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# ON THE EXISTENCE AND UNIQUENESS OF THE WEAKENED CLASSICAL SOLUTION ON THE AXIS OF THE THIRD CENTRALLY SYMMETRIC MIXED PROBLEM FOR THE THREE-DIMENSIONAL NONHOMOGENEOUS GENERAL HYPERBOLIC EQUATION OF THE SECOND ORDER

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ABSTRACT. We demonstrate the existence and uniqueness of the weakened classical solution on the axis of the third centrally symmetric mixed problem for the three-dimensional non-homogeneous general hyperbolic equation of the second order with the minimum conditions on the initial data.

## 1. INTRODUCTION AND POSITION OF THE PROBLEM

In the cylinder $\overline{P} = \overline{G} \times [0, T]$ with $G = \{x \in \mathbb{R}^3 / \ |x| = r < R\}$, we consider the following mixed problem:

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} + \sum_{i=1}^{3} a_i(x,t)\frac{\partial^2 u}{\partial x_i \partial t} - \triangle u(x,t) + \sum_{i=1}^{3} b_i(x,t)\frac{\partial u}{\partial x_i} \\ + c(x,t)\frac{\partial u}{\partial t} + q(x,t)u = f(x,t),\end{aligned} \tag{1.1}$$

with initial conditions

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$$\begin{cases} u(x,0) &= \varphi(|x|) \\ \dfrac{\partial u(x,0)}{\partial t} &= \psi(|x|), \end{cases} \tag{1.2}$$

and the boundary condition of the third type

$$\left( \frac{\partial u(x,t)}{\partial n} + \frac{1}{|x|} u(x,t) \right) \Big|_{\Gamma} = 0, \tag{1.3}$$

where $\varphi$ and $\psi$ are defined over the whole ball $\overline{G}$, $\dfrac{\partial}{\partial n}$ is the derivative at the point $(x,t)$ following the external normal $n$ of the lateral surface

$$\Gamma = \left\{ (x,t) \in \overline{P} : |x| = R, 0 \le t \le T \right\}$$

of the closed cylinder $\overline{P}$.

Because of the central symmetry, the normal derivative is equal to the radial differential operator i.e. $\dfrac{\partial}{\partial n} = \dfrac{\partial}{\partial r}$ with $|x| = r$.

We assume that the coefficients $a_i$, $b_i$, $c$, $q$ of the equation (1.1) and the right hand side $f$ are real and continuous in the closed cylinder $\overline{P}$ and that their first derivatives $\dfrac{\partial a_i(x,t)}{\partial x}$, $\dfrac{\partial b_i(x,t)}{\partial x}$, $\dfrac{\partial c(x,t)}{\partial x}$, $\dfrac{\partial q(x,t)}{\partial x}$ and $\dfrac{\partial f(x,t)}{\partial x}$ are bounded in $\overline{P}$ i.e.

$$\begin{cases} a_i, b_i, c, q, f \in C(\overline{P}), \\ \dfrac{\partial a_i(x,t)}{\partial x}, \dfrac{\partial b_i(x,t)}{\partial x}, \dfrac{\partial c(x,t)}{\partial x}, \quad \dfrac{\partial q(x,t)}{\partial x} \text{ and } \dfrac{\partial f(x,t)}{\partial x} \in L_\infty(\overline{P}). \end{cases} \tag{1.4}$$

The nature of central symmetry is manifested in the operator $\triangle$, in the geometry of the closed cylinder $\overline{P}$ in which we study the problem, in the coefficients of the equation (1.1) and in the functions $f(x,t)$ and $u(x,t)$ as follows:

$$\begin{cases} a_i(x,t) &= x_i a(r,t), \\ b_i(x,t) &= x_i b(r,t), \\ c(x,t) &= c(r,t), \\ q(x,t) &= q(r,t), \\ f(x,t) &= f(r,t), \\ u(x,t) &= u(r,t). \end{cases} \tag{1.5}$$

The coefficients $a_i(x,t)$ and $b_i(x,t)$ as well as the function $f$ obey the conciliation condition at the origin of the axis of symmetry of the equation (1.1) and to the boundary conditions of the domain of study (1.3) that is to say

$$\begin{cases} \sum_{i=1}^{3} a_i(x,t)\big|_\Gamma = \sum_{i=1}^{3} b_i(x,t)\big|_\Gamma = f(x,t)|_\Gamma = 0,\ 0 \le t \le T, \\ \sum_{i=1}^{3} a_i(x,t)\big|_{|x|=0} = \sum_{i=1}^{3} b_i(x,t)\big|_{|x|=0} = f(x,t)|_{|x|=0} = 0.\ 0 \le t \le T. \end{cases} \tag{1.6}$$

With the variables $(r,t)$, the differential properties of the coefficients of the equation and of the second member $f$ are expressed as follows:

$$\begin{cases} a(r,t), b(r,t), c(r,t), q(r,t), f(r,t) \in C(\overline{Q}) \\ \dfrac{\partial a(r,t)}{\partial r}, \dfrac{\partial b(r,t)}{\partial r}, \dfrac{\partial c(r,t)}{\partial r}, \dfrac{\partial q(r,t)}{\partial r} \text{ and } \dfrac{\partial f(r,t)}{\partial r} \in L_\infty(\overline{Q}) \\ \qquad \text{with} \quad Q = (0,R) \times (0,T). \end{cases} \tag{1.7}$$

For the non-homogeneous general hyperbolic equation (1.1), let us pose the following problem: determine the function $u(x,t)$ belonging to the class $C^2_{\{r=0\}}(\overline{P})$ which transforms the equation (1.1) into an identity in $P \backslash \{0\} \times [0,T]$ satisfying the initial conditions (1.2) and the condition at the limits (1.3) on the lateral surface $\Gamma$.

Based on the requirement imposed on the unknown function $u(x,t)$, let us give the following definition.

**Definition 1.1.** *By classical solution weakened on the axis* $r = 0$ *of the mixed problem* (1.1)*,* (1.2)*,* (1.3)*, the function* $u(x,t) \in C^2_{\{r=0\}}(\overline{P})$ *transforming the equation* (1.1) *into an identity in the open cylinder from which we exclude the lateral axis* $r = 0$ *that is to say (i.e)* $(P \setminus \{0\} \times [0,T])$ *and checking the conditions* (1.2)*,* (1.3) *in the usual sense.*

Such a definition makes it possible to formulate the problem posed for the equation (1.1) in a more laconic way: find the weakened classical solution on the axis $r = 0$ of the mixed problem (1.1), (1.2), (1.3).

To solve the problem thus posed, we must define the necessary and sufficient conditions to impose on the initial data $\varphi$, $\psi$ and on the second member $f$ so that the solution $u(x,t)$ of the third mixed problem (1.1), (1.2), (1.3) is classic everywhere but weakened on the axis of symmetry.

## 2. On the necessary and sufficient conditions for the existence of the weakened classical solution on the axis of the third centrally symmetric mixed problem for the three-dimensional nonhomogeneous general hyperbolic equation of the second order

We consider in the cylinder $\overline{P} = \overline{G} \times [0, T]$, the mixed problem (1.1), (1.2), (1.3). By passing in spherical coordinates, our problem reduces to the following mixed problem whose equation depends on a single space variable with Bessel operator in the main part.

$$\frac{\partial^2 u}{\partial t^2} + ra(r,t)\frac{\partial^2 u}{\partial r \partial t} - \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right) + rb(r,t)\frac{\partial u}{\partial r} + c(r,t)\frac{\partial u}{\partial t} + q(r,t)u = f(r,t), \ (r,t) \in Q, \tag{2.1}$$

with initial conditions

$$\begin{cases} u(r,0) &= \varphi(r) \\ \dfrac{\partial u(r,0)}{\partial t} &= \psi(r), \ 0 \leq r \leq R, \end{cases} \tag{2.2}$$

and the boundary condition of the third type

$$\left(\frac{\partial u(r,t)}{\partial r} + \frac{1}{r}u(r,t)\right)\Bigg|_{r=R} = 0, \ 0 \leq t \leq T. \tag{2.3}$$

The equation (2.1) is one-dimensional hyperbolic of the second order with source term.

