ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **13** (2024), no.3, 399–431
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.13.3.9

ADVANCED DEVELOPMENT OF CONVERGENCE RATE ANALYSIS BY IMPROVED LANDWEBER METHOD BASED ON LOGARITHMIC CONDITION FOR NONLINEAR INVERSE PROBLEMS

Ly Van An

ABSTRACT. In this article, we use the Landweber method to analyze the degree of convergence based on the conditions of the logarithmic function for nonlinear misalignment problems. The regularization parameters are chosen according to the difference principle. That is the main result in this paper.

1. INTRODUCTION

The Landweber iteration or Landweber algorithm is an algorithm to solve ill-posed linear inverse problems, and it has been extended to solve non-linear problems that involve constraints. The method was first proposed in the 1950 by Louis Landweber, [1] and it can be now viewed as a special case of many other more general methods [2].

To formulate the problem, we consider the inverse potential energy problem as follows:

$$\begin{cases} \Delta u = H_{\Omega_1} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma. \end{cases} \tag{1.1}$$

[1] *corresponding author*
2020 *Mathematics Subject Classification.* 65J20, 65R30.
Key words and phrases. Landweber iteration method, nonlinear operator; regularization; logarithmic functional condition.
Submitted: 13.08.2024; *Accepted:* 28.08.2024; *Published:* 11.09.2024.

I determine the form of an unknown domain Ω_1 by measuring the Neumann boundary values of u on $\partial\Omega$.

In there H_{Ω_1} is the characteristic function of the domain $\Omega_1 \subset \Omega = \{x \in \mathbb{R}^n : |x| < R\}$. To study the next problems, we consider the following nonlinear operator equation:

$$F(x) := y. \tag{1.2}$$

In there:

(1) Here we assume that $\mathbf{X}, \mathbf{Y}$ are Hilbert spaces and the unknown x includes the information of the domain $\Omega_1 \subset \Omega$,
(2) y is the derivative of u on the boundary, $\frac{\partial u}{\partial v}\Big|_{\Gamma}$,
(3) v is the outer normal vector on Γ,
(4) $F : G(F) \subseteq \mathbf{X} \to \mathbf{Y}$ is a nonlinear operator on domain $G(F) \subset \mathbf{X}$.

Note: For convenience in this article, the indices of inner products $< \cdot, \cdot >$ and norms $\| \cdot \|$ are neglected but they can always be identified from the context in which they appear. Due to the nonlinearity of equation (1.2), we assume all over that equation (1.2) has a solution x^+ which needs not to be unique. We have the disturbed data y^δ with

$$\left\| y^\delta - y \right\| \leq \delta, \tag{1.3}$$

where $\delta > 0$ is a noise level. If one solves equation (1.2) by traditional numerical method, high oscillating solutions may occur. Thus, one needs a regularization to minimize the approximation and data error.

Recently, we have been researching improved regularization methods, (see [3], [4])

$$x^\delta(t) = F'(x^\delta(t))\Big[y^\delta - F(x^\delta(t))\Big] - \Big(x^\delta(t) - \hat{x}\Big), 0 < t \leq T. \tag{1.4}$$

where the term $\hat{x} - x^\delta(t), x^\delta(0) = \hat{x}$.

A discrete version analogue to equation (1.4) is successfully developed (see [5]), where the whole family of Runge-Kutta methods is applied and one obtaines an optimal convergence rate under Hölder-type sourcewise condition if the Fréchet derivative is properly scaled and locally Lipschitz continuous.

It is well known that, for many applications such as the inverse potential problem and the inverse scattering problem (see [6]), the Holder type source condition in general is not fulfilled even if a solution is very smooth. It is applicable only for mildly ill-posed problems (see [7], [8]). Therefore, the convergence rate analysis of an explicit Euler method presented by

$$x^{\delta}_{n_i+1} = x^{\delta}_{n_i} + F'(x^{\delta}_{n_i})^*\Big[y^{\delta} - F(x^{\delta}_{n_i})\Big] - \gamma_{n_i}\big(x^{\delta} - x_0\big), \forall i = 1, 2, \dots, k. \tag{1.5}$$

