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ADJUNCTION OF ROOTS TO PROFINITE GROUPS

G. Mantika[1], R.M. Fomekong Ndjouo, and D. Tieudjo

ABSTRACT. In this paper, when G' is a group obtained by adjoining a nth-root of g to a given group G, where n is a nonzero natural number and g is an element of G of infinite order, we compute the profinite completion $\widehat{G'}$ of G'. Also, given G a profinite group in which any subgroup of finite index is open, n a nonzero natural number, g an element of G, and x an element not belonging to G, we point out necessary and sufficient condition under which the group obtained by adjoining roots to the profinite group G remains again profinite. Our proofs make use of theoretico-combinatorial methods.

1. INTRODUCTION AND RESULTS

We recall that a *profinite group* G is a topological, compact, Hausdorff and totally disconnected group. Any profinite group is isomorphic to a closed subgroup of a direct product of finite groups. So, profinite groups are very large. They are very rich since they have algebraic and topological properties. Today, profinite groups have been generalized to pro-$\mathcal{C}$ groups and free pro-$\mathcal{C}$ constructions, where $\mathcal{C}$ is a class of groups which is closed under taking subgroups, quotients and isomorphic images. When $\mathcal{C}$ is the class of all finite groups, all finite p-groups, all solvable

[1]*corresponding author*
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groups and all finite nilpotent groups, then we talk about profinite groups, pro-p groups, pro-solvable groups and pro-nilpotent groups respectively. See [13, 16, 18].

Let G be a profinite group. When G is devoid with its topological structure, we also denote by G the remaining group structure and we call it *abstract group*.

For any abstract group G, write $\mathcal{N}_G$ for the set of all normal subgroups in G with finite index. *The profinite completion* of G, denoted by $\widehat{G}$, is the inverse limit of a projective system of finite groups, i.e., $\widehat{G} = \varprojlim_{N \in \mathcal{N}_G} G/N$, with $\mathcal{N}_G$ a directed set. $\widehat{G}$ is a profinite group. See [13, 16, 18]. If the set $\mathcal{N}_G$ is used as a basis of neighbourhoods of the identity, the effect is to introduce a topology on G, *the profinite topology*. Let G be an abstract group endowed with its profinite topology. Then, for any subgroup H, the profinite closure $\overline{H}$ in G is the intersection of all the subgroups in G of finite index that contain H. When the trivial subgroup is closed for the profinite topology of the group G, we say that G is *residually finite*. Equivalently, a group G is residually finite if for any $g \neq 1_G$ there exists a normal subgroup K of G of finite index not containing g. A subgroup H of G is *finitely separable* if it is closed in the profinite topology of G. Equivalently, a subgroup H of a group G is said to be finitely separable if for any element g not belonging in the subgroup H, there exists a normal subgroup N of finite index such that $g \notin NH$. This means that, for any $a \in G \setminus H$, there exists a homomorphism φ from G onto a finite group such that $\varphi(a) \notin \varphi(H)$.

Equations over groups is an old and well-established area of group theory in view of the abundant scientific production on the subject. See [1, 6–9, 11, 14]. An equation with the variable x over a group G is an equality of the form

$$w(x) = 1, \tag{1.1}$$

where $w(x)$ belongs in $G * \langle x \rangle$, the free product of G and the cyclic group $\langle x \rangle$. See [10, 11]. There may or may not exist solution of equation (1.1) in G. If there exists one, it need not be unique. If equation (1.1) does not have solution in G, the question arises whether there is an over group H containing G in which this equation has solution. When such group H exists, we say that equation (1.1) is solvable over G. See [11]. Using the theory of generalized free product with amalgamation, B.H. Neumann studied conditions on G and $w(x)$ such that equation (1.1) is solvable over G. He obtained that the over group H is generated by a solution of (1.1) and elements of G. We say that H is obtained by adjoining a solution of (1.1) to G. See [12].

Since B.H. Neumann introduced the first systematic investigation of the problem of adjoining roots to groups ([12]), many authors have studied the subject in different directions $[1, 6, 8, 9, 11, 14]$. L. Louder in [9] analyzed limit groups obtained from other limit groups by adjoining roots. In [1], R.B.J.T. Allenby proved that under some conditions a nilpotent group B of a given class can be embedded in a group G having the same solubility lenght as B. A. Menshov and V. Roman'kov studied embeddings to some unitriangular groups arising under adjunction of roots. See [11]. Recently, using a twisted group algebra, T. Lawson constructively adjoined formal radicals in a symmetrical ∞-category. See [8]. In [12], B.H. Neumann also focused on a particular form of equation (1.1). That is equation:

$$x^n = g, \tag{1.2}$$

where n is a positive integer and g is an element of a group G. Any solution of the equation (1.2) is called a nth-root of g. Also, there may or may not exist a nth-root of g in G. If g does not have a nth-root in G, again the question arises whether we can adjoin one. More precisely, we pose the following question: given an element g in a group G, does there exist a group $G^\star$ containing an isomorphic replica of G (denoted again by G) as a subgroup such that in $G^\star$, g has a nth-root x? [10]. The motivation for the study of adjunction of roots to groups goes back to the fact that the affirmative answer to the previous question plays a role in the solution for the word problem for groups with a single defining relation. See $[10, 12]$. In [17], D. Tieudjo proved that, given A a $\mathcal{K}$-residual group for a root-class $\mathcal{K}$ and a nonzero natural number n, the group $G = A \underset{g=x^n}{\star} \langle x \rangle$, obtained by adjoining roots to A, is $\mathcal{K}$-residual if and only the infinite cyclic group $< g >$ generated by element g is $\mathcal{K}$-separable in A.

B.H. Neumann proved (see [12]) that, given a group G, n a nonzero natural number and an element $g \in G$, there exists exactly one group $G^\star$ with the following properties:

(1) $G^\star$ contains G and is generated by the elements of G and an additional element x such that $x^n = g$;
(2) all groups sharing property 1. with $G^\star$ are quotient groups of $G^\star$.