It is obvious that the problems (1.1), (1.2), (1.3) and (2.1), (2.2), (2.3) are equivalent in the whole domain where the problem is placed except on the axis $|x| = r = 0$, because the functional definition of the change in spherical coordinates is equal to zero in the case of central symmetry, only for $r = 0$.

**Remark 2.1.** *If we were looking for the solution to the problem* (1.1)*,* (1.2)*,* (1.3) *belonging to the class* $C^2_{\{r=0\}}(\overline{P})$ *in the sense of the definition 1.1, then it would be sufficient to consider one of the following conditions*

$$\lim_{|x|\to 0} |x| \triangle u(x,t) = 0, \tag{2.4}$$

$$\lim_{|x|\to 0} |x| \frac{\partial^2 u}{\partial t^2}(x,t) = 0. \tag{2.5}$$

For the rest, we use the condition (2.4).

If the function $u(x,t)$ is a classical solution (weakened or usual) of the problem (1.1), (1.2) with the boundary condition

$$u(x,t)\Big|_{\Gamma} = 0, \tag{2.6}$$

then from the equation (1.1), from the condition (2.6) as well as from the conditions

$$\sum_{i=1}^{3} a_i(x,t)\Bigg|_{\Gamma} = \sum_{i=1}^{3} b_i(x,t)\Bigg|_{\Gamma} = 0, \quad 0 \le t \le T, \tag{2.7}$$

we obtain the conciliation conditions,

$$\triangle u\Big|_{\Gamma} = f(|x|,t)\Big|_{\Gamma} = 0 \tag{2.8}$$

by crossing the limit.

According to the definition 1.1 and the remark 2.1, the solution of the singular problem (2.1), (2.2), (2.3) precisely the function $u(r,t)$ which transforms the equation (2.1) into an identity in $Q$ and verifying (2.2), (2.3) is searched in the class of functions $u(r,t) \in C^1(\overline{Q}) \cap C^2((0,R] \times [0,T])$, for which the following limit equalities are true:

$$\lim_{r\to 0} r\left(\frac{\partial^2 u(r,t)}{\partial r^2} + \frac{2}{r}\frac{\partial u(r,t)}{\partial r}\right) = 0, \tag{2.9}$$

$$\lim_{r\to 0} r\frac{\partial^2 u}{\partial r \partial t}(r,t) = 0. \tag{2.10}$$

Furthermore, the solution of the problem (2.1)-(2.3) must respect the conditions (2.8) which for the variables $r$ and $t$ are of the form

$$\lim_{r\to R} \left(\frac{\partial^2 u(r,t)}{\partial r^2} + \frac{2}{r}\frac{\partial u(r,t)}{\partial r}\right) = \lim_{r\to R} f(r,t) = 0, \tag{2.11}$$

$$\lim_{r\to R} \frac{\partial^2 u(r,t)}{\partial t^2} = 0. \tag{2.12}$$

The condition (2.12) is an additional conciliation condition.

For this equation (2.1), if we make the following change of variable for $r \neq 0$

$$v(r,t) = ru(r,t), \tag{2.13}$$

then we multiply each member of the equality obtained by $r$, the equation (2.1) takes the form

$$\begin{aligned} \mathscr{L}v(r,t) \equiv \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial r^2} + A(r,t)\frac{\partial^2 v}{\partial r \partial t} + B(r,t)\frac{\partial v}{\partial r} + C(r,t)\frac{\partial v}{\partial t} \\ + D(r,t)v = f_1(r,t), \qquad (r,t) \in Q = (0,R) \times (0,T) \end{aligned} \tag{2.14}$$

where we placed

$$\begin{cases} A(r,t) = ra(r,t), \\ B(r,t) = rb(r,t), \\ C(r,t) = c(r,t) - a(r,t), \\ D(r,t) = q(r,t) - b(r,t), \\ f_1(r,t) = rf(r,t). \end{cases} \tag{2.15}$$

Thus, outside the axis $r = 0$, assuming that $u$ is bounded and taking into account the change of variable (2.13), the problem (2.1), (2.2), (2.3) is equivalent to the following problem

$$\begin{aligned} \mathscr{L}v(r,t) \equiv \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial r^2} + A(r,t)\frac{\partial^2 v}{\partial r \partial t} + B(r,t)\frac{\partial v}{\partial r} + C(r,t)\frac{\partial v}{\partial t} \\ + D(r,t)v = f_1(r,t), \qquad (r,t) \in Q = (0,R) \times (0,T), \end{aligned} \tag{2.16}$$

with initial conditions

$$v(r,0) = \Phi(r), \quad \frac{\partial v}{\partial t}(r,0) = \Psi(r), \qquad 0 \leq r \leq R, \tag{2.17}$$

and the boundary conditions of the first type on the edge $r = 0$ and of the second type on the limit $r = R$ respectively

$$v(0,t) = 0, \quad \frac{\partial v}{\partial r}(R,t) = 0, \quad 0 \leq t \leq T. \tag{2.18}$$

Here we placed

$$\begin{cases} \Phi(r) = r\varphi(r), \\ \Psi(r) = r\psi(r), \quad 0 \leq r \leq R. \end{cases} \tag{2.19}$$

It should also be remembered that the necessary and sufficient conditions for the existence of the weakened classical solution $u(x,t)$ on the axis $r=0$ of the first mixed problem with central symmetry for the non-homogeneous general hyperbolic equation second order three-dimensional (see Siliadin [6]) are the following conditions on the initial functions $\varphi$ and $\psi$

$$\varphi(r) \ \in \ C^1[0,R]\cap C^2(0,R], \ \varphi(R)=\triangle\varphi(R)=0, \ \lim_{r\to 0} \ r\triangle(r)=0; \tag{2.20}$$

$$\psi(r) \ \in \ C[0,R]\cap C^1(0,R], \ \psi(R)=0, \ \lim_{r\to 0} \ \frac{d\psi(r)}{dr}=0. \tag{2.21}$$

Conditions (2.20) and (2.21), it follows that the initial functions $\varPhi$ and $\varPsi$ defined by the formula (2.19) verify according to the article [3], the conditions

$$\varPhi(r) \ \in \ C^2[0,R], \ \varPhi(0)=\varPhi(R)=0, \ \frac{d^2\varPhi(0)}{dr^2}=\frac{d^2\varPhi(R)}{dr^2}=0, \tag{2.22}$$

$$\varPsi(r) \ \in \ C^1[0,R], \ \varPsi(0)=\varPsi(R)=0. \tag{2.23}$$

The following paragraph contains the demonstration of the existence of the weakened classical solution on the axis $r=0$ of the third mixed problem (1.1), (1.2), (1.3) when the functions $\varphi$ and $\psi$ satisfy the conditions (2.20), (2.21). To do this, we will rely on the diagram which was exposed during the proof of the sufficiency of the theorem (see KOLANI [2]).

Let's move on to the resolution demonstration.