The Fr$\acute{e}$chet derivative is properly scaled and locally Lipschitz continuous, $\big\|F'(x^+)\big\| \leq 1$ and $\upsilon_i = \gamma_{n_i}^{-1}, i = 1, 2, \dots, k.$

Next, we consider the equation

$$f = f_{\beta}, \qquad f_{\beta}(\lambda) := \begin{cases} \Big(ln\frac{e}{\lambda}\Big)^{-k\beta} & \text{for} \quad 0 < \lambda \leq 1, \\ 0 & \text{for} \quad \lambda = e, \end{cases} \tag{1.6}$$

with $2 \geq \beta \geq 1, k \in \mathbb{N}^*$ and the usual sourcewise representation.

$$x^+ - x_0 = f\Big(F'(x^+)^*F'(x^+)\Big)w, \quad w \in \mathbf{X}. \tag{1.7}$$

The method in equation (1.6) is also known as the modified Landweber method [9] which has the rate $O(\sqrt{\delta})$ under the Holder-type source condition and general discrepancy principle. As usual, the Fr$\acute{e}$chet derivative of F needs to be scaled. Furthermore, we assume a nonlinearity condition of F in a ball $B_{\rho}(x_0) = \big\{x \in \mathbf{X} : \big\|x - x_0\big\| \leq \rho\big\}, \rho > 0$, which is given in Assumption 1. It is well known that, without the additional assumption on the nonlinear operator, the convergence rate cannot be provided. The following assumption has been used in many works (see [10]), i.e., there exists a bounded linear operator $R : \mathbf{Y} \to \mathbf{Y}$ and $G : \mathbf{X} \to \mathbf{Y}$ such that

$$F'(\hat{x}) = R\big(\hat{x}, x\big)F'(x) + G\big(\hat{x}, x\big), \tag{1.8}$$

$$\Big\|I - R\big(\hat{x}, x\big)\Big\| \leq C_R, \tag{1.9}$$

$$\Big\|G\big(\hat{x}, x\big)\Big\| \leq C_G\Big\|F'(x^+)(\hat{x} - x)\Big\|. \tag{1.10}$$

with nonnegative constants C_R and C_G. The essence of this article is that we analyze the convergence of the iteration based on equation (1.7) and equation (1.8) to reconstruct the solution domain Ω_1 for Math problem (1.1).

Layout of the article: In the Preliminaries section, some properties are reiterated:

- Describe the Landweber iterative method for the inverse problem
- Some basic assumptions as the basis for convergence analysis and basic theorems.

Section3 Basis for building convergence.

Section4 Convergence Analysis.

2. Preliminaries

2.1. **Describe the Landweber iterative method for the inverse problem.** Let $F = (F_0, \dots, F_{p-1})$ and $y = (y_0, \dots, y_{p-1})$ then the Landweber iteration for solving

$$F_j(x) = y_j, \quad j = 1, \dots, p-1, \tag{2.1}$$

reads as follows

$$\begin{aligned} x_{k+1} &= x_k^\delta - F_j'(x_k^\delta)^*(F(x_k^\delta) - y^\delta) \\ &= x_k^\delta - \sum_{j=1}^{p-1} F_j'(x_k^\delta)^*(F(x_k^\delta) - y^\delta), \quad k = 1, \dots . \end{aligned} \tag{2.2}$$

Let $B_r(x_0)$ be an open ball of radius r containing x_*.

I. The conditions $\mathbf{A_I}$

(1) F is Frèchet differentiable on $B_r(x_0)$

(2) $F'(x) \leq 1$ for $x \in B_r(x_0)$

(3) $\left\|F(x) - F(\hat{x}) - F'(x)(x - \hat{x})\right\| \leq \left\|F(x) - F(\hat{x})\right\|, \quad \eta < \frac{1}{2}, x, \hat{x} \in B_r(x_0)$

are strong enough to ensure at least local convergence to a solution of

$$F_j(x) = y_j, j = 1, \dots, p-1. \tag{2.3}$$

II. The conditions $\mathbf{A_{II}}$

If y^δ does not belong to the range of F, then the iterates x_k^δ of (2.2) cannot converge but still allow a stable approximation of x_* provided the iteration is stopped after $k_* = k_*^\delta$ steps acording to the generalized discrepancy principle

$$\left\|y^\delta - F(x_{k_*}^\delta)\right\| \leq \tau\delta \leq \left\|y^\delta - F(x_k^\delta)\right\|, 0 \leq k \leq k_0, \quad \text{for } \tau > 2\frac{1+\eta}{1-2\eta} > 2. \tag{2.4}$$