In this paper, when G' is a group obtained by adjoining a nth-root of g to a given group G, where n is a nonzero natural number and g is an element of G of infinite order, we compute the profinite completion $\widehat{G'}$ of G'. That is:

Theorem 1.1. *Let G be a group. Let n be a nonzero natural number and g an element of G of infinite order such that $x^n = g$. Let $G' = G \underset{<g>=<x^n>}{*} \langle x\rangle$ be the group obtained by the adjunction of a nth-root x of g to G. If the following conditions are satisfied:*

(1) *the cyclic group $\langle g\rangle$ is closed in the profinite topology on G, and*
(2) *the profinite topology on G' induces the profinite topologies on G, $\langle x\rangle$, $\langle g\rangle$ and $\langle x^n\rangle$, then $G^* = \widehat{G} \underset{\widehat{<g>}=\widehat{<x^n>}}{\coprod} \widehat{<x>}$, the amalgamated free profinite product of the profinite completions of G and $\langle x\rangle$.*

In [7], the authors showed that under some conditions, by adjoining roots to certains classes of conjugacy separable groups the resulting groups are again conjugacy separable groups. In this paper we investigate on profinite groups. We obtain necessary and sufficient condition under which a group obtained by adjoining root to a profinite group remains again profinite. We prove:

Theorem 1.2. *Let G be a profinite group in which any subgroup of finite index is open, n a nonzero natural number, g an element of G, and x an element not belonging to G. Assume that the closures $\overline{<g>}$ and $\overline{<x^n>}$ of the cyclic subgroups $\langle g\rangle$ and $\langle x^n\rangle$, in the respective profinite groups G and $\widehat{<x>}$, are topologically isomorphic. Then the following conditions are equivalent:*

(1) *$\langle g\rangle$ is closed in the profinite topology of G;*
(2) *there exists a unique profinite group G^* containing G and x, in which $g = x^n$.*

Since in any finitely generated profinite group the open subgroups are subgroups of finite index, it follows that:

Corollary 1.1. *Let G be a finitely generated profinite group, n a nonzero natural number, g an element of G, and x an element not belonging to G. Assume that the* *closures $\overline{< g >}$ and $\overline{< x^n >}$ of the cyclic subgroups $\langle g \rangle$ and $\langle x^n \rangle$, in the respective profinite groups G and $\widehat{< x >}$, are topologically isomorphic. Then there exists a unique profinite group $G^* = G \coprod_{\overline{<g>}=\overline{<x^n>}} \widehat{< x >}$ containing G and x, in which $g = x^n$.*

2. Preliminary notions and results

In this section, we recall definitions and properties of some notions we will use. One can refer to [10, 13] for more details.

2.1. Free groups.

Definition 2.1. *Let X be a nonempty set. Let $F(X)$ be a group and let $l : X \to F(X)$ be a map. We say that the family $(F(X), l)$ is a free group having basis X, or simply $F(X)$ is a free group on X, if for any map $\varphi : X \to G$, where G is a group, there exists a unique homomorphism $\bar{\varphi} : F(X) \to G$ such that the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{\;l\;} & F(X) \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \bar{\varphi}} \\ & & G \end{array}$$

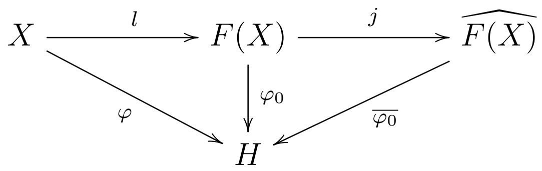
For any abstract group G, write $\mathcal{N}_G$ for the set of all normal subgroups in G with finite index. *The profinite completion* of G, denoted by $\widehat{G}$, is the inverse limit of a projective system of finite groups, i.e., $\widehat{G} = \varprojlim_{N \in \mathcal{N}_G} G/N$, with $\mathcal{N}_G$ a directed set. $\widehat{G}$ is a profinite group. Particularly, when G is a free group on X denoted $F(X)$, every element of $F(X)$ can uniquely be written as the product of finitely many elements from X and $X^{-1} = \{x^{-1}, x \in X\}$. In what follows, we compute the profinite completion of a free group on a set X. That is:

Proposition 2.1. *Let X be a nonempty set and let $(F(X), l)$ be the free group having basis X. Then:*

(1) *if X is finite, then the profinite completion of $F(X)$ is the free profinite group with basis the discrete space X;*
(2) *if X is discrete and infinite, then the profinite completion of $F(X)$ is the free profinite group with basis $X \cup \{*\}$ the Alexandroff-compactification of the discrete space X.*

Proof. X is a nonempty set. Then two cases arise:

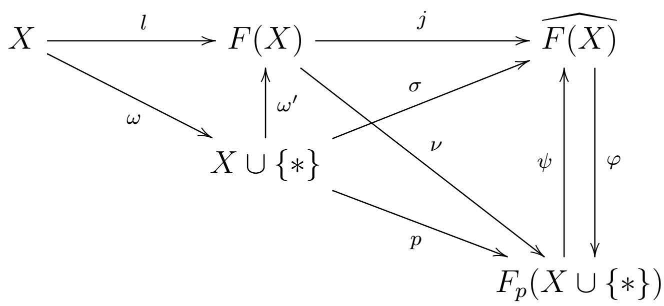
(1) X is a discrete and finite set. Let $\mathcal{N}$ be the set of all normal subgroups of finite index in $F(X)$. Let $\widehat{F(X)} = \varprojlim_{N\in\mathcal{N}} F(X)/N$ be the profinite completion of the abstract group $F(X)$. Let $j : F(X) \to \widehat{F(X)}$ be the canonical continuous homomorphism onto a dense subgroup of $\widehat{F(X)}$, where $F(X)$ is endowed with the profinite topology. Since X is a discrete space, the map $l : X \to F(X)$ is continuous. Thus the composition $\rho : X \to F(X) \to \widehat{F(X)}$ is continuous. Since the discrete and finite space X is profinite, let prove that $(\widehat{F(X)}, \rho)$ satisfies the universal property. Consider $\varphi : X \to H$ a continuous map into a discrete and finite group H. $F(X)$ is the free abstract group with basis the set X. So there exists a unique homomorphism $\varphi_0 : F(X) \to H$ that extends φ. This situation is illustrated by the following diagram



Since l and φ are continuous, so is φ_0. By application of the universal property of the profinite completion $\widehat{F(X)}$ to H, there is a unique continuous homomorphism $\overline{\varphi_0} : \widehat{F(X)} \to H$ such that $\overline{\varphi_0} j = \varphi_0$. Easily $\overline{\varphi_0} j l = \varphi_0 l = \varphi$ since $\varphi_0 l = \varphi$. Then $\widehat{F(X)}$ is the free profinite group with basis the discrete space X.

