Advances in Mathematics: Scientific Journal **13** (2024), no.3, 387–397
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.13.3.8

EQUALITY BETWEEN THE ALGEBRAS OF DIFFERENTIAL OPERATORS AND ENDOMORPHISMS IN FINITE DIMENSION

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ABSTRACT. In this paper, we show that in characteristic zero, the condition *finite local $k-$rational* is not necessary for the algebra of differential operators on a commutative and unitary $k-$algebra to be equal to that of endomorphisms. But the *finite dimension* is necessary for the algebra of differential operators to be an algebra of finite dimension.

1. INTRODUCTION

Let A be a commutative and unitary algebra over characteristic field k. Differential operators on a given $A-$module M was introduced in 1967 by A. Grotendieck in [1].

In 1992 R. C Canning and M. P Holland discovered in [5] that, in characteristic zero, the *finite local $k-$rational* condition is sufficient for the algebra of differential operators on a $k-$algebra to be equal to that of endomorphisms.

In this paper, we show that this condition is not necessary to get this equality. But the *finite dimension* is necessary for the algebra of differential operators to be an algebra of finite dimension.

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2020 *Mathematics Subject Classification.* Primary 13N10, Secondary 16S32.
Key words and phrases. Differential operators, Weyl Algebras, tensor product.
Submitted: 11.07.2024; *Accepted:* 26.07.2024; *Published:* 24.08.2024.

This paper is organized as follow: in section 2, we give some definitions, notations, basics properties and theorems of differential operators, tensor product and *finite locale* $k-$*rational*.

In section 3, we first demonstrate that it is necessary for the dimension of the algebra to be finite for the algebra of differential operators to be of finite dimension. Morever, we use the properties of the tensor product of algebras to construct a two-sided ideal of tensor product of two algebras. This allows us to prove that:

the tensor product of two finite local $k-$*rational algebras is not necessarily a finite local* $k-$*rational algebra*.

Then, thanks to our results on the tensor product of algebras of differential operators in [2], we prove that: *for all finite local* $k-$*rational algebras,* $A_1, \cdots, A_n$,

$$\mathcal{D}(A_1 \otimes A_2 \otimes \cdots \otimes A_n) = End_k(A_1 \otimes \cdots \otimes A_n).$$

We deduce from this theorem that the *finite local* $k-$*rational* condition is not necessary so that in finite dimension, the algebras of the operators differentials and endomorphisms coincide. Finally, using this result, we construct $k-$algebras A for which we have $\mathcal{D}(A) = End_k(A)$.

2. Preliminaries

In this part, we give some definitions, notations basics properties and theorems of differential operators, tensor product and *finite local* $k-$*rational*.

Definition 2.1 (A ring of differential operators)**.** *Let* A *be a commutative algebra over a field* k*. A ring of differential operators* $\mathcal{D}(A)$ *on* A *is defined as:*

$$\mathcal{D}(A) = \bigcup_{n \in \mathbb{N}} \mathcal{D}^n(A),$$

where

$$\mathcal{D}^0(A) = \{u \in End_k(A) : [u, a] = ua - au = 0, \forall a \in A\} \simeq End_A(A)$$

and

$$\mathcal{D}^n(A) = \{u \in End_k(A) : [u, a] = ua - au \in \mathcal{D}^{n-1}(A), for\ all\ a \in A\},$$

ua and au are elements of $End_k(A)$ defined by:

$$\forall x \in A, ua(x) = u(ax) \text{ and } au(x) = a(u(x)).$$

All element $u \in \mathcal{D}^n(A)$ is called differential operator of order n.

Proposition 2.1. *([3])*

1. *$id_A \in \mathcal{D}(A)$ and $\mathcal{D}^0(A)$ is identified to A by:$a \to \varphi_a : x :\mapsto ax$.*
2. *$End_k(A)$ is a $(\mathcal{D}(A), \mathcal{D}(A))-$bimodule.*
3. *let $n \geq 0$ be an integer, $\mathcal{D}^n(A)$ is a (A, A)-subbimodule of $End_k(A)$*
4. *$\mathcal{D}(A)$ is a $(A, A)-$subbimodule of $End_k(A)$.*
5. *$\mathcal{D}(A)$ is a subalgebra of $End_k(A)$.*

Proposition 2.2. *([4]) Let M and N be two $A-$modules.*

1. *If $(m_i)_{i\in I}$ and $(n_j)_{j\in J}$ are two generating families of M and N respectively, $(m_i \otimes n_j)_{i\in I;j\in J}$ is a generating family of $M \otimes N$.*
2. *Furthermore assume that M and N are two free $A-$modules. If $(m_i)_{i\in I}$ and $(n_j)_{j\in J}$ are two bases of M and N respectively, then $(m_i \otimes n_j)_{i\in I;j\in J}$ is a basis of $M \otimes_A N$.*