When speaking of convergence rates to a solution of (2.1) of an iterative method $x_{k+1} = U(x_k)$ for solving an illposed problem we understand:

(2.5) $$(a) \quad if \quad \delta = 0 \quad \text{the rate of} \quad \left\|x_* - x_k\right\| \to 0 \quad \text{as} \quad k \to \infty.$$

(2.6) $$(b) \quad if \quad \delta > 0 \quad \text{therate of} \quad \left\|x_* - x_{k_*(\delta)}\right\| \to 0 \quad \text{as} \quad \delta \to \infty$$

Under the general assumptions A_I the rate of convergence of $x_k \to x_*$ as $k \to \infty$ (with precise data, i.e. $\delta = 0$) or $x^{\delta}_{k_*(\delta)} \to x_*$ as $\delta \to 0$ (with perturbed data) will, in general, be arbitrarily slow. This is known for linear ill-posed problems $Kx = y$ where the rate of convergence is almost completely determined by the tuple $(v; \|f\|)$ in the source-wise representation

(2.7) $$x_* - x_0 = (K^*K)^v f, \quad v > 0, f \in X$$

cf. Example 3.1 and Theorem 7.3 in (see [11]). The same parameters also determine the rate of convergence of Tikhonov regularization (see [12]); the corresponding numbers

(2.8) $$x_* - x_0 = (F'(x_*)^* F'(x_*))^v f, \quad v > 0, f \in X$$

play the same role in Tikhonov regularization for nonlinear problems (see [13]) and in many iterative regularization methods (see [14]). In contrast to Tikhonov regularization, assumption (2.8) (with $\|$f$\|$ being sufficiently small) is not enough to obtain convergence rates for the Landweber iteration; we need further properties of F: we require

(2.9) $$F(x) = R_x F'(x_*), x \in B_r(x_0),$$

where $\{R_x : x \in B_r(x_0)\}$ is a family of bounded linear operators $R_x : Y \to Y$ with

(2.10) $$\|R_x - I\| \leq C\|x - x_*\| x \in B_r(x_0),$$

and C is a positive constant. Note that in the linear case $R_x \equiv I$.

Therefore, (2.9) may be interpreted as a further restriction of the "non-linearity" of F. In particular, (2.9) implies that

$$\mathcal{N}(F'(x_*)) \subset \mathcal{N}(F'(x)), \quad x \in B_r(x_0).$$

It is not diffcult to see that (2.9), (2.10) Deduce condition 3 of A_I pseudosecretion with $\hat{x} = x_*$ for r being sufficiently small.

Theorem 2.1. *Assume that problem* (2.1)

$$F_j(x) = y_j, j = 1, \dots, p-1, \tag{2.11}$$

has a solution in $B_r(x_0)$*, that* y^δ *satisfies*

$$\left\| y_j^\delta - y_j \right\| \le \delta j \in \{0, 1, \dots, p-1\}, \tag{2.12}$$

and that F *fulfils*

(1) $F'(x) \le 1$ *for* $x \in B_r(x_0)$*,*
(2) $\|F(x) - F(\hat{x}) - F'(x)(x - \hat{x}) \le \eta \|F(x) - F(\hat{x})\|, \eta < \frac{1}{2}, x, \hat{x} \in B_r(x_0)$*,*
(3) $F(x) = R_x F'(x_*)$*,* $x \in B_r(x_0)$*, where* $\{R_x : x \in B_r(x_0)\}$ *is a family of bounded linear operators* $R_x : Y \to Y$ *with* $\|R_x - I\| \le C\|x - x_*\|, x \in B_r(x_0)$*.*