(2) Now X is discrete and infinite. It holds automatically that X is a Hausdorff and locally compact space. Let $X\cup\{*\}$ be the Alexandroff-compactification of X and let $\omega : X \to X \cup \{*\}$ be a continuous map such that its corestriction $X \to X$ is a homeomorphism. Consider $\widehat{F(X)}$ the profinite completion of the abstract group $F(X)$ and $j : F(X) \to \widehat{F(X)}$ the canonical continuous homomorphism onto a dense subgroup of $\widehat{F(X)}$, where $F(X)$ is endowed with the profinite topology. Then $\widehat{F(X)}$ is the free profinite group with basis $X \cup \{*\}$. Indeed: let $(F_p(X \cup \{*\}), p)$ be the free profinite group with basis $X \cup \{*\}$. On one hand, $X \cup \{*\}$ is a pointed space and the topological groups $F(X)$ and $\widehat{F(X)}$ can be seen as pointed spaces with

distinguished point the identity 1. See [13]. Then, let $\omega' : X \cup \{*\} \to F(X)$ and let $\sigma : X \cup \{*\} \to \widehat{F(X)}$ be continuous maps between these pointed spaces such that $\sigma(*) = 1$, $\omega'(*) = 1$, $\omega'\omega = l$, $\sigma\omega' = j$ and $\sigma\omega = jl$. By the definition of $F_p(X \cup \{*\})$, there exists a unique continuous homomorphism $\psi : F_p(X \cup \{*\}) \to \widehat{F(X)}$ satisfying $\psi p = \sigma$. On the other hand, the composition of continuous maps $p\omega : X \to X \cup \{*\} \to F_p(X \cup \{*\})$ is continuous. Therefore there exists a unique continuous homomorphism $\nu : F(X) \to F_p(X \cup \{*\})$ with $\nu l = p\omega$. Thus, by the definition of the profinite completion $\widehat{F(X)}$ of $F(X)$, there exists a unique continuous homomorphism $\varphi : \widehat{F(X)} \to F_p(X \cup \{*\})$ such that $\varphi j = \nu$. All this situation is illustrated by the following commutative diagram



Then we have: $\psi\varphi j = \psi\nu = j = id_{\widehat{F(X)}} j$. Since $j(F(X))$ is dense in $\widehat{F(X)}$, it follows that $\psi\varphi = id_{\widehat{F(X)}}$. Also, $\varphi\psi p = \varphi\sigma = \varphi j\omega' = \nu\omega' = p = id_{F_p(X \cup \{*\})} p$. We obtain from the fact that $p(X \cup \{*\})$ generates topologically $F_p(X \cup \{*\})$ that $\varphi\psi = id_{F_p(X \cup \{*\})}$. Thus φ is a topological isomorphism and the result is obtained.

□

Therefore, a profinite completion of a free group of finite rank is finitely generated.

2.2. **Free product of groups with amalgamation.**

Definition 2.2. *Let H be a subgroup of a group G_1 and let K be a subgroup of a group G_2 such that H is isomorphic to K through the isomorphism $\varphi : H \to K$. The free product of groups G_1 and G_2 amalgamating subgroups H and K through the isomorphism φ is a group generated by the disjoint union of all the generators of* *groups G_1 and G_2, and defined by all the relators of groups G_1 and G_2, together with all the relations of the form $\varphi(h) = k$, for all $h \in H$ and $k \in K$.*

Let G_1 and G_2 be two abstract groups, and let H and K be their respective subgroups such that H is isomorphic to K through the isomorphism $\varphi : H \to K$. We denote by $G = G_1 \underset{\substack{H=K\\ \varphi}}{\star} G_2$ (or simply $G = G_1 \underset{H=K}{*} G_2$ when there is no confusion) the free product of groups G_1 and G_2 amalgamating isomorphic subgroups H and K via the isomorphism φ. Relatively to the isomorphism φ, subgroups H and K can be identified. We then write $G_1 \underset{H}{\star} G_2$ the free product of groups G_1 and G_2 over subgroup H, meaning that H is the common subgroup of groups G_1 and G_2 (indeed, $K = \varphi(H)$, where φ is the known isomorphism). See [10] for more details.

An element g in G can be written in a form $g = g_1 g_2 \cdots g_r$ $(r \geq 1)$ where for any $i = 1, 2, \cdots, r$, element g_i belongs to one of the free factor G_1 or G_2, and if $r > 1$, any successive g_i and g_{i+1} do not belong to the same factor G_1 or G_2 (nor to the amalgamated subgroups H and K). We say that g is written in a *reduced form*. In general, an element of the group $G = G_1 \underset{H=K}{*} G_2$ can have more than one reduced form. But any two reduced forms of an element g have the same number of components, which we will call the length of the element g.

Also we have the following usefull notions.