2.1. **Strongly generalized solution.** Consider the following sets of functions:

$$D_0=\left\{u(r,t) \ \in C^3(\overline{Q}) : u\Big|_{r=0}=u\Big|_{r=R}=0, \frac{\partial^2 u}{\partial r^2}\Big|_{r=0}=\frac{\partial^2 u}{\partial r^2}\Big|_{r=R}=0\right\},$$
$$D_1=\left\{\varPhi(r) \ \in C^3[0,R] : \varPhi(0)=\varPhi(R)=0, \frac{d^2\varPhi(0)}{dr^2}=\frac{d^2\varPhi(R)}{dr^2}=0\right\},$$
$$D_2=\left\{\varPsi(r) \ \in C^2[0,R] : \varPsi(0)=\varPsi(R)=0\right\},$$
$$D_3=\left\{f_1(r,t) \ \in C^1(\overline{Q}) : f_1(0,t)=f_1(R,t)=0\right\},$$
$$D_4=\left\{u(r,t)\in C^3(\overline{Q}) : u(0,t)=\frac{\partial^2 u(0,t)}{\partial r^2}=0, \frac{\partial u(R,t)}{\partial r}=0\right\},$$
$$D_5=\left\{\varPhi(r)\in C^3[0,R] : \varPhi(0)=\frac{d^2\varPhi(0)}{dr^2}=0, \frac{d\varPhi(R)}{dr}=0\right\},$$
$$D_6=\left\{\varPsi(r)\in C^2[0,R] : \varPsi(0)=0, \frac{d\varPsi(R)}{dr}=0\right\},$$

$$D_7(\mathscr{L}) = \left\{ v \in C^2(\overline{Q}) : v(0,t) = 0, \frac{\partial v(R,t)}{\partial r} = 0 \right\},$$

$$D(L) = \left\{ v \in C^2(\overline{Q}) : v(0,t) = v(R,t) = 0, \frac{\partial^2 v(0,t)}{\partial r^2} = \frac{\partial^2 v(R,t)}{\partial r^2} = 0 \right\}.$$

We have the following assertion.

**Lemma 2.1.** *For any function* $v(r,t) \in D_4$ *transforming the equation* (2.16) *into an identity in the rectangle* $(0,R) \times (0,T)$ *and take the point* $t = 0$*, the value* $v(r,0) = \Phi(r) \in D_5$*,* $\frac{\partial v(r,0)}{\partial t} = \Psi(r) \in D_6$*, we have the inequality*

$$\begin{aligned} &\sup_{0 \le t \le T} \left\{ \int_0^R \left[ \left( \frac{\partial^2 v}{\partial r^2} \right)^2 + \left( \frac{\partial v}{\partial r} \right)^2 + v^2 + \left( \frac{\partial^2 v}{\partial r \partial t} \right)^2 + \left( \frac{\partial v}{\partial t} \right)^2 \right] dr \right\} \\ &\le c_3 \int_0^R \left[ \left( \frac{d^2 \Phi}{dr^2} \right)^2 + \left( \frac{d\Phi}{dr} \right)^2 + \Phi^2 + \left( \frac{d\Psi}{dr} \right)^2 + \Psi^2 \right] dr \\ &+ c_4 \int_0^T \int_0^R \left[ \left( \frac{\partial f_1}{\partial r} \right)^2 + f_1^2 \right] dr dt \end{aligned} \tag{2.24}$$

*where the constants* $c_3$ *and* $c_4$ *do not depend on* $v$*,* $\Phi$ *and* $\Psi$ *and are defined by*

$$\begin{aligned} &c_3 = c_1 e^{c_5 T} \quad \text{with} \quad c_5 = \max\{6M + 1, (5 + 2T)M + 1\}; \quad c_1 = 2e^{c_0 T} \\ &\qquad \text{where} \quad c_0 = \max\{6M; (5 + 2T)M\}.T; \quad c_4 = e^{c_5 T} \end{aligned} \tag{2.25}$$

*and*

$$\begin{aligned} M = \max_{\overline{Q}} \{ &R|A(r,t)|; \ |A(r,t)|; \ |C(r,t)|; \ |D(r,t)|; \\ &R\left|\frac{\partial A}{\partial r}(r,t)\right|; \ \left|\frac{\partial A}{\partial r}(r,t)\right|; \ \left|\frac{\partial C}{\partial r}(r,t)\right|; \ \left|\frac{\partial D}{\partial r}(r,t)\right| \}. \end{aligned} \tag{2.26}$$

*Proof.* Let us denote by $E_4$ the Banach space which is the completeness of the set of functions $D_3$ by the norm

$$||v||_{E_4} = \left( \sup_{0 \le t \le T} \left\{ \int_0^R \left( \left( \frac{\partial^2 v}{\partial r^2} \right)^2 + \left( \frac{\partial v}{\partial r} \right)^2 + v^2 + \left( \frac{\partial^2 v}{\partial r \partial t} \right)^2 + \left( \frac{\partial v}{\partial t} \right)^2 \right) dr \right\} \right)^{\frac{1}{2}}.$$

(According to the left part of the inequality (2.24)). By the symbol $E_5$, let us designate the Hilbert space formed of all the elements $\mathcal{F} = \{f_1, \Phi, \Psi\}$ for which we have the following finite standard

$$||\mathcal{F}||_{E_5} = (\mathcal{F},\mathcal{F})_{E_5}^{\frac{1}{2}} = \left(c_3\left(||\Phi||^2_{W_2^2(0,R)} + ||\Psi||^2_{W_2^1(0,R)}\right) + c_4|||f_1|||^2\right)^{\frac{1}{2}},$$

where

$$|||f_1|||^2 = \int_0^T\int_0^R\left(\left(\frac{\partial f_1}{\partial r}\right)^2 + f_1^2\right)drdt.$$

Here the scalar product is defined as follows. For any two elements $\mathcal{F}_i = \{\Phi_i,\Psi_i\}$, $\Phi_i(r) \in E_6(0,R)$, $\Psi_i(r) \in E_7(0,R)$ $(i = 1,2,\quad)$ where the symbols $E_6(0,R)$, $E_7(0,R)$ designate the Hilbert spaces that we obtain by the completeness of the sets $D_5$ and $D_6$ according to the norms of Sobolev spaces $W_2^2(0,R), W_2^1(0,R)$ respectively, $(\mathcal{F}_1,\mathcal{F}_2)_{E_5} = (\Phi_1,\Phi_2)_{W_2^2(0,R)} + (\Psi_1,\Psi_2)_{W_2^1(0,R)}$. Everywhere after completeness, we must understand differentiability in the sense of distributions (generalized functions). Any function $v \in D_7(\mathscr{L}) = \left\{C^2(\overline{Q}) : v(0,t) = 0, \dfrac{\partial v}{\partial r}(R,t) = 0\right\}$ can be approximated in the norm $C^2(\overline{Q})$ by the functions of $D_4$. This is why, for each function $v(r,t) \in D_7(\mathscr{L})$ satisfying the equation (2.16) in Q and the initial conditions (2.17) on $[0,R]$, the inequality (2.24) is verified. The initial functions of (2.17) satisfy (2.22), (2.23). The sets of functions $D_5$, $D_6$ defined respectively by the conditions (2.22), (2.23) are dense.

Then we have the following lemma.

**Lemma 2.2.** *For any function $v \in D_7(\mathscr{L})$, we have the following inequality*

$$||v||^2_{E_4} \le c_3||\mathcal{F}||^2_{E5}, \tag{2.27}$$

*where the constant $c_3$ does not depend on $v$ and is defined using the formula* (2.25).

The proof of the inequality (2.27) follows from the successive closures of the inequality (2.24) according to the norms of the spaces $C^2(\overline{Q})$ and $C^2[0,R]\times C^1[0,R]$ then after according to the norms of the spaces $E_4$ and $E_5$.