If $x_* - x_0$ *satisfies*

$$x_* - x_0 = (F'(x_*)^* F'(x_*))^v f, v > 0, f \in X,$$

with some $0 < v \le \frac{1}{2}$ *and* $\|f\|$ *being sufficiently small, then there exists a positive constant* c_**, depending on* v

$$\|x_* - x_k^\delta\| \le C_* \|f\| (k+1)^{-v} \tag{2.13}$$

and

$$\|y^\delta - F(x_k^\delta)\| \le 4.C_* \|f\| (k+1)^{-v-1/2} \tag{2.14}$$

for all $0 \le k \le k_*$*. For* $\delta = 0$ (2.1) *and* (2.13) *holds for all* $k \ge 0$*. Furthermore, for* $\delta > 0$

$$k_* \le C_1 (\|f\|/\delta)^{2/(2v+1)} \tag{2.15}$$

and

$$\|x_* - x_{k_*}^\delta\| \le C_2 \|f\|^{1/(2v+1)} \delta^{2v/(2v+1)} \tag{2.16}$$

for some constsnts $C_1, C_2 > 0$*, depending on* v *only.*

2.2. **Basis for convergence analysis.**

Assumption1 : Suppose that $\beta > 0$ and $m \in \mathbb{N}_0, k \in \mathbb{N}^*$. The real-valued function

$$\widetilde{f}(\gamma) = (1-\gamma)^m (\ln \frac{e}{\gamma})^{-k\beta}$$

defined on $[0,1]$ satisfies $\widetilde{f}(\gamma) \leq C(\ln(m+e))^{-k\beta}$ with C independent of m.

Moreover, for each $r \in \mathbb{R}$. The real-valued function

$$\widetilde{g}(\gamma) = (1-\gamma)^m \gamma^{\frac{1}{2k}} (\ln \frac{e}{\gamma})^{-r}$$

defined on $[0,1]$ satisfies $\widetilde{g}(\gamma) \leq C(m+1)^{-\frac{1}{2k}}(\ln(m+e))^{-r} \leq C(m+1)^{-\frac{1}{2k}}(\ln(m+1))^{-r}$ with C independent of m,

$$\widetilde{g}^2(m^{-s}) = \left(1 - \frac{1}{m^s}\right)^m \left(1 - s\ln\frac{1}{m}\right)^{-k\beta} \leq \big(\ln(m+e)\big)^{-k\beta}, \quad s \geq 1.$$

Assumption2 :

Suppose $\beta \geq 1, C > 0$ and $\delta > 0, k \in \mathbb{N}^*, k \geq 2$ be sufficiently small such that

$$1 \geq (-\ln(\delta C))^{-kp} \geq \delta.$$

Let

$$\int_0^1 exp\big(-((1-\ln(\lambda))^{-kp})^{\frac{-1}{k\beta}}\big)(1-\ln(\lambda))\big\|dE_\lambda w\big\|^k = C\delta. \tag{2.17}$$

Then

$$\int_0^1 (1-\ln(\lambda))^{-k\beta}\big\|E_\lambda w\big\|^k \leq C(-\ln(\delta))^{-k\beta} \tag{2.18}$$

with a generic constant C.

Assumption3 :

Suppose $\beta \geq 1, k \in \mathbb{N}^*, k \geq 2$. Then, there exists a constant M, which is independent of m, such that

$$\sum_{j=0}^{k-1} \left(\frac{j+1}{k+1}\right)^{\frac{-1}{k}} \left(\frac{j+1}{k+1}\right)^{\frac{-1}{k}} \frac{1}{k+1} \left(\frac{\ln(k+2)^{k\beta}}{\ln(k-j+1)}\right) \leq M, \tag{2.19}$$

$$\big(\ln(k+2)\big)^{-k\beta} \sum_{j=0}^{k-1} \left(\frac{j+1}{k+1}\right)^{-1} \left(\frac{j+1}{k+1}\right)^{\frac{-1}{k}} \frac{1}{k+1} \left(\frac{\ln(k+2)}{\ln(k-j+1)}\right)^{k\beta} \leq M. \tag{2.20}$$

Moreover, there exists a constant M (independent of k) such that

$$\sum_{j=0}^{m-1} (j+1)^{-1/2k}\big(\ln(j+1)\big)^{\beta}\big(m-j+1\big)^{-1/2k}\big(\ln\big(m-j+1\big)\big)^{-k\beta} \leq M. \tag{2.21}$$