Definition 2.3. *Let $G = G_1 \underset{\substack{H=K\\ \varphi}}{\star} G_2$ be the free product of groups G_1 and G_2 with amalgamated subgroups H and K via the isomorphism φ.*

(1) *Let R and S be normal subgroups of finite index in groups G_1 and G_2 respectively. The subgroups R and S are (H, K, φ)-compatible if the following equality holds: $\varphi(R \cap H) = S \cap K$.*
(2) *A family $(R_i)_{i \in I}$ of subgroups of a group G is called a filtration if $\bigcap_{i \in I} R_i = \{1\}$. And the family $(R_i)_{i \in I}$ is a H-filtration if it is a filtration, and in addition we have $\bigcap_{i \in I} HR_i = H$.*

Concerning free profinite products of profinite groups with amalgamation, we have according to [13]:

Definition 2.4. *Let H be a closed subgroup of a profinite group G_1 and let K be a closed subgroup of a profinite group G_2. Let $\sigma : H \to G_1$ and $\tau : K \to G_2$ be the* *inclusion maps and let $\varphi : H \to K$ be an isomorphism of topological groups. The free profinite product of the profinite groups G_1 and G_2 with amalgamated subgroups H and K is a family $(G, \varphi_1, \varphi_2)$ where G is a profinite group and $\varphi_1 : G_1 \to G$, $\varphi_2 : G_2 \to G$ are continuous homomorphisms satisfying:*

(1) *$\varphi_1\sigma = \varphi_2\tau\varphi$ and*
(2) *If G' is a profinite group with continuous homomorphisms $\psi_1 : G_1 \to G'$ and $\psi_2 : G_2 \to G'$ such that $\psi_1\sigma = \psi_2\varphi\tau$, then there exists a unique continuous homomorphism $\psi : G \to G'$ such that $\psi\varphi_1 = \psi_1$ and $\psi\varphi_2 = \psi_2$.*

We denote by $G_1 \coprod_{H=K} G_2$ the free profinite product of the profinite groups G_1 and G_2 with amalgamated subgroups H and K.

Let G_1 and G_2 be two profinite groups with respective closed subgroups H and K. A concrete free profinite product G of the profinite groups G_1 and G_2 with amalgamated subgroups H and K can be obtained by taking $G = \widehat{\tilde{G}} = \varprojlim_{N\in\mathcal{N}} \tilde{G}/N = \widehat{G_1 \underset{H=K}{*} G_2}^{\mathcal{N}}$, the profinite completion of $\tilde{G}$ with respect to $\mathcal{N}$, where $\tilde{G} = G_1 \underset{H=K}{*} G_2$ is the free abstract product of G_1 and G_2 with amalgamated subgroups H and K, considered as abstract groups and $\mathcal{N} = \{N \lhd_f G = G_1 \underset{H=K}{*} G_2 : N \cap G_i \text{ is open in } G_i,\ i = 1, 2\}$. See [13].

$G_1 \coprod_{H=K} G_2$ is said to be proper if the continuous homomorphisms $G_1 \to G_1 \coprod_{H=K} G_2$ and $G_2 \to G_1 \coprod_{H=K} G_2$ are one to one. There are examples of free profinite product of profinite groups with amalgamated subgroups which are not proper. See [13].

3. Proof of results

3.1. **Proof of Theorem 1.1.** When considering the root class consisting of all finite groups, then the characterization of residually finite group after a nth-root has been adjoined to it obtained by D. Tieudjo in [17, Theorem 3.4] can be rewritten as follow:

Lemma 3.1. *[17, Theorem 5.1] Let G be a residually finite group. Let n be a nonzero natural number and let g be an element of G of infinite order such that $x^n = g$. The group $G' = G \underset{<g>=<x^n>}{*} \langle x\rangle$ obtained after a nth- root has been adjoined to G is residually finite if and only if the infinite cyclic subgroup $\langle g\rangle$ generated by g is closed in the profinite topology of G.*

Let now prove Theorem 1.1.

Proof of Theorem 1.1. Assume that the cyclic group $\langle g\rangle$ is closed in the profinite topology on G. Then $G' = G \underset{<g>=<x^n>}{*} \langle x\rangle$ is residually finite by Lemma 3.1. If we assume in addition that the profinite topology on G' induces on G, $\langle x\rangle$, $\langle g\rangle$ and $\langle x^n\rangle$ their profinite topologies, then by [13], $\widehat{G'} = \widehat{G} \underset{\widehat{<g>}=\widehat{<x^n>}}{\coprod} \widehat{<x>}$, that is, the profinite completion $\widehat{G'}$ of G' and the proper amalgamated free profinite product of the profinite completions of G and $\langle x\rangle$ coincide. And Theorem 1.1 is demonstrated.

3.2. **Proof of Theorem 1.2.** We first prove the following Lemma.

Lemma 3.2. *Let G be a profinite group in which any subgroup of finite index is open, n a nonzero natural number, g an element of G of infinite order and x an element not belonging to G such that $x^n = g$. Let $\overline{<g>}$ and $\overline{<x^n>}$ be the closures of the cyclic subgroups $\langle g\rangle$ and $\langle x\rangle$, in the respective profinite groups G and $\widehat{<x>}$. The abstract group $G' = G \underset{\substack{<g>=<x^n>\\ \varphi}}{*} \langle x\rangle$ is residually finite if, and only if so is $G^* = G \underset{\substack{\overline{<g>}=\overline{<x^n>}\\ \varphi}}{*} \widehat{<x>}$.*

Proof. Assume that $G^* = G \underset{\substack{\overline{<g>}=\overline{<x^n>}\\ \varphi}}{*} \widehat{<x>}$ is residually finite. Let prove that G' is residually finite. Since $\langle x\rangle$ is residually finite, we get clearly that $\langle x\rangle \subseteq \widehat{\langle x\rangle}$. Also, the inclusions $\langle g\rangle \subseteq \overline{\langle g\rangle}$, $\langle x^n\rangle \subseteq \overline{\langle x^n\rangle}$ hold since $\overline{\langle g\rangle}$ and $\overline{\langle x^n\rangle}$ are finitely generated by Proposition 2.1, such that G' can be viewed as a subgroup of G^*. Therefore G' is residually finite as a subgroup of the residually finite group G^*.