Let us put the problem (2.16), (2.17), (2.18) into conformity with the operational equation

$$Lv = \mathcal{F}, \tag{2.28}$$

where the operator $L$ admits in $E_4$ the dense definition domain $D(L) = D_7(\mathscr{L})$ and acts according to the law $Lv = \left\{\mathscr{L}v,\ v\big|_{t=0},\ \frac{\partial v}{\partial t}\big|_{t=0}\right\}$ of the Banach space $E_4$ in the Hilbertian space $E_5$.

From lemma 2.2, it follows that for any function $v \in D_7(\mathscr{L})$ we have the inequality

$$||v||_{E_4} \leq c_2||Lv||_{E5} \tag{2.29}$$

where the constant $c_2 = \sqrt{c_1}$ with $c_1$ defined in the formulas (2.25).

In a standard way, we demonstrate that the operator $L$ admits a closure which we designate by $\overline{L}$.

Let's look at the operational equation

$$\overline{L}v = \mathcal{F}, \quad \mathcal{F} = \{f_1, \Phi, \Psi\} \in E_5. \tag{2.30}$$

**Definition 2.1.** *The solution of the equation* (2.30) *is called the highly generalized solution of the mixed problem* (2.16)*,* (2.17)*,* (2.18).

Using the passage to the limit, we extend the inequality (2.29) to the strongly generalized solution. We obtain that for any element of the domain of definition $D(\overline{L})$ of the operator $\overline{L}$, we have the inequality

$$||v||_{E_4} \leq c_2||\overline{L}v||_{E5}. \tag{2.31}$$

From the inequality (2.31), it follows that the highly generalized solution of the mixed problem (2.16), (2.17), (2.18) is unique and $R(\overline{L}) = \overline{R(L)}$. Therefore, if $R(\overline{L})$ is dense in the space $E_5$, then the strongly generalized solution of the mixed problem (2.16) -(2.18) exists whatever the second member $\mathcal{F} \in E_5$. The special case $(n = 1,\ m = 1,\ \mathcal{A}(t) \equiv \mathcal{A})$ of the work of Radino and Yurchuk [5], obtained for the Cauchy problem in general form, demonstrates that the mixed problem (2.16)-(2.18) admits a strongly generalized solution. And immediately from theorem 2 of the article [5] it follows the density of the set of values of the operator L: $\overline{R(L)} = E_5$. Thus, we have the following lemma.

**Lemma 2.3.** *If the conditions of lemma 2.2 are verified, then for all* $\mathcal{F} = \{f_1, \Phi, \Psi\} \in E_5$ *there exists a unique strongly generalized solution* $v = (\overline{L})^{-1}\mathcal{F} = (\overline{L^{-1}})\mathcal{F}$ *of the problem* (2.16)*,* (2.17)*,* (2.18).

□

2.2. **Search for the regularity of the highly generalized Solution.** We consider the mixed problem (2.16), (2.17), (2.18) in the domain $\overline{Q} = [0, R] \times [0, T]$.

Based on (1.4), we also assume that the functions $A$, $B$, $C$, $D$, $f_1$, $\dfrac{\partial A}{\partial r}$, $\dfrac{\partial A}{\partial t}$ are continuous in $\overline{Q} = [0, R] \times [0, T]$, the functions $\dfrac{\partial^2 A}{\partial t^2}$, $\dfrac{\partial B}{\partial r}$, $\dfrac{\partial C}{\partial r}$, $\dfrac{\partial D}{\partial r}$ are bounded in $\overline{Q}$ and satisfy the conditions for reconciling the coefficients and the boundary conditions

$$\begin{cases} A(R, t) = B(R, t) = f_1(R, t) = 0, \ 0 \le t \le T \\ A(0, t) = B(0, t) = f_1(0, t) = 0, \ 0 \le t \le T \end{cases} \tag{2.32}$$

which follow from (1.6).

Under the above conditions, it appears that the function $v(r, t)$ is of class $C^2(\overline{Q})$ and that it is also a classical solution of the problem (2.16), (2.17), (2.18) then from the equation (2.16) and the conditions (2.17), (2.18), (2.32), it follows that the functions initials $\Phi$ and $\Psi$ check according to [8] the conditions (2.22), (2.23).

Thus, the conditions (2.22), (2.23) are the necessary conditions for the existence of the classical solution $v(r, t)$ of the problem (2.16), (2.17), (2.18).

We will now demonstrate that these conditions (2.22), (2.23) are also sufficient conditions for the existence of the classical solution of the problem (2.16)-(2.18).

From the definition of a strongly generalized solution, (see Morou [4], Siliadin [7]) there exists a sequence of functions $v_n \in C^2(\overline{Q})$ such that $v_n \to v$ following the norm of the first member of the inequality (2.24) and whose terms are classical solutions of the problems

$$\begin{aligned} \mathscr{L} v_n(r, t) \equiv \frac{\partial^2 v_n}{\partial t^2} - \frac{\partial^2 v_n}{\partial r^2} + A(r, t)\frac{\partial^2 v_n}{\partial r \partial t} + B(r, t)\frac{\partial v_n}{\partial r} + C(r, t)\frac{\partial v_n}{\partial t} \\ + D(r, t)v_n = f_{1,n}(r, t), \qquad (r, t) \in Q = (0, R) \times (0, T) \end{aligned} \tag{2.33}$$

with initial conditions

$$v_n(r, 0) = \Phi_n(r), \quad \frac{\partial v_n}{\partial t}(r, 0) = \Psi_n(r), \qquad 0 \le r \le R \tag{2.34}$$

and boundary conditions

$$v_n(0, t) = 0, \quad \frac{\partial v_n}{\partial r}(R, t) = 0, \quad 0 \le t \le T \tag{2.35}$$

while $\Phi_n \to \Phi$ and $\Psi_n \to \Psi$ following the norms of the Sobolev spaces $W_2^2(0,R)$ and $W_2^1(0,R)$ respectively. (see Morou [4], Siliadin [7]) If $v_n \in C^2(\overline{Q})$ is a classical solution, then by applying the condition (2.34) to the equation (2.33) and by virtue of conditions (2.22) and (2.23) and the conciliation and boundary conditions (2.32), it follows that $v_n$ satisfy the conditions

$$\frac{\partial^2 v_n(0,t)}{\partial r^2} = \frac{\partial^2 v_n(R,t)}{\partial r^2} = 0. \tag{2.36}$$

**Theorem 2.1.** *For the highly generalized solution $v(r,t)$ to be a classical solution of the problem* (2.16)-(2.18)*, it is sufficient that the initial functions $\Phi$ and $\Psi$ respectively verify the conditions* (2.22)*,* (2.23) *and that the second member $f_1$ of the equation* (2.16) *satisfies the following conditions:*

$$f_1(r,t) \in C([0,R]\times[0,T]), \tag{2.37}$$

$$f_1(r,t)\Big|_{\Gamma} = 0, \text{ with } \Gamma = \partial\overline{Q}, \tag{2.38}$$

$$\int_0^t \overline{f_1}\left(h_1(g_1(r,t),0),\ \tau\right) d\tau \in C^1(\overline{Q}), \tag{2.39}$$

$$\int_0^t \overline{f_1}\left(h_2(g_2(r,t),0),\ \tau\right) d\tau \in C^1(\overline{Q}), \tag{2.40}$$

*where $\overline{f_1}$ denotes the extension of $f_1$ constructed as follows: first, we extend the function $f_1(r,t)$ of the segment $[0,R]\times[0,T]$ on the segment $[-R,0]\times[0,T]$ in an odd way then after we extend it in an even way compared to the axis $r=R$ of the segment $[-R,R]\times[0,T]$ on the segment $[R,3R]\times[0,T]$. The function thus obtained is extended periodically by period $4R$ of the segment $[-R,3R]\times[0,T]$ on the plane $\mathbb{R}^1\times[0,T]$.*