Assumption4 :

Suppose that x is a solution of

$$x\big(\ln x\big)^{k\beta} = \frac{k}{\delta^2}. \tag{2.22}$$

Then, $\hat{x}$ satisfies

$$\hat{x} = o\bigg(\frac{(-\ln\delta)^{-k\beta}}{\delta^2}\bigg). \tag{2.23}$$

Assumption5 :

There exist positive constants $C_L, C_R,$ and C_r and linear bounded operator R_x : $Y \to Y$ such that, for $x_{n_1}, x_{n_2}, \ldots, x_{n_k} \in B_\rho(x_0)$, the following condition holds

$$F'(x_{n_i}) = R_{x_{n_i}} F'(x^+), \tag{2.24}$$

$$\big\|R_{x_{n_i}} - I\big\| \leq C_L \big\|x_{n_i} - x^+\big\|, \tag{2.25}$$

$$\Big|\big\|R_{x_{n_i}}\big\| - \big\|I\big\|\Big| \geq C_R, \tag{2.26}$$

$$\big\|R_{x_{n_i}}\big\| b \leq C_r, \quad i = 1, 2, \ldots, k. \tag{2.27}$$

Here x^+ is the exact solution of equation $F(x_{n_i}) = y$, $i = 1, \ldots, k$.

Theorem 2.2. *Assuming that hypothesis Assumption 5 is satisfied, then we have the following*

$$\sum_{i=1}^{k} \big\|(1-\gamma_i)I - \widehat{R}_{x^\delta_{n_i}}\big\| \leq \frac{1}{k} P \sum_{i=1}^{k} \big\|e_{n_i}\big\|, \tag{2.28}$$

with $P_R > 0$ being a positive constant for $e_{n_i} = x^+ - x^\delta_{n_i}$, $i = 1 \to k$.

Theorem 2.3. *Suppose that the following conditions are satisfied*

$$F'(x_{n_i}) = R_{x_{n_i}} F'(x^+), \tag{2.29}$$

$$\big\|R_{x_{n_i}} - I\big\| \leq c_L \big\|x_{n_i} - x^+\big\|, \forall i = 1, \ldots, k. \tag{2.30}$$

Then

$$\big\|F(x^\delta_{n_i}) - F(x^+) - F'(x^+)(x^\delta_{n_i} - x^+)\big\| \leq \frac{1}{k} c_L \big\|e_{n_i}\big\| \big\|P e_{n_i}\big\|, \tag{2.31}$$

$\forall i = 1, \ldots, k$, $\forall x_{n_1}, x_{n_2}, \ldots, x_{n_k} \in B_\rho(x_0)$, for $P = F'(x^+), e_{n_i} = x^+ - x^\delta_{n_i}, \forall i = 1, \ldots, k$.

3. Basis for building convergence

In this section we give two main lemmas for convergence analysis

Lemma 3.1. *Let B be a linear operator with $\|B\| \leq \ln e$. For $n_1, n_2, \ldots, n_k \in \mathbb{N}^* \setminus \{1\}$, $e_0 = f(\lambda)w$ for f given by*

$$f = f_\beta, \qquad f_\beta(\lambda) := \begin{cases} \left(\ln \frac{e}{\lambda}\right)^{-k\beta} & \text{for} \quad 0 < \lambda \leq 1, \\ 0 & \text{for} \quad \lambda = e, \end{cases} \tag{3.1}$$

and $\beta > 0, k \in \mathbb{N}^*$, there exist positive constants q_1 and q_2 such that

$$\Big\| \prod_{j_k=0}^{n_k-1} \prod_{j_{k-1}=0}^{n_{k-1}-1} \cdots \prod_{j_1=0}^{n_1-1} (1-\gamma_{j_1}) \cdots (1-\gamma_{j_{k-1}})$$

$$\cdot (1-\gamma_{j_k})(I - B^*B)^{n_1+n_2+\ldots+n_k} e_0 \Big\| \leq q_1 (\prod_{j=1}^{k} \ln(n_j + e))^{-\beta} \|w\| \tag{3.2}$$