Conversely, let $G' = G \underset{\substack{<g>=<x^n>\\ \varphi}}{*} \langle x\rangle$ be residually finite. Let establish that so is the group $G^* = G \underset{\substack{\overline{<g>}=\overline{<x^n>}\\ \varphi}}{*} \widehat{<x>}$. By [13, Theorem 9.2.4] it follows that there exist an indexing set Λ and the families of normal subgroups of finite index filtered from below $\mathcal{N}_G = \{N_{1\lambda}, \lambda \in \Lambda\}$ and $\mathcal{N}_{<x>} = \{N_{2\lambda}, \lambda \in \Lambda\}$ of G and $\langle x\rangle$ respectively, which are simultaneously $(\overline{\langle g\rangle}, \overline{\langle x^n\rangle})$-filtrations and $(\langle g\rangle, \langle x^n\rangle, \varphi)$-compatible.

Let $\overline{\mathcal{N}}_G = \{\overline{N}_{1\lambda}; \lambda \in \Lambda\}$ and $\overline{\mathcal{N}}_{\widehat{\langle x\rangle}} = \{\overline{N}_{2\lambda}; \lambda \in \Lambda\}$ be the families obtained by considering for any $\lambda \in \Lambda$, $\overline{N}_{1\lambda}$ and $\overline{N}_{2\lambda}$ to be the closure of $N_{1\lambda}$ and $N_{2\lambda}$ in the respective profinite groups G and $\widehat{\langle x\rangle}$. By [5, Proposition 91], $\overline{\mathcal{N}}_G$ and $\overline{\mathcal{N}}_{\widehat{\langle x\rangle}}$

are families of normal subgroups of finite index in G and $\widehat{\langle x\rangle}$ respectively. $\overline{\mathcal{N}}_G$ and $\overline{\mathcal{N}}_{\widehat{\langle x\rangle}}$ are therefore basis of neighborhoods of the identities of the respective profinite topologies on G and $\widehat{\langle x\rangle}$. Since any subgroup of finite index in G is open, it follows that the profinite topology of G and the topology of G as profinite group coincide. Consequently, $\bigcap\limits_{\overline{N}_{1\lambda}\in\overline{N}_G} \overline{N}_{1\lambda} = 1$ using the fact that G is Hausdorff as profinite group.

Similarly, the profinite group $\widehat{\langle x\rangle}$ is finitely generated by Proposition 2.1. Therefore, its profinite topology and its topology as profinite group coincide. Thus, $\bigcap\limits_{\overline{N}_{2\lambda}\in\overline{N}_{\widehat{\langle x\rangle}}} \overline{N}_{1\lambda} = 1$ since $\widehat{\langle x\rangle}$ is Hausdorff as profinite group. We obtain then that the families $\overline{\mathcal{N}}_G$ and $\overline{\mathcal{N}}_{\widehat{\langle x\rangle}}$ are filtrations.

(1) Let prove that the families $\overline{\mathcal{N}}_G$ and $\overline{\mathcal{N}}_{\widehat{\langle x\rangle}}$ are $(\overline{\langle g\rangle}, \overline{\langle x^n\rangle})$-filtrations. It suffices to prove that $\bigcap\limits_{\lambda\in\Lambda} \overline{N}_{1\lambda}\overline{\langle g\rangle} = \overline{\langle g\rangle}$ and $\bigcap\limits_{\lambda\in\Lambda} \overline{N}_{2\lambda}\overline{\langle x^n\rangle} = \overline{\langle x^n\rangle}$ since $\overline{\mathcal{N}}_G$ and $\overline{\mathcal{N}}_{\widehat{\langle x\rangle}}$ are already filtrations as seen previously. Now, recalling that $\overline{\langle g\rangle}$ is a closed subgroup of G and $\overline{\mathcal{N}_G} = \{\overline{N}_{1\lambda}; \lambda\in\Lambda\}$ a family of closed subsets of G filtered from below, it follows that $\bigcap\limits_{\lambda\in\Lambda} \overline{N}_{1\lambda}\overline{\langle g\rangle} = \overline{\langle g\rangle}$ by [13, Proposition 2.1.4].

Similarly, $\bigcap\limits_{\lambda\in\Lambda} \overline{N}_{2\lambda}\overline{\langle x^n\rangle} = \overline{\langle x^n\rangle}$.

So that the families $\overline{\mathcal{N}}_G$ and $\overline{\mathcal{N}}_{\widehat{\langle x\rangle}}$ are $(\overline{\langle g\rangle}, \overline{\langle x^n\rangle})$-filtrations.

(2) Let now show that the families $\overline{\mathcal{N}}_G$ and $\overline{\mathcal{N}}_{\widehat{\langle x\rangle}}$ are $(\overline{< g >}, \overline{< x^n >}, \overline{\varphi})$-compatible. That is, for any $\lambda\in\Lambda$, $\overline{\varphi}(\overline{N}_{1\lambda}\cap\overline{\langle g\rangle}) = \overline{N}_{2\lambda}\cap\overline{\langle x^n\rangle}$. Consider the following commutative diagram:

$$\begin{array}{ccc}
\langle g\rangle & \xrightarrow{\quad\varphi\quad} & \langle x^n\rangle \\
\big\downarrow{\scriptstyle l_1} & & \big\downarrow{\scriptstyle l_2} \\
\overline{\langle g\rangle} & \xrightarrow{\quad\overline{\varphi}\quad} & \overline{\langle x^n\rangle}
\end{array}$$

We get the following identities:

$\overline{\varphi}(\overline{N_{1\lambda}} \cap \overline{\langle g\rangle}) = \overline{\varphi}(N_{1\lambda} \cap \overline{\langle g\rangle}),$ since $N_{1\lambda}$ is open and then closed ;

$= \overline{\varphi}(\overline{N_{1\lambda} \cap \langle g\rangle}),$ by [3, Lemma 3];

$= \overline{\overline{\varphi}(N_{1\lambda} \cap \langle g\rangle)},$ since $\overline{\varphi}$ is an isomorphism between topological groups;

$= \overline{\overline{\varphi}(l_1(N_{1\lambda} \cap \langle g\rangle))},$ since l_1 is one to one using the fact that $\langle g\rangle$ is residually finite as a free group ;

$= \overline{l_2 \circ \varphi(N_{1\lambda} \cap \langle g\rangle)},$ since $\overline{\varphi} \circ l_1 = i_2 \circ \varphi;$

$= \overline{l_2(N_{2\lambda} \cap \langle x^n\rangle)},$ since $\varphi(N_{1\lambda} \cap \langle g\rangle = N_{2\lambda} \cap \langle x^n\rangle$;

$= \overline{N_{2\lambda} \cap \langle x^n\rangle},$ since l_2 is one to one using the fact that $\langle x^n\rangle$ is residually finite as a free group ;

$= N_{2\lambda} \cap \overline{\langle x^n\rangle},$ by [3, Lemma 3];

$= \overline{N_{2\lambda}} \cap \overline{\langle x^n\rangle},$ since $\overline{N_{2\lambda}}$ is open and then closed .