Proposition 2.3. *([4]) Let R be a ring, A and B two $R-$algebras. There exist on the $R-$module $A \otimes_R B$ an unique $R-$algebra structure such that: for all $((a, b), (a', b'^2, (a \otimes b)(a' \otimes b') = aa' \otimes bb'$. The $R-$algebra $A \otimes_R B$ is called the tensor product of the $R-$algebras A and B.*

Theorem 2.1. *([2]) For all algebras A and B, of finite type on a field k of characteristic $p \geq 0$, $\mathcal{D}(A) \otimes \mathcal{D}(B) \simeq \mathcal{D}(A \otimes B)$.*

Definition 2.2. *(A finite local $k-$rational algebra) An algebra A on a field k of characteristic zero is called finite local $k-$rational algebra when it is a $k-$vector space of finite dimension possessing a nilpotent element whose nilpotence index is equal to its dimension.*

Example 1. *$\frac{\mathbb{R}[x]}{(x^2)}$ is $\mathbb{R}-$finite local $k-$rational algebra.*

Theorem 2.2. *([5]) Let A be an algebra on a field k of characteristic zero. If A is finite local $k-$rational, $\mathcal{D}(A) = End_k(A)$.*

3. Coincidence of algebras of differential operators and endomorphisms

In this part, we show that the *finite local $k-$rational* condition is not necessary so that in finite dimension, the algebras of differential operators and endomorphisms coincide.

Therefore, The algebras of differential operators on a $k-$algebra A which is a subalgebra of $End_k(A)$ is finite dimension. The proposition below indicates the converse is true.

Proposition 3.1. *Let A be a commutative and unitary $k-$algebra. If $\mathcal{D}(A)$ is finite dimension, then A is finite dimension.*

Proof. Let $n \in \mathbb{N} \setminus \{0\}$ and A be a commutative and unitary $k-$algebra. Suppose the dimension of $\mathcal{D}(A)$ is finite and equal to n. Then, $\mathcal{D}(A)$ admits a base. Let $(d_i)_{i\in\overline{1;n}}$ be a basis of $\mathcal{D}(A)$. For all $x \in A$, we have: $x = \varphi_x(1)$.

Yet $\varphi_x \in \mathcal{D}(A)$, so there exists $(\alpha_i)_{i\in\overline{1;n}} \in k^n$ such that: $x = \sum_{i=1}^{n} \alpha_i \; d_i(1)$. Thus, the following $(d_i(1))_{i\in\overline{1;n}}$ generates A.

It follows that the dimension of A is finite. □

In the following, A and B are *finite local $k-$rational* algebras of respective dimensions n and m such that: $A = vect\left((x^i)_{i\in\overline{0;n-1}}\right)$; $B = vect\left((y^j)_{j\in\overline{0;m-1}}\right)$, and

$$F = vect\left(\left(x^i \otimes y^j\right)_{\substack{i\in\overline{0,n-1}\\ j\in\overline{0,m-1}\\ (i,j)\neq(0,0)}}\right).$$

Lemma 3.1. *Let $(A,+,.)$ be a commutative algebra on a field k and $a_1, \cdots, a_p$ be non-zero elements of A. We have: for all $n \in \mathbb{N}^*$,*

$$(a_1 + \cdots + a_p)^n = \sum_{i_1+\cdots+i_p=n} \left(\beta_{i_1} + \cdots + i_n a_1^{i_1} \cdots a_p^{i_p}\right),$$

for all $i_1 + \cdots + i_p = n$, $\beta_{i_1} + \cdots + i_n \in k$.

Proof. By induction on $p \geq 1$. Let $n \in \mathbb{N}^*$:

- $p = 1$, it is obvious.
- Now, assume that the result is true for an integer $p \geq 1$ and prove it for $p+1$.

- We have

$$\begin{aligned}((a_1+\cdots+a_{p+1})^n &= (a_1+\cdots+a_p)+a_{p+1})^n \\ &= \sum_{q=0}^{n} C_n^q (a_1+\cdots a_p)^q (a_{p+1})^{n-q} \\ &= \sum_{q=0}^{n} C_n^q \left(\sum_{i_1+\cdots+i_n=q} (\beta_{i_1}+\cdots+i_n a_1^{i_1}\cdots a_p^{i_p}) \right) (a_{p+1})^{n-q} \\ &= \sum_{i_1+\cdots+i_p+i_{p+1}=n} \beta_{i_1}+\cdots+i_n a_1^{i_1}\cdots a_p^{i_p} a_{p+1}^{i_{p+1}}\end{aligned}$$

And we have the lemma.