and

$$\Big\| B \prod_{j_k=0}^{n_k-1} \prod_{j_{k-1}=0}^{n_{k-1}-1} \cdots \prod_{j_1=0}^{n_1-1} (1-\gamma_{j_1}) \cdots (1-\gamma_{j_{k-1}})$$

$$\cdot (1-\gamma_{j_k})(I - B^*B)^{n_1+n_2+\ldots+n_k} e_0 \Big\| \leq q_2 \big(\prod_{i=1}^{k} (n_i + e) \big)^{\frac{-1}{2k}} \big(\prod_{i=1}^{k} \ln(n_i + e) \big)^{-\beta} \|w\|, \tag{3.3}$$

$0 < \gamma_{j_i} \leq 1$, $J_i = 0, 1, 2, \ldots, n_i - 1$,*and* $i = 1, 2, \ldots, k$

Proof. By assumption 1 and equations (3.1), $q_1, q_2 > 0$, we have

$$\Big\| \prod_{j_k=0}^{n_k-1} \prod_{j_{k-1}=0}^{n_{k-1}-1} \cdots \prod_{j_1=0}^{n_1-1} (1-\gamma_{j_1}) \cdots (1-\gamma_{j_{k-1}})(1-\gamma_{j_k})$$

$$\cdot (I - B^*B)^{n_1+n_2+\ldots+n_k} e_0 \Big\|$$

$$\leq \prod_{j_k=0}^{n_k-1} \prod_{j_{k-1}=0}^{n_{k-1}-1} \cdots \prod_{j_1=0}^{n_1-1} (1-\gamma_{j_1}) \cdots (1-\gamma_{j_{k-1}})(1-\gamma_{j_k}) \tag{3.4}$$

$$\cdot \big\| (I - B^*B)^{n_1+n_2+\ldots+n_k} f(BB^*) \big\| \|w\|$$

$$\leq \sup_{\lambda(0,1]} \left|(1-\lambda)^{n_1+n_2+\ldots+n_k}(1-\ln\lambda)^{-k\beta}\right|\|w\|$$
$$\leq q_1\big(\prod_{i=1}^{k}\ln(n_i+e)\big)^{-k\beta}\|w\|$$

and

$$\Big\|B\prod_{j_k=0}^{n_k-1}\prod_{j_{k-1}=0}^{n_{k-1}-1}\cdots\prod_{j_1=0}^{n_1-1}(1-\gamma_{j_1})\cdots(1-\gamma_{j_{k-1}})(1-\gamma_{j_k})$$
$$\cdot(I-B^*B)^{n_1+n_2+\ldots+n_k}e_0\Big\|$$
$$\leq \prod_{j_k=0}^{n_k-1}\prod_{j_{k-1}=0}^{n_{k-1}-1}\cdots\prod_{j_1=0}^{n_1-1}(1-\gamma_{j_1})\cdots(1-\gamma_{j_{k-1}})$$
$$\cdot(1-\gamma_{j_k})\|(I-B^*B)^{n_1+n_2+\ldots+n_k}(BB^*)^{\frac{1}{2}}f(BB^*)\|\|w\| \tag{3.5}$$
$$\leq \sup_{\lambda\in(0,1]}\left|(1-\lambda)^{n_1+n_2+\ldots+n_k}\lambda^{\frac{1}{2}}(1-\ln\lambda)^{-k\beta}\right|\|w\|$$
$$\leq q_2\big(\prod_{i=1}^{k}(n_i+1)\big)^{-1/2k}\big(\prod_{i=1}^{k}\ln(n_i+e)\big)^{-k\beta}\|w\|.$$

