F(\hat{x}_{n_*} - F(x_*)) - He_{n_*}\Big\| + \Big\|F(\hat{x}_{n_*} - F(x_*))\Big\| + \frac{\delta}{k}\\ &\leq \eta\Big\|F(\hat{x}_{n_*} - F(x_*))\Big\| + \Big\|y - F(\hat{x}_{n_*})\Big\| + \frac{\delta}{k}\\ &\leq (\eta+1)\Big\|y - F(\hat{x}_{n_*})\Big\| + \frac{\delta}{k}\\ &\leq \frac{((\eta+1)(1+\tau)+1)\delta}{k}. \end{aligned}\tag{4.48}$$

From $\frac{1}{m\beta+1} + \frac{m\beta}{m\beta+1} = 1$,

$$\Big\|H(H^*H)^\beta Q_{n_*}\Big\| \leq \Big(\frac{(\eta+1)(1+\tau)+1)\delta}{k}\Big)^{\frac{m\beta}{m\beta+1}}\Big((\hat{M}_\beta+1)\|\gamma\|\Big)^{\frac{1}{m\beta+1}}$$

$$\leq M\Big(\frac{\delta}{k}\Big)^{\frac{m\beta}{m\beta+1}}\|\gamma\|^{\frac{1}{m\beta+1}} \tag{4.49}$$

where M is some positive constant and note that when $n_* = 0$, $\Big\|e_{n_*}\Big\| \leq M\|\gamma\|^{\frac{1}{m\beta+1}}\Big(\frac{\delta}{k}\Big)^{\frac{m\beta}{m\beta+1}}$ and when $n_* > 0$, I apply (4.37) with $n = n_* - 1$ to obtain

$$\frac{\delta}{k} \leq \eta\|\gamma\|(n_*)^{-\beta-1/2}. \tag{4.50}$$

Now I put $\delta_1 = \frac{\delta}{k}$,

$$\Big\|n_*\Big\| \leq M_1\Big(\frac{\big\|\gamma\big\|}{\delta_1}\Big)^{\frac{m}{m\beta+1}}. \tag{4.51}$$

By making use of this result we get,

$$\begin{aligned}\Big\|e_{n_*}\Big\| &\leq \Big\|H(H^*H)^\beta Q_{n_*}\Big\| + \sqrt{n_*}\delta_1\\ &\leq M\big\|\gamma\big\|^{\frac{1}{m\beta+1}}\delta_1^{\frac{m\beta}{m\beta+1}} + M_1\Big(\big\|\gamma\big\|^{\frac{m}{m\beta+1}}\Big)^{1/m}\Big(\delta_1^{-\frac{m\beta}{m\beta}}\Big)^{1/m}\delta_1\\ &\leq M\big\|\gamma\big\|^{\frac{1}{m\beta+1}}\delta_1^{\frac{m\beta}{m\beta+1}} + M_1\big\|\gamma\big\|^{\frac{1}{m\beta+1}}\delta_1^{\frac{m\beta}{m\beta}}\\ &\leq M_2\big\|\gamma\big\|^{\frac{1}{m\beta+1}}\delta_1^{\frac{m\beta}{m\beta+1}} \qquad (4.52)\end{aligned}$$

□

5. Conclude

In this article, I introduce lemmas such as Lemma 4.1 and Lemma 4.2 to analyze convergence on the Inverse Math problem using Algorithm Landweber. That is the main result in this paper.

6. Conflicts of Interest

The author declares no conflicts of interest.

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FACULTY OF MATHEMATICS TEACHER EDUCATION, TAY NINH UNIVERSITY, TAY NINH, VIETNAM.
Email address: lyvanan145@gmail.com, lyvananvietnam@gmail.com