*Proof.* Let us denote by $\overline{\Phi}$, $\overline{\Phi_n}$, $\overline{\Psi}$, $\overline{\Psi_n}$, the extensions of the functions $\Phi$, $\Phi_n$, $\Psi$, $\Psi_n$ respectively obtained as follows: first, we extend $\Phi$, $\Phi_n$, $\Psi$ and $\Psi_n$ of the segment $[0,R]$ on the segment $[-R,0]$ in an odd way, then afterwards we extend them in an even way with respect to the axis $r=R$ of the segment $[-R,R]$ on the segment $[R,3R]$. The functions thus obtained are extended periodically by period $4R$ of the segment $[-R,3R]$ on any axis $\mathbb{R}^1$. It is clear that $\overline{\Phi}$, $\overline{\Phi_n}$, $\overline{\Psi}$ and $\overline{\Psi_n}$ are odd with respect to the points $2mR$, $m = 0, \pm 1, \pm 2, ...$ and even with respect to the points $(2m+1)R$, $m = 0, \pm 1, \pm 2, ...$.

By the symbols $\hat{A}(r,t) = \tilde{r}\hat{a}(r,t)$, $\hat{B}(r,t) = \tilde{r}\hat{b}(r,t)$, $\hat{C}(r,t) = \hat{c}(r,t) - \hat{a}(r,t)$, $\hat{D}(r,t) = \hat{q}(r,t) - \hat{b}(r,t)$ we have respectively designated the extensions of the functions $A(r,t)$, $B(r,t)$, $C(r,t)$, $D(r,t)$ of $\overline{Q}$ on the half-plane $\mathbb{R}^1 \times [0,T]$ : first in an even way with respect to the axis $r = 0$ of $\overline{Q}$ on $[-R,0] \times [0,T]$, after in an even way with respect to the axis $r = R$ of $[-R,R] \times [0,T]$ on $[R,3R] \times [0,T]$ and finally periodically of period $4R$ of $[-R,3R] \times [0,T]$ on the half-plane $\mathbb{R}^1 \times [0,T]$. By $\tilde{r}$ let us denote the periodic extension of period $4R$ of the function $r$ obtained by the even extension with respect to $r = 0$ and the odd extension of $[-R,R]$ on $(R,3R)$ with respect to $r = R$, subsequently extended on the axis $\mathbb{R}^1$ of period $4R$. The functions $\overline{v}$, $\overline{v}_n$ are $4R$-periodic following $r$ on $\mathbb{R}^1 \times [0,T]$ having a structure following the variable $r$, analogous to the extension $\overline{\Phi}$, $\overline{\Phi_n}$, $\overline{\Psi}$, $\overline{\Psi_n}$ as functions depending on a single variable.

Differential properties of $A(r,t)$, $B(r,t)$, $C(r,t)$, $D(r,t)$ indicated in (1.7) and the procedure extension of these functions on the half-plane $\mathbb{R}^1 \times [0,T]$, it follows that $\hat{A}(r,t)$, $\hat{B}(r,t)$, $\hat{C}(r,t)$, $\hat{D}(r,t)$ are continuous, bounded and admit piecewise continuous first derivatives following $r$ on the half-plane $\mathbb{R}^1 \times [0,T]$. From the conditions (2.22), (2.23), it follows that

$$\overline{\Phi},\ \overline{\Phi_n} \in C^2(\mathbb{R}^1); \qquad \overline{\Psi},\ \overline{\Psi_n} \in C^1(\mathbb{R}^1) \tag{2.41}$$

and $\overline{v_n} \in C^2([0,R] \times [0,T])$ is classical solution of the following Cauchy problem

$$\begin{aligned} \mathscr{L}\overline{v_n}(r,t) \equiv \frac{\partial^2 \overline{v_n}}{\partial t^2} - \frac{\partial^2 \overline{v_n}}{\partial r^2} + \hat{A}(r,t)\frac{\partial^2 \overline{v_n}}{\partial r \partial t} + \hat{B}(r,t)\frac{\partial \overline{v_n}}{\partial r} + \hat{C}(r,t)\frac{\partial \overline{v_n}}{\partial t} \\ + \hat{D}(r,t)\overline{v_n} = \overline{f_{1n}}(r,t), \qquad (r,t) \in Q = (0,R) \times (0,T) \end{aligned} \tag{2.42}$$

$$\overline{v_n}(r,0) = \overline{\Phi_n}(r), \quad \frac{\partial \overline{v_n}}{\partial t}(r,0) = \overline{\Psi_n}(r), \qquad 0 \le r \le R \tag{2.43}$$

and boundary conditions

$$\overline{v_n}(0,t) = 0, \quad \frac{\partial \overline{v_n}}{\partial r}(R,t) = 0. \quad 0 \le t \le T. \tag{2.44}$$

If we admit the equation (2.42) in the form

$$\left[\frac{\partial}{\partial t}+\left(\frac{\hat{A}}{2}-\sqrt{\frac{\hat{A}^2}{4}+1}\right)\frac{\partial}{\partial r}-\left(\frac{\partial}{\partial t}+\frac{\hat{A}}{2}\frac{\partial}{\partial r}\right)ln\sqrt{\frac{\hat{A}^2}{4}+1}+\frac{1}{2}\frac{\partial \hat{A}}{\partial r}\right] \times\left[\frac{\partial \overline{v_n}}{\partial t}+\left(\frac{\hat{A}}{2}+\sqrt{\frac{\hat{A}^2}{4}+1}\right)\frac{\partial \overline{v_n}}{\partial r}\right]=\overline{F_1}_n(r,t) \tag{2.45}$$

where

$$\overline{F_1}_n(r,t)=\overline{F_n}(r,t)+\overline{f_1}(r,t) \tag{2.46}$$

with

$$\begin{aligned}\overline{F_n}=&\left[\left(\frac{\partial}{\partial t}+\left(\frac{\hat{A}}{2}-\sqrt{\frac{\hat{A}^2}{4}+1}\right)\frac{\partial}{\partial r}\right)\left(\frac{\hat{A}}{2}+\sqrt{\frac{\hat{A}^2}{4}+1}\right)\frac{\partial \overline{v_n}}{\partial r}\right]\\ &+\left[-\left(\left(\frac{\partial}{\partial t}-\frac{\hat{A}}{2}\frac{\partial}{\partial r}\right)ln\sqrt{\frac{\hat{A}^2}{4}+1}\right)+\frac{1}{2}\frac{\partial \hat{A}}{\partial r}\right]\\ &\times\left[\frac{\partial \overline{v_n}}{\partial t}+\left(\frac{\hat{A}}{2}+\sqrt{\frac{\hat{A}^2}{4}+1}\right)\frac{\partial \overline{v_n}}{\partial r}\right]-\hat{C}\frac{\partial \overline{v_n}}{\partial t}-\hat{B}\frac{\partial \overline{v_n}}{\partial r}-\hat{D}\overline{v_n}\end{aligned} \tag{2.47}$$

and $\overline{f_1}$, the extension of $f_1$ constructed in paragraph 2.2.