Thus, $\overline{\varphi}(\overline{N_{1\lambda}} \cap \overline{\langle g\rangle}) = \overline{N_{2\lambda}} \cap \overline{\langle x^n\rangle}$. And the families $\overline{\mathcal{N}}_G$ and $\overline{\mathcal{N}}_{\widehat{\langle x\rangle}}$ are $(\overline{\langle g\rangle}, \overline{\langle x^n\rangle}, \overline{\varphi})$-compatible.

Finally, the families $\overline{\mathcal{N}}_G$ and $\overline{\mathcal{N}}_{\widehat{\langle x\rangle}}$ are simultaneously $(\overline{\langle g\rangle}, \overline{\langle x^n\rangle})$-filtrations and $(\overline{\langle g\rangle}, \overline{\langle x^n\rangle}, \overline{\varphi})$-compatible. Consequently, G^* is residually finite by [2, Proposition 2] and Lemma 3.2 is proven.

□

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let G be a profinite group in which any subgroup of finite index is open, n a nonzero natural number, g an element of G, and x an element not belonging to G. Let $\overline{\varphi} : \overline{< g >} \to \overline{< x^n >}$ be the topological isomorphism between the closures $\overline{< g >}$ and $\overline{< x^n >}$ of the cyclic subgroups $< g >$ and $< x >$, in the respective profinite groups G and $\widehat{< x >}$.

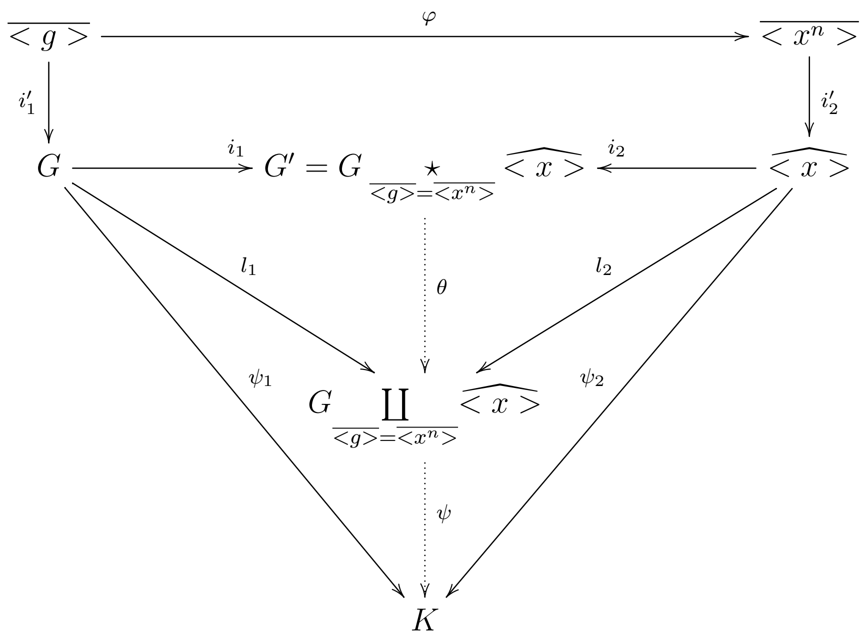
(1) Suppose that $< g >$ is closed in the profinite topology of G. Let show that there exists a unique profinite group G^* containing G and x, in which $x^n = g$. To do it we will give an explicite construction of such group G^*.

Let $G' = G \underset{\overline{<g>}=\overline{<x^n>}}{\star} \widehat{< x >}$ be the free abstract product of the groups G and $\widehat{< x >}$ with amalgamated subgroups $\overline{< g >}$ and $\overline{< x^n >}$, considering here that the groups are devoid with their topological structures. Set

$$\mathcal{N} = \{N \lhd_f G' : gN \cap G \text{ is open in } G, \text{ and } gN \cap \widehat{<x>} \\ \text{is open in } \widehat{<x>}, \text{ for all } g \in G'\},$$

and $G^* = \varprojlim_{N \in \mathcal{N}} G'/N$. Let show that $G^* = G \coprod_{\overline{<g>} = \overline{<x^n>}} \widehat{<x>} = \widehat{G'}$ is the desired profinite group. Considering that the abstract group $G' = G \underset{\overline{<g>} = \overline{<x^n>}}{\star} \widehat{<x>}$ is endowed with the profinite topology, the canonical maps $i_1 : G \to G'$, $i'_1 : \overline{<g>} \to G$, $i_2 : \widehat{<x>} \to G'$, $i'_2 : \overline{<x^n>} \to \widehat{<x>}$ and $\theta : G' \to G^*$ given by the definition of the profinite completion $\widehat{G'} = G^*$ of group G', are continuous. Therefore i_1, i_2 and θ induce the continuous homomorphism $l_1 = \theta \circ i_1 : G \to G^*$ and $l_2 = \theta \circ i_2 : \widehat{<x>} \to G'$.

Let K be a profinite group with continuous homomorphisms $\psi_1 : G \to K$ and $\psi_2 : \widehat{<x>} \to K$ such that $\psi_1 i'_1 = \psi_2 i'_2 \varphi$. This situation is illustrated by the following commutative diagram:



Let determine a unique continuous homomorphism $\psi : G^* \to K$ such that $\psi l_1 = \psi_1$ and $\psi l_2 = \psi_2$.