□

Lemma 3.2. *F is a two-sided ideal of $A\otimes B$.*

Proof. According to Proposition 2.2, $A\otimes B$ is an algebra of finite dimension equal to mn and

$$A\otimes B = vect\left(\left(x^i\otimes y^j\right)_{\substack{i\in\overline{0,n-1}\\ j\in\overline{0,m-1}}} \right).$$

We have

(1) $0_{A\otimes B}\in F$ and $F\subset A\otimes B$
(2) Let

$$z = \sum_{\substack{i\in\overline{0,n-1}\\ j\in\overline{0,m-1}\\ (i,j)\neq(0,0)}} \alpha_{ij}(x^i\otimes y^j)$$

and

$$z' = \sum_{\substack{r\in\overline{0,n-1}\\ t\in\overline{0,m-1}\\ (r,t)\neq(0,0)}} \beta_{rt}(x^r\otimes y^t)$$

be elements of F. We have:

$$z - z' = \sum_{\substack{i\in\overline{0,n-1}\\ j\in\overline{0,m-1}\\ (i,j)\neq(0,0)}} \alpha_{ij}(x^i \otimes y^j) \sum_{\substack{r\in\overline{0,n-1}\\ t\in\overline{0,m-1}\\ (r,t)\neq(0,0)}} \beta_{rt}(x^r \otimes y^t)$$

$$= \sum_{\substack{I\in\overline{0,n-1}\\ J\in\overline{0,m-1}\\ (I,J)\neq(0,0)}} (\alpha_{IJ} - \beta_{IJ})\,(x^I \otimes y^J)$$

Thus, $z - z' \in F$.

(3) Let

$$z = \sum_{\substack{i\in\overline{0,n-1}\\ j\in\overline{0,m-1}}} \alpha_{ij}(x^i \otimes y^j) \in A \otimes B$$

and

$$z' = \sum_{\substack{r\in\overline{0,n-1}\\ t\in\overline{0,m-1}\\ (r,t)\neq(0,0)}} \beta_{rt}(x^r \otimes y^t) \in F$$

We have:

$$zz' = \alpha_{00}z' + \left(\sum_{\substack{i\in\overline{0,n-1}\\ j\in\overline{0,m-1}\\ (i,j)\neq(0,0)}} (\alpha_{ij}x^i) \otimes y^j\right)\left(\sum_{\substack{r\in\overline{0,n-1}\\ t\in\overline{0,m-1}\\ (r,t)\neq(0,0)}} (\beta_{rt}x^r) \otimes y^t\right)$$

yet $\alpha_{00}z' \in F$ and

$$\left(\sum_{\substack{i\in\overline{0,n-1}\\ j\in\overline{0,m-1}\\ (i,j)\neq(0,0)}} (\alpha_{ij}x^i) \otimes y^j\right)\left(\sum_{\substack{r\in\overline{0,n-1}\\ t\in\overline{0,m-1}\\ (r,t)\neq(0,0)}} (\beta_{rt}x^r) \otimes y^t\right)$$

$$= \sum_{\substack{i\in\overline{0,n-1}\\ j\in\overline{0,m-1}\\ (i,j)\neq(0,0)}} \left[((\alpha_{ij}x^i) \otimes y^j)\left(\sum_{\substack{r\in\overline{0,n-1}\\ t\in\overline{0,m-1}\\ (r,t)\neq(0,0)}} (\beta_{rt}x^r) \otimes y^t\right)\right]$$

$$= \sum_{\substack{i\in\overline{0,n-1}\\ j\in\overline{0,m-1}\\ (i,j)\neq(0,0)}} \left[\sum_{\substack{r\in\overline{0,n-1}\\ t\in\overline{0,m-1}\\ (r,t)\neq(0,0)}} ((\alpha_{ij}x^i)\otimes y^j)((\beta_{rt}x^r)\otimes y^t) \right]$$

$$= \sum_{\substack{i\in\overline{0,n-1}\\ j\in\overline{0,m-1}\\ (i,j)\neq(0,0)}} \left[\sum_{\substack{r\in\overline{0,n-1}\\ t\in\overline{0,m-1}\\ (r,t)\neq(0,0)}} (\alpha_{ij}\beta_{rt})x^{i+r}\otimes y^{j+t} \right] \in F$$

because for every fixed couple (i,j) and every couple $(r,t) \in \overline{0,n-1} \times \overline{0,m-1}$, we have: $(i+r, j+t) \neq (0,0)$.