Then using d'Alembert's formula associated with Duhamel's principle, we obtain the solution of the mixed problem (2.42)-(2.44) in the form

$$\begin{aligned}\overline{v_n}(r,t)=&\frac{\overline{\Phi_n}\left(h_2\left(g_2(r,t),0\right)\right)+\overline{\Phi_n}\left(h_1(g_1(r,t),0)\right)}{2}\\ &+\frac{1}{2}\int_{h_1(g_1(r,t),0)}^{h_2(g_2(r,t),0)}\left(\frac{\overline{\Psi_n}(\xi)+\frac{\hat{A}}{2}(\xi,0)\overline{\Phi'_n}(\xi)}{\sqrt{\frac{\hat{A}^2}{4}(\xi,0)+1}}\right)d\xi\\ &+\frac{1}{2}\int_0^t\int_{h_1(g_1(r,t),\xi)}^{h_2(g_2(r,t),\xi)}\left(\frac{\overline{F_1}_n(\xi,\tau)}{\sqrt{\frac{\hat{A}^2}{4}(\xi,\tau)+1}}\right)d\xi d\tau,\end{aligned} \tag{2.48}$$

where $g_1(r,t) = k_1$ and $g_2(r,t) = k_2$ are the characteristic functions of the equation (2.42), while the functions $h_1(y,\tau)$ and $h_2(y,\tau)$ are for all $\tau$ inverse functions to the functions $g_1(r,t)$ and $g_2(r,t)$ respectively.

As $\hat{A},\ \hat{B} \in C^1(\mathbb{R}\times[0,T])$ then $g_1,\ g_2 \in C^2(\mathbb{R}\times[0,T])$ and $h_1, h_2 \in C^2(\mathbb{R}\times[0,T])$.

It should be noted that the special case where $A(r,t) = 0,\ g_1(r,t) = r - t$, $g_2(r,t) = r + t$, $h_1(y,\tau) = y + \tau$, $h_2(y,\tau) = y - \tau$, the formula (2.48) is the well-known one of d'Alembert for the vibration equation of a string.

When $(r,t) \in \overline{Q}$, $\overline{v_n}(r,t) = v_n(r,t)$. In the formula (2.48), let $(r,t) \in Q$ and passing to the limit when $n \to \infty$, then the strongly generalized solution $v(r,t)$ takes the form

$$
\begin{aligned}
v(r,t) &= \frac{\overline{\Phi}\left(h_2\left(g_2(r,t),0\right)\right) + \overline{\Phi}\left(h_1(g_1(r,t),0)\right)}{2} \\
&+ \frac{1}{2}\int_{h_1(g_1(r,t),0)}^{h_2(g_2(r,t),0)} \left( \frac{\overline{\Psi}(\xi) + \frac{\hat{A}}{2}(\xi,0)\overline{\Phi'}(\xi)}{\sqrt{\frac{\hat{A}^2}{4}(\xi,0)+1}} \right) d\xi \\
&+ \frac{1}{2}\int_0^t \int_{h_1(g_1(r,t),\xi)}^{h_2(g_2(r,t),\xi)} \left( \frac{\overline{F_1}(\xi,\tau)}{\sqrt{\frac{\hat{A}^2}{4}(\xi,\tau)+1}} \right) d\xi d\tau
\end{aligned} \tag{2.49}
$$

with

$$
\begin{aligned}
\overline{F_1}(r,t) &= \overline{F}(r,t) + \overline{f_1}(r,t) \\
&= \left[ \left( \frac{\partial}{\partial t} + \left( \frac{\hat{A}}{2} - \sqrt{\frac{\hat{A}^2}{4}+1} \right) \frac{\partial}{\partial r} \right) \left( \frac{\hat{A}}{2} + \sqrt{\frac{\hat{A}^2}{4}+1} \right) \frac{\partial \overline{v}}{\partial r} \right] \\
&+ \left[ - \left( \left( \frac{\partial}{\partial t} - \frac{\hat{A}}{2}\frac{\partial}{\partial r} \right) ln\sqrt{\frac{\hat{A}^2}{4}+1} \right) + \frac{1}{2}\frac{\partial \hat{A}}{\partial r} \right] \\
&\times \left[ \frac{\partial \overline{v}}{\partial t} + \left( \frac{\hat{A}}{2} + \sqrt{\frac{\hat{A}^2}{4}+1} \right) \frac{\partial \overline{v}}{\partial r} \right] - \hat{C}\frac{\partial \overline{v}}{\partial t} - \hat{B}\frac{\partial \overline{v}}{\partial r} - \hat{D}\overline{v}\overline{f_1}(r,t).
\end{aligned} \tag{2.50}
$$

Let us now show that the function $v(r,t)$ is twice continuously differentiable in $\overline{Q}$ i.e. $v(r,t) \in C^2(\overline{Q})$.

Note that the first two terms of the right hand side of the formula (2.49) do not depend on the function $v(r,t)$. So from the properties of composition and the integral, it immediately follows that these terms belong to $C^2(\overline{Q})$. Let us now study the belonging to a certain class of the third term of second member of the formula (2.49). Given the conditions (2.37)-(2.40) on the function $f_1$ and like $v_n \to v$ following the norm of first member of the inequality (2.24) then the sequences of functions

$$\int_0^t \frac{\overline{F_{1n}}\left(h_i(g_i(r,t),\tau)\right).h_i'(g_i(r,t),\tau).\frac{\partial g_i}{\partial r}}{\sqrt{\dfrac{\hat{A}^2\left(h_i(g_i(r,t),\tau),\tau\right)}{4}+1}}d\tau \;\in\; C^1(\overline{Q});\;\; i=1,2$$

converge according to the norm of the space $C^1(\overline{Q})$ towards the functions

$$\int_0^t \frac{\overline{F_1}\left(h_i(g_i(r,t),\tau)\right).h_i'(g_i(r,t),\tau).\frac{\partial g_i}{\partial r}}{\sqrt{\dfrac{\hat{A}^2\left(h_i(g_i(r,t),\tau),\tau\right)}{4}+1}}d\tau \;\in\; C^1(\overline{Q});\;\; i=1,2.$$

By virtue of this, we conclude that the third term belongs to $C^2(\overline{Q})$ and therefore $v(r,t) \in C^2(\overline{Q})$. A simple transfer of $v$ in the equation and in the conditions proves that it is a classical solution of the mixed problem (2.16)-(2.18). Which proves the theorem. □

2.3. **Existence of the classical weakened solution on the axis of the third centrally symmetric mixed problem for the three-dimensional non-homogeneous general hyperbolic equation of the second-order.** In the cylinder $\overline{P}$ for the non-homogeneous equation (1.1), let us pose the following problem: define in $\overline{P}$ the solution $u(x,t)$ of this equation belonging to the class of functions $C^2_{\{r=0\}}(\overline{P})$ which satisfy the initial conditions (1.2) and the boundary condition of the third type (1.3).

**Definition 2.2.** *By classical solution weakened on the axis $r=0$ of the mixed problem* (1.1), (1.2), (1.3), *we will understand the class function $C^2_{\{r=0\}}(\overline{P})$ transforming the equation* (1.1) *into an identity in the open cylinder with lateral axis $r=0$ $(C \setminus \{0\} \times [0,T])$ and checking the conditions* (1.2), (1.3) *in the usual sense.*

The problem thus posed solves the following:

**Theorem 2.2.** *For the existence (of the unique) weakened classical solution* $u(x,t)$ *on the axis* $|x| = r = 0$ *of the third mixed problem* (1.1), (1.2), (1.3)

$$u(x,t) = \frac{\overline{\Phi}\left(h_2\left(g_2(|x|,t),0\right)\right) + \overline{\Phi}\left(h_1(g_1(|x|,t),0)\right)}{2|x|} + \frac{1}{2|x|}\int_{h_1(g_1(|x|,t),0)}^{h_2(g_2(|x|,t),0)} \left( \frac{\overline{\Psi}(\xi) + \frac{\hat{A}}{2}(\xi,0)\overline{\Phi'}(\xi)}{\sqrt{\frac{\hat{A}^2}{4}(\xi,0)+1}} \right) d\xi + \frac{1}{2|x|}\int_0^t \int_{h_1(g_1(|x|,t),\xi)}^{h_2(g_2(|x|,t),\xi)} \left( \frac{\overline{F_1}(\xi,\tau)}{\sqrt{\frac{\hat{A}^2}{4}(\xi,\tau)+1}} \right) d\xi d\tau \tag{2.51}$$