By the universal property applied to the amalgamated product G', it comes that there exists a unique homomorphism $\lambda : G^* \to K$ satisfying the equalities $\lambda i_1 = \psi_1$ and $\lambda i_2 = \psi_2$. It is easily seen that λ is continuous.

Using now the fact that $G \coprod_{\overline{<g>}=\overline{<x^n>}} \widehat{<x>}$ is the profinite completion of $G \star_{\overline{<g>}=\overline{<x^n>}} \widehat{<x>}$, it follows that there exists a unique continuous homomorphism $\psi : G \coprod_{\overline{<g>}=\overline{<x^n>}} \widehat{<x>} \to K$ such that $\lambda = \psi\theta$. Clearly, we have the equalities $\psi l_1 = \psi_1$ and $\psi l_2 = \psi_2$. Indeed: $\psi_1 = \lambda i_1 = \psi\theta i_1 = \psi l_1$ and $\psi_2 = \lambda i_2 = \psi\theta i_2 = \psi l_2$.

Thus, $G^* = G \coprod_{\overline{<g>}=\overline{<x^n>}} \widehat{<x>}$.

Let now show that the profinite group $G^* = G \coprod_{\overline{<g>}=\overline{<x^n>}} \widehat{<x>}$ contains G and x. It suffices to prove that the amalgamated free profinite product G^* is proper.

Since G is profinite, it is residually finite. And the amalgamated product $G \ast_{<g>=<x^n>} <x>$ is residually finite using the fact that the cyclic group $<g>$ is closed in the profinite topology of G. Thus, the abstract group $G' = G \ast_{\overline{<g>}=\overline{<x^n>}} \widehat{<x>}$ is residually finite by Lemma 3.2. Consequently, the homomorphism $\theta\colon G \ast_{\overline{<g>}=\overline{<x^n>}} \widehat{<x>} \to G \coprod_{\overline{<g>}=\overline{<x^n>}} \widehat{<x>}$ is one to one. And by [13, Theorem 9.2.4] the amalgamated free profinite product G^* is proper. Thus, G^* contains G and x, and we have $x^n = g$. The unicity of G^* is given by this specific construction.

(2) Conversely, assume that there exists a unique profinite group G^* containing G and x, in which $g = x^n$. Let prove that $<g>$ is closed in the profinite topology of G. We shall establish that the amalgamated abstract free product $G' = G \ast_{\overline{<g>}=\overline{<x^n>}} \widehat{<x>}$ is residually finite and conclude by Lemma 3.2. So, let $1 \neq g' \in G'$. g' can be written in the reduced form as $g' = g'_1 x_{i_1}^{\varepsilon_1} \cdots g'_m x_{i_m}^{\varepsilon_m}$ with $g'_j \in G$, $x_{i_j} \in \widehat{<x>}$, $\varepsilon_j = \pm 1$ for any $j \in \{1, \cdots, m\}$, $m \geq 1$, and if $m > 1$, any successive g'_j and x_{i_j} do not belong to the same factor G or $\widehat{<x>}$ (nor to the amalgamated subgroups $\overline{<g>}$ and $\overline{<x^n>}$.

We need to determine a subgroup N of G' of finite index such that $g' \notin N$.

First, assume that $m = 1$. Then $g' \in G$ or $g' \in \widehat{<x>}$.

Let $g' \in \widehat{<x>}$. By hypotesis $G^* = G \coprod_{\overline{<g>}=\overline{<x^n>}} \widehat{<x>}$ contains G and x. Thus, the canonical homomomorphisms $l_1\colon G \to G^*$ and $l_2\colon \widehat{<x>} \to G^*$ are one to one. Consequently, the amalgamated free profinite product G^* is proper. Therefore, again by [13, Theorem 9.2.4], there exist an indexing set Λ and the families of normal subgroups of finite index filtered from below $\mathcal{N}_G = \{N_{1\lambda}, \lambda \in \Lambda\}$ and $\mathcal{N}_{\widehat{<x>}} = \{N_{2\lambda}, \lambda \in \Lambda\}$ of G and $\widehat{<x>}$ respectively, which are simultaneously $(\overline{\langle g\rangle}, \overline{\langle x^n\rangle})$-filtrations and $(\overline{<g>}, \overline{<x^n>}, \varphi)$-compatible. Since the profinite group $\widehat{<x>}$ is residually finite, it follows that there exists $\mu \in \Lambda$ such that $g' \notin N_{2\mu}$. Set

$$P = \frac{G}{N_{1\mu}} \underset{\frac{\overline{<g>}N_{1\mu}}{N_{1\mu}} = \frac{\overline{<x^n>}N_{2\mu}}{N_{2\mu}}}{*} \frac{\widehat{<x>}}{N_{2\mu}}.$$

The canonical homomomorphisms $G \to \dfrac{G}{N_{1\mu}}$ and $\widehat{<x>} \to \dfrac{\widehat{<x>}}{N_{2\mu}}$ extends to the homomorphism $\theta : G' \to P$ with $\theta(g') = g'N_{2\mu} \neq N_{2\mu}$, clearly onto. Since P is residually finite by [[2], Theorem 2.] as an amalgamated free product of finite groups, it contains a subgroup N' of finite index satisfying $\theta(g') \notin N'$. Thus, $N = \theta^{-1}(N')$ is a subgroup N of G' of finite index such that $g' \notin N$ and G' is residually finite.

We prove similarly that, if $g' \in G$, then G' is residually finite.