According to 1), 2) and 3), F is left-sided ideal of $A\otimes B$. Since $A\otimes B$ is a commutative algebra, then F is a two-sided ideal.

□

Theorem 3.1. *The tensor product of two finite local $k-$algebras is not necessarily a finite local $k-$algebra.*

Proof. Let A and B be finite a local $k-$algebras of respective dimensions n and m such that $(n,m) \neq (0,0)$, $A = vect\left((x^i)_{i\in\overline{0,n-1}}\right)$ and $B = vect\left((y^j)_{j\in\overline{0,m-1}}\right)$, where $x^n = 0$ and $y^m = 0$.

According to Proposition 2.3, $A \otimes B$ is an algebra of finite dimension equal to nm and $A\otimes B = \langle x^i \otimes y^j\rangle_{\substack{i\in\overline{0,n-1}\\ j\in\overline{0,m-1}}}$

- Prove that for all $z \in A\otimes B, z^{nm-1} = 0$.

Case 1: $z \in F = vect\left((x^i\otimes y^j)_{\substack{i\in\overline{0,n-1}\\ j\in\overline{0,m-1}\\ (i,j)\neq(0,0)}}\right)$

In this case,

$$z = \sum_{\substack{i\in\overline{0,n-1}\\ j\in\overline{0,m-1}\\ (i,j)\neq(0,0)}} \alpha_{ij}(x^i\otimes y^j) = \sum_{\substack{i\in\overline{0,n-1}\\ j\in\overline{0,m-1}\\ (i,j)\neq(0,0)}} (\alpha_{ij}x^i)\otimes y^j.$$

By setting $e_{ij} = (\alpha_{ij} x^i \otimes y^j$, we obtain

$$z = \sum_{\substack{i \in \overline{0,n-1} \\ j \in \overline{0,m-1} \\ (i,j) \neq (0,0)}} e_{ij}.$$

By Lemma 3.1, we have:

$$\begin{aligned} z^{nm-1} &= \sum_{\sum lij = mn-1} \left(\beta_{ij} \prod_{\substack{i \in \overline{0,n-1} \\ j \in \overline{0,m-1} \\ (i,j) \neq (0,0)}} (e_{ij})^{l_{ij}} \right) \\ &= \sum_{\sum lij = mn-1} \left(\beta_{ij} \prod_{\substack{i \in \overline{0,n-1} \\ j \in \overline{0,m-1} \\ (i,j) \neq (0,0)}} ((\alpha_{ij} x^i) \otimes y^j)^{l_{ij}} \right) \\ &= \sum_{\sum lij = mn-1} \left(\beta_{ij} \prod_{\substack{i \in \overline{0,n-1} \\ j \in \overline{0,m-1} \\ (i,j) \neq (0,0)}} (\alpha_{ij})^{l_{ij}} (x^{il_{ij}} \otimes y^{jl_{ij}}) \right) \\ &= \sum_{\sum lij = mn-1} \left(\beta_{ij} \prod_{\substack{i \in \overline{0,n-1} \\ j \in \overline{0,m-1} \\ (i,j) \neq (0,0)}} (\alpha_{ij})^{l_{ij}} \prod_{\substack{i \in \overline{0,n-1} \\ j \in \overline{0,m-1} \\ (i,j) \neq (0,0)}} (x^{il_{ij}} \otimes y^{jl_{ij}}) \right). \end{aligned}$$

By setting $\lambda_{ij} = \beta_{ij} \prod_{\substack{i \in \overline{0,n-1} \\ j \in \overline{0,m-1} \\ (i,j) \neq (0,0)}} (\alpha_{ij})^{l_{ij}}$, we obtain:

$$z = \sum_{\substack{\sum lij = mn-1 \\ i \in \overline{0,n-1} \\ j \in \overline{0,m-1} \\ (i,j) \neq (0,0)}} \lambda_{ij} \left(x^{\sum il_{ij}} \otimes y^{\sum jl_{ij}} \right).$$

Since $Sup\{\sum il_{ij}, \sum jl_{ij}\} \geq mn-1$ and $mn-1 \geq Sup\{m,n\}$, then $x^{\sum il_{ij}} \otimes y^{\sum jl_{ij}} = 0, for\ all (i,j) \in \overline{0,n-1} \times \overline{0,m-1}$ where $(i,j) \neq (0,0)$.

It follows that $z^{mn-1} = 0$. Thus, for all $z \in F$, z is nilpotent with nilpotency index strictly less than nm.