*it is necessary and sufficient that the initial functions* $\varphi$ *and* $\psi$ *satisfy all the requirement* (2.20), (2.21) *and the right hand side* $f$ *of the equation* (1.1) *satisfies the following conditions*

$$f(x,t) \in C\left(P \setminus \{0\} \times [0,T]\right), \tag{2.52}$$

$$\lim_{|x|\to 0} f_1(|x|,t) = 0, \quad 0 \le t \le T, \tag{2.53}$$

$$f(x,t)\Big|_{\Gamma} = 0, \tag{2.54}$$

$$\int_0^t \overline{f_1}\left(h_1(g_1(|x|,t),0),\ \tau\right) d\tau \in C^1(\overline{Q}), \tag{2.55}$$

$$\int_0^t \overline{f_1}\left(h_2(g_2(|x|,t),0),\ \tau\right) d\tau \in C^1(\overline{Q}). \tag{2.56}$$

By the symbols $\overline{\Phi}$, $\overline{\Psi}$, $\overline{F_1}$, $\overline{f_1}$, we have designated the respective extensions functions $\Phi$, $\Psi$, $F_1$, $f_1$ whose extension structure was defined in §2.2.

The uniqueness observed here of the solution arises from the energy type inequality that we will construct subsequently.

*Proof.* Consider further the following sets: $E_1$ Banach space which is the completeness of the set of functions $D_0$, $E_2 = W^2_{2,0}(0,R) \times W^1_{2,0}(0,R)$; $E_3 = C\left([0,T], W^1_{2,0}(0,R)\right) \times W^2_{2,0}(0,R) \times W^1_{2,0}(0,R)$; $E_5$, are Hilbert spaces; $E_4$ Banach space which is the completeness of the set of functions $D_3$. $E_6(0,R)$ and $E_7(0,R)$ designate the Hilbert spaces that we obtain by the completeness of the sets $D_5$ and $D_6$ according to the norms of Sobolev spaces $W_2^2(0,R)$ and $W_2^1(0,R)$.

Suppose that for the functions $\varphi$ and $\psi$ the conditions (2.20) and (2.21) are verified, and the function $f$ verifies the conditions (2.52)-(2.56) . Let us demonstrate that if the required conditions are verified, there exists the classical weakened solution $u(x,t)$ on the axis $r=0$ of the mixed problem (1.1), (1.2), (1.3). It is defined by the formula (2.51). For this, it is sufficient to demonstrate that the mixed problem (2.1), (2.2), (2.3) in $\overline{Q}$ admits the solution

$$\begin{aligned} u(r,t) &= \frac{\overline{\Phi}\left(h_2\left(g_2(r,t),0\right)\right)+\overline{\Phi}\left(h_1(g_1(r,t),0)\right)}{2r} \\ &+ \frac{1}{2r}\int_{h_1(g_1(r,t),0)}^{h_2(g_2(r,t),0)}\left(\frac{\overline{\Psi}(\xi)+\frac{\hat{A}}{2}(\xi,0)\overline{\Phi'}(\xi)}{\sqrt{\frac{\hat{A}^2}{4}(\xi,0)+1}}\right)d\xi \\ &+ \frac{1}{2r}\int_0^t\int_{h_1(g_1(r,t),\xi)}^{h_2(g_2(r,t),\xi)}\left(\frac{\overline{F_1}(\xi,\tau)}{\sqrt{\frac{\hat{A}^2}{4}(\xi,\tau)+1}}\right)d\xi d\tau \end{aligned} \tag{2.57}$$

which belongs to the function class $C^1(\overline{Q})\cap C^2((0,R]\times[0,T])$ and checks the conditions (2.9)-(2.11), if the conditions (2.20) and (2.21) are verified as well as the condition

$$f(r,t)\ \in\ C\left((0,R]\times[0,T]\right), \tag{2.58}$$

$$\lim_{r\to 0} f_1(r,t)=0,\quad 0\le t\le T, \tag{2.59}$$

$$\lim_{r\to R} f(r,t)=0,\ 0\le t\le T, \tag{2.60}$$

$$\int_0^t \overline{f_1}\left(h_1(g_1(r,t),0),\ \tau\right)d\tau\in\ C^1(\overline{Q}), \tag{2.61}$$

$$\int_0^t \overline{f_1}\left(h_2(g_2(r,t),0),\ \tau\right)d\tau\in\ C^1(\overline{Q}). \tag{2.62}$$

By changing the variable (2.13) in the mixed problem (2.1), (2.2), (2.3) , of the unknown function $u$, we then obtain the auxiliary problem (2.16), (2.17), (2.18) not containing a singularity with the boundary condition of the first type on the limit $r = 0$ and the boundary condition of the second type on the edge $r = R$. Subsequently, let us carry out the proof following the diagram exposed during the proof of the sufficiency of Theorem (see KOLANI [2]). We have the following result

**Lemma 2.4.** *For the solution* $v(r,t) \in D_0$ *of the mixed problem* (2.16)*,* (2.17)*,* (2.18) *for the non-homogeneous equation with initial functions* $\Phi(r) \in D_1$*,* $\Psi(r) \in D_2$ *and the right hand side* $f_1(r,t) \in D_3$*, we have the following inequality:*

$$\sup_{0\le t\le T}\left\{\int_0^R\left[\left(\frac{\partial^2 v}{\partial r^2}\right)^2+\left(\frac{\partial v}{\partial r}\right)^2+v^2+\left(\frac{\partial^2 v}{\partial r\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2\right]dr\right\}$$
$$\le c_3\int_0^R\left[\left(\frac{d^2\Phi}{dr^2}\right)^2+\left(\frac{d\Phi}{dr}\right)^2+\Phi^2+\left(\frac{d\Psi}{dr}\right)^2+\Psi^2\right]dr \tag{2.63}$$
$$+\,c_4\int_0^T\int_0^R\left[\left(\frac{\partial f_1}{\partial r}\right)^2+f_1^2\right]drdt$$

*where the constants* $c_3$ *and* $c_4$ *do not depend on* $v$*,* $\Phi$ *and* $\Psi$ *and are defined by the formula* (2.25)*, the constant* $M$ *by the formula* (2.26)*.*

From the a priori energy inequality (2.63) follows the uniqueness of the solution of the mixed problem (2.16), (2.17), (2.18).

By the symbol $E_3$ let us designate the Hilbert space which is made up of all the elements $\mathcal{F} = \{f_1, \Phi, \Psi\}$ for which Standard

$$||\mathcal{F}||_{E_3} = \left(c_3\left(||\Phi||^2_{W_2^2(0,R)} + ||\Psi||^2_{W_2^1(0,R)}\right) + c_4|||f_1|||^2\right)^{\frac{1}{2}}$$

where $|||f_1|||^2 = \displaystyle\int_0^T\int_0^R\left(\left(\frac{\partial f_1}{\partial r}\right)^2 + f_1^2\right)drdt,$ is finished.

By the symbol $E_1$ let us first designate the Banach space defined at the beginning of our demonstration as well as $D(\mathscr{L})$.