Assume now that $m > 1$. Then, for any $j \in \{1, \cdots, m\}$, $g'_j \in G \setminus \overline{<g>}$ and $x_{i_j}^{\varepsilon_j} \in \widehat{<x>} \setminus \overline{<x^n>}$. Again using the fact that $G^* = G \coprod_{\overline{<g>}=\overline{<x^n>}} \widehat{<x>}$ contains G as subgroup and x as element, it follows that the homomorphisms $l_1\colon G \to G^*$ and $l_2\colon \widehat{<x>} \to G^*$ are one to one. And again, the free amalgamated product G^* is proper. Also, by [13, Theorem 9.2.4], there exist an indexing set Λ and the families of normal subgroups filtered from below $\mathcal{N}_G = \{N_{1\lambda}, \lambda \in \Lambda\}$ and $\mathcal{N}_{\widehat{<x>}} = \{N_{2\lambda}, \lambda \in \Lambda\}$ of finite index of $\mathcal{N}_G$ and $\mathcal{N}_{\widehat{<x>}}$ which are simultaneously $(\overline{\langle g\rangle}, \overline{\langle x^n\rangle})$-filtrations and $(\overline{<g>}, \overline{<x^n}>, \varphi)$-compatible. Using now the fact that $\overline{<g>}$ and $\overline{<x^n>}$

are respective closed subgroups of G and $\widehat{<x>}$, it follows by [13, Proposition 2.1.4] that $\overline{<g>} = \cap_{\lambda\in\Lambda}\overline{<g>N_{1\lambda}}$ and $\overline{<x^n>} = \cap_{\lambda\in\Lambda}\overline{<x^n>N_{2\lambda}}$. Thus, there exists $\lambda \in \Lambda$ satisfying $g'_j \notin \overline{<g>}N_{1\lambda}$ and $x_{i_j}^{\varepsilon_j} \notin \overline{<x^n>}N_{2\lambda}$. Consequently, $\psi(g') = g'_1N_{1\lambda}x_{i_1}^{\varepsilon_1}N_{2\lambda}\dots g'_mN_{1\lambda}x_{i_m}^{\varepsilon_m}N_{2\lambda} \neq 1$ where

$$\psi\colon G^* \to \frac{G}{N_{1\lambda}} \underset{\frac{\overline{<g>}N_{1\lambda}}{N_{1\lambda}} = \frac{\overline{<x^n>}N_{2\lambda}}{N_{2\lambda}}}{*} \frac{\widehat{<x>}}{N_{2\lambda}}$$

is the homomorphism induced by the canonical homomorphisms $G \to \dfrac{G}{N_{1\lambda}}$ and $\widehat{<x>} \to \dfrac{\widehat{<x>}}{N_{2\lambda}}$. Since $\dfrac{G}{N_{1\lambda}} \underset{\frac{\overline{<g>}N_{1\lambda}}{N_{1\lambda}} = \frac{\overline{<x^n>}N_{2\lambda}}{N_{2\lambda}}}{*} \dfrac{\widehat{<x>}}{N_{2\lambda}}$ is residually finite as an amalgamated free product of finite groups, it follows that it contains a subgroup N' of finite index such that $\psi(g') \notin N'$. Also, ψ is a homomorphism which is onto. Indeed, if $g'_1N_{1\lambda}x_{i_1}^{\varepsilon_1}N_{2\lambda}\dots g'_mN_{1\lambda}x_{i_m}^{\varepsilon_m}N_{2\lambda} \in \dfrac{G}{N_{1\lambda}} \underset{\frac{\overline{<g>N_{1\lambda}}}{N_{1\lambda}} = \frac{\overline{<x^n>N_{2\lambda}}}{N_{2\lambda}}}{*} \dfrac{\widehat{<x>}}{N_{2\lambda}}$, then we clearly obtain

$$\psi(g'_1x^{\varepsilon_1}\cdots g'_mx_{i_m}^{\varepsilon_m}) = g'_1N_{1\lambda}x_{i_1}^{\varepsilon_1}N_{2\lambda}\cdots g'_mN_{1\lambda}x_{i_m}^{\varepsilon_m}N_{2\lambda}.$$

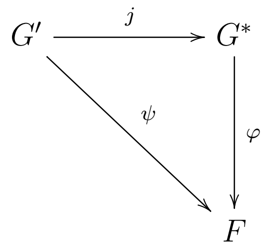
$N = \psi^{-1}(N')$ is a subgroup of G' of finite index with $g' \notin N$, and G' is residually finite. Theorem 1.2 is demonstrated.

Any profinite group containing G and x in which $g = x^n$, is a quotient group of such G^*. That is:

Remark 3.1. *Let G be a profinite group in which any subgroup of finite index is open, n a nonzero natural number, g an element of G and x an element not belonging to G. Assume that the closures $\overline{<g>}$ and $\overline{<x^n>}$ of the cyclic subgroups $<g>$ and $<x>$, in the respective profinite groups G and $\widehat{<x>}$, are topologically isomorphic. If $<g>$ is closed in the profinite topology of G, then any profinite group containing G and x in which $g = x^n$, is a quotient group of G^*.*

Indeed: Let F be a profinite group containing the profinite group G as a closed subgroup, and let x be such that in F we have $g = x^n$. Let determine a continuous homomorphism $\varphi : G^* \to F$ which is onto.

Since F contains G and x, and in F we have $g = x^n$, it follows that there exists an onto homomorphism $\psi : G' \to F$ using the definition of G'(as abstract group). Using also the fact that F is a profinite group and $G^* = \widehat{G'}$, it follows by [13, Lemma 3.2.1] that there exists a unique continuous homomorphism $\varphi : G^* \to F$ such that $\varphi \circ j = \psi$, where j is the canonical continuous homomorphism between G' and G^*. This situation can be illustrated by the following commutative diagram:



Since $\psi = \varphi \circ j$ is onto, so is φ. Consequently F is the quotient of G^*.

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DEPARTMENT OF MATHEMATICS OF HIGHER TEACHER'S TRAINING COLLEGE
THE UNIVERSITY OF MAROUA
P.O. BOX 55, MAROUA,
CAMEROON.
Email address: gilbertmantika@yahoo.fr

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE OF THE FACULTY OF SCIENCE
THE UNIVERSITY OF MAROUA
P.O. BOX 814, MAROUA,
CAMEROON.
Email address: ndjouofomekong@yahoo.com

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE OF THE NATIONAL SCHOOL OF AGRO-INDUSTRIAL SCIENCE
THE UNIVERSITY OF NGAOUNDERE
P.O. BOX 455, NGAOUNDERE
CAMEROON.
Email address: tieudjo@yahoo.com