Case 2: $z \notin F$, that's to say

$$z = \sum_{\substack{i \in \overline{0,n-1} \\ j \in \overline{0,m-1}}} \alpha_{ij}(x^i \otimes y^j) = \alpha_{00} + \sum_{\substack{i \in \overline{0,n-1} \\ j \in \overline{0,m-1} \\ (i,j) \neq (0,0)}} \alpha_{ij}(x^i \otimes y^j)$$

with $\alpha_{00} \in k \setminus \{0\}$. By setting

$$z' = \sum_{\substack{i \in \overline{0,n-1} \\ j \in \overline{0,m-1} \\ (i,j) \neq (0,0)}} \alpha_{ij}(x^i \otimes y^j),$$

we obtain $z' \in F$ and $z = \alpha_{00} + z'$.

- Show that for all $t \in \mathbb{N}, z^t \neq 0$:
We have

$$z^t = (\alpha_{00} + z'^t = \alpha_{00}^t + \sum_{r=1}^{t} C_t^r \alpha_{00}^{t-r}(z'^r.$$

Since F is an ideal of $A \otimes_k B$ (by Lemma 3.2), then

$$\sum_{r=1}^{t} C_t^r \alpha_{00}^{t-r}(z'^r \in F,$$

which means that $\sum_{r=1}^{t} C_t^r \alpha_{00}^{t-r}(z'^r \notin k \setminus \{0\}$.

Therefore $z' = \alpha_{00}^t + \sum_{r=1}^{t} C_t^r \alpha_{00}^{t-r}(z'^r$ is different from $0_{A \otimes_k B}$. It follows that: for all $t \in \mathbb{N}, z^t \neq 0$. So, any elements $z \notin F$ is nilpotent. According to these cases 1 and 2, $A \otimes B$ is not a finite local $k-$rational algebra.

□

Theorem 3.2. *k is a field of characteristic zero. Let $A_1, \cdots, A_n$ be $k-$finite local$k-$rational algebras.*

$$\mathcal{D}(A \otimes \cdots \otimes A_n) = End_k(A \otimes \cdots \otimes A_n).$$

Proof. Let $m_1, \cdots, m_n$ be the respective dimensions of *finite local* $k-$*rational* algebras $A_1, \cdots, A_n$. We deduce from Theorem 2.1 that:

$$\mathcal{D}(A_1 \otimes \cdots \otimes A_n) \simeq \mathcal{D}(A_1) \otimes \cdots \otimes \mathcal{D}(A_n) \tag{3.1}$$

Since for all $i \in \overline{1,n}$, A_i is a finite local $k-$rational algebra, then, according to Theorem 2.2, (3.1) becomes:

$$\mathcal{D}(A_1 \otimes \cdots \otimes A_n) \simeq End_k(A_1) \otimes \cdots \otimes End_k(A_n). \tag{3.2}$$

We have:

$$dim_k[End_k(A_1 \otimes \cdots \otimes A_n) = dim_k[End_k(A_1) \otimes \cdots \otimes End_k(A_n)] = \prod_{j=1}^{n} m_j^2$$

So, we deduce from (3.2) that:

$$dim_k[\mathcal{D}(A_1 \otimes \cdots \otimes A_n)] = dim_k[End_k(A_1 \otimes \cdots \otimes A_n)$$

As $\mathcal{D}(A_1 \otimes \cdots \otimes A_n)$ is an subalgebra of $End_k(A_1 \otimes \cdots \otimes A_n)$, then:

$$\mathcal{D}(A_1 \otimes \cdots \otimes A_n) = End_k(A_1 \otimes \cdots \otimes A_n.)$$

□

Corollary 3.1. *The finite local* $k-$*rational condition is not necessary for the algebra of differential operators on a finite-dimensional algebra to be equal to its algebra of endomorphisms.*

Proof. Let A and B be finite local $k-$rational algebras such that $(dim_k A, dim_k B) \neq (1,1)$. According to Theorem 3.1,$A \otimes B$ is not a finite local $k-$rational algebra.

However, we have $\mathcal{D}(A \otimes B) = End_k(A \otimes B)$, according to Theorem 3.2. □

Application

Construction of algebras A **such that:** $\mathcal{D}(A) = End_k(A)$

These results permit to construct algebras which are not necessarily *finite local* $k-$*rationals* including the algebras of differential operators and endomorphisms coincide.

Indeed, if a *finite local $k-$rational*, then for any $k-$algebra

$$A_n = \bigotimes_{i=1}^{n} A_i$$

where $A_i = A, \forall i \in \overline{2, n}$, we have $\mathcal{D}(A_n) = End_k(A_n)$.

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