**Lemma 2.5.** *For any function $v \in D(\mathscr{L})$ satisfying the equation* (2.16) *and the conditions* (2.17)*,* (2.18)*, we have the following inequality:*

$$||v||^2_{E_1} \leq ||\mathcal{F}||^2_{E_3}. \tag{2.64}$$

*Proof.* The sets $D_1$, $D_2$, $D_3$ are dense in the sets designated by the relations (2.22), (2.23) and $D_3 = \{f_1(r,t) \in C^1(\overline{Q}) : f_1(0,t) = f_1(R,t) = 0\}$ respectively. Then the inequality (2.63) remains true and for any function $v(r,t) \in D(\mathscr{L})$ and from the fact that such a function can be approximated in the norm $C^2(\overline{Q})$ by the functions $D_0$ verifying the equation (2.16) and the conditions (2.17), (2.18) and the assertion of the lemma follows immediately from the completeness according to the norms $E_1$ and $E_3$ of the inequality (2.63). □

Let the operator $L$ have the domain of definition $D(L)$ defined at the start of the demonstration and acting according to the law

$$Lv \equiv \left\{\mathscr{L}v, \quad v\Big|_{t=0}, \frac{\partial v}{\partial t}\Big|_{t=0}\right\} : E_1 \longrightarrow E_3.$$

It is clear that the operator $L$ admits a completeness (see for example Radino and Yurchuk [5] page 338) which we designate by $\overline{L}$.

**Definition 2.3.** *The solution of the equation*

$$\overline{L}v = \mathcal{F}, \ \mathcal{F} \in E_3, \tag{2.65}$$

*will be called a highly generalized solution of the mixed problem* (2.16)*,* (2.17)*,* (2.18).

**Lemma 2.6.** *Suppose that the conditions of lemma 2.5 are verified, then for any function $v \in D(\overline{L})$, we have the inequality*

$$||v||^2_{E_1} \leq ||\overline{L}v||^2_{E_3}. \tag{2.66}$$

*Proof.* If in the first and second mixed problem (respectively on the limit $r = 0$ and the limit $r = R$) (2.16), (2.17), (2.18) we match the operational equation

$$Lv = \mathcal{F} \tag{2.67}$$

from Lemma 2.5, it follows that for any function $v$ belonging in the set $E_1$ to the set $D(\overline{L})$, we have the following inequality

$$||v||_{E_1} \leq ||Lv||_{E_3}. \tag{2.68}$$

Passing to the limit and tending the inequality (2.68) to the strongly generalized solution, as a result we obtain the inequality (2.66). Which proves lemma 2.6. □

We have the following assertion.

**Lemma 2.7.** *If the conditions of lemma 2.6 are verified, then for $\mathcal{F} \in E_3$ there exists a unique strongly generalized solution*

$$v = (\overline{L})^{-1}\mathcal{F} = \overline{(L^{-1})}\mathcal{F}$$

*of the mixed problem* (2.16)*,* (2.17)*,* (2.18) *which verifies the inequality* (2.66).

It is sufficient to demonstrate that the set of values $R(L)$ the operator $L$ is dense in $E_3$. That is, for a certain element $V = \{g_1, \Phi_1, \Psi_1\} \in E_3$ and for any function $v \in D(L)$ we have the following equality:

$$\begin{aligned}(Lv, V)_{E_3} = (\mathscr{L}v, g_1)_{C([0,T],\ W^1_{2,0}(0,R))} + \left(v\Big|_{t=0}, \Phi_1\right)_{W^2_2(0,R)} \\ + \left(\frac{\partial v}{\partial t}\Big|_{t=0}, \Psi_1\right)_{W^1_2(0,R)} = 0.\end{aligned} \tag{2.69}$$

The proof for $V \equiv 0$ follows from the details of the extension method following a parameter exposed during the proof of Theorem 2 of the article [5].

Investigations of the regularity of the strongly generalized solution $v$ based on the results of the article [1] (see § 3.2) show that the strongly generalized solution of the mixed problem (2.16), (2.17), (2.18) is of the form

$$+\frac{1}{2}\int_{h_1(g_1(r,t),0)}^{h_2(g_2(r,t),0)} \left( \frac{\overline{\Psi}(\xi) + \frac{\hat{A}}{2}(\xi,0)\overline{\Phi'}(\xi)}{\sqrt{\frac{\hat{A}^2}{4}(\xi,0)+1}} \right) d\xi \tag{2.70}$$

$$+\frac{1}{2}\int_0^t \int_{h_1(g_1(r,t),\xi)}^{h_2(g_2(r,t),\xi)} \left( \frac{\overline{F}_1(\xi,\tau)}{\sqrt{\frac{\hat{A}^2}{4}(\xi,\tau)+1}} \right) d\xi d\tau$$

and is a classical solution, i.e. belongs to $C^2(\overline{Q})$.

The demonstration according to which the function $u(r,t) = \frac{v(r,t)}{r}$ where $v(r,t)$ is defined by the formula (2.70) is a classic solution weakened on the axis $r = 0$ of the mixed problem (1.1), (1.2), (1.3) differs very little from the research carried out during the demonstration of the sufficiency of theorem of the article [2]. Which demonstrates sufficiency.

The necessity of the conditions (2.20), (2.21) on the functions $\varphi$ and $\psi$ is demonstrated in an analogous way as in the case of the homogeneous equation (see Theorem [2]).

The differentiation according to the variables $x$ and $t$ of the formula (2.51) establishing the weakened classical solution $u(x,t)$ on the axis $r = 0$, demonstrates the necessity of the conditions (2.55), (2.56) for the function $f$. From the equation (1.1) and the fact that the solution $u(x,t)$ belongs to the class $C^2_{\{r=0\}}(\overline{P})$ and that the coefficients $a_i$, $b_i$, verify the conciliation conditions (1.6), it follows the necessity of the conditions (2.52)-(2.54).

Therefore, if the function $u(x,t)$ defined by the equality (2.51) is a weakened classical solution on the axis $r = 0$ of the mixed problem (1.1), (1.2) , (1.3) then the functions $\varphi$ and $\psi$ check the conditions (2.20),(2.21) and the function $f$ checks the conditions (2.52)-(2.56). Which proves the theorem 2.2. □

2.4. **Uniqueness of the weakened classical solution on the axis of the third centrally symmetric mixed problem for the three-dimensional nonhomogeneous general hyperbolic equation of the second order.** If the classical weakened solution $u(x,t)$ on the axis $r = 0$ of the third mixed problem (1.1), (1.2), (1.3) exists, then it is unique.

In other words, the strongly generalized solution $v(r,t) = ru(r,t)$ verifies the energy inequality a priori

$$\sup_{0\le t\le T}\left\{\int_0^R\left[\left(\frac{\partial^2 v}{\partial r^2}\right)^2+\left(\frac{\partial^2 v}{\partial t^2}\right)^2+\left(\frac{\partial v}{\partial r}\right)^2\right.\right.$$
$$\left.\left.+v^2+\left(\frac{\partial^2 v}{\partial r\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2\right]dr\right\}$$
$$\le d\left\{\int_0^R\left[\left(\frac{d^2\Phi}{dr^2}\right)^2+\left(\frac{d\Phi}{dr}\right)^2+\Phi^2+\left(\frac{d\Psi}{dr}\right)^2+\Psi^2\right]dr\right.$$
$$\left.+\int_0^T\int_0^R\left(\frac{\partial f_1}{\partial r}\right)^2 drdt+\sup_{0\le t\le T}\int_0^R f_1^2 dr\right\} \tag{2.71}$$

Here the constant $d$ does not depend on $v$, $\Phi$, $\Psi$ and $f_1$. From the inequality (2.71) flows the uniqueness of the highly generalized solution $v(r,t)$ and therefore of the weakened classical solution $u(r,t)$ on the axis $r = 0$ of the third mixed problem (1.1), (1.2), (1.3).

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