□

Lemma 3.2. *Let B be a linear operator with $\|B\| \leq \ln e$. For $n_1, n_2, \ldots, n_k \in \mathbb{N}^* \setminus \{1\}$, $e_0 = f(\lambda)w$ for f given by*

$$f = f_\beta, f_\beta(\lambda) := \begin{cases} \big(\ln\frac{e}{\lambda}\big)^{-k\beta} & \textit{for} 0<\lambda\leq 1 \\ 0 & \textit{for} \lambda = e \end{cases} \tag{3.6}$$

and $\beta = 2\phi, \phi \in [1/2, 1], k \in \mathbb{N}^*$, there exist positive constants q_3 and q_4 such that

$$\Big\|\big(\sum_{j_1=0}^{n_1-1}\gamma_{n_1-j_1-1}(I-B^*B)^{j_1}\prod_{i=1}^{j_1}(1-\gamma_{n_1-i})$$
$$+\sum_{j_2=0}^{n_2-1}\gamma_{n_2-j_2-1}(I-B^*B)^{j_2}\prod_{i=1}^{j_2}(1-\gamma_{n_2-i}) \tag{3.7}$$

$$+\cdots+\sum_{j_k=0}^{n_k-1}\gamma_{n_k-j_k-1}(I-B^*B)^{j_k}\prod_{i=1}^{j_k}(1-\gamma_{n_k-i})\big)e_0\Big\|$$
$$\leq q_3\big(\ln(\sum_{j=1}^{k}n_j+e)\big)^{-\beta}\|w\|$$

$$\Big\|B\big(\sum_{j_1=0}^{n_1-1}\gamma_{n_1-j_1-1}(I-B^*B)^{j_1}\prod_{i=1}^{j_1}(1-\gamma_{n_1-i})$$
$$+\sum_{j_2=0}^{n_2-1}\gamma_{n_2-j_2-1}(I-B^*B)^{j_2}\prod_{i=1}^{j_2}(1-\gamma_{n_2-i}) \tag{3.8}$$
$$+\cdots+\sum_{j_k=0}^{n_k-1}\gamma_{n_k-j_k-1}(I-B^*B)^{j_k}\prod_{i=1}^{j_k}(1-\gamma_{n_k-i})\big)e_0\Big\|$$
$$\leq q_4(n_1+n_2+\cdots+n_k+1)^{-1/2k}(\ln(\sum_{j=1}^{k}n_j+e))^{-\beta}\|w\|,$$

$$\gamma_i=\frac{1}{2k}(i+t_0)^{-\phi},t_0\in[2,+\infty),i=0,1,\ldots,n_j,j=1,\ldots,k.$$

Proof. First we prove (3.7). Without loss of generality we assume that $n_1=n_2=\cdots=n_k=n_h,(n_h<\sum_{j=1}^{k}n_j)$. So from (3.7) we get

$$\Big\|\sum_{j=0}^{n_h-1}\gamma_{n_h-j_h-1}(I-B^*B)^{j}\prod_{i=1}^{j}(1-\gamma_{n_h-i})\Big\|\leq\frac{1}{k}(\ln(n_h\cdot k+e))^{-\beta})\|w\| \tag{3.9}$$

From the problem hypothesis we have $\beta=2\phi$ deduce that $\frac{\beta}{\phi}=2\leq\frac{\ln n_h}{\ln(\ln(n_h-1+e))},\forall n_h\geq 2$. So, $\frac{\beta}{\phi}\ln(\ln(n_h-1+e))\leq\ln 2\leq\ln n_h\leq\ln t_0\leq\ln(t_0+\sum_{j=1}^{k}n_j-1)$. We have

$$\ln(\ln(n_h-1+e))^{\frac{\beta}{\phi}}\leq\ln(t_0+\sum_{j=1}^{k}n_j-1)$$
$$\Leftrightarrow(\ln(n_h-1+e))^{\frac{\beta}{\phi}}\leq(t_0+\sum_{j=1}^{k}n_j-1)$$

$$\Leftrightarrow (\ln(n_h - 1 + e))^{-\beta} \geq (t_0 + \sum_{j=1}^{k} n_j - 1)^{-\phi}. \tag{3.10}$$

By (3.9) we have

$$\begin{aligned}\left\|\gamma_{n_h-1}(I - B^*B)^0 e_0\right\| &\leq \frac{1}{n_h}(t_0 + \sum_{j=1}^{k} n_j - 1)^{-\phi} \sup_{\lambda\in(0,1]} \left|(1-\lambda)^{-\beta}\right| \left\|w\right\| \\ &\leq \frac{1}{n_h}(t_0 + \sum_{j=1}^{k} n_j - 1)^{-\beta}. \end{aligned} \tag{3.11}$$

Note that (3.10) is still true for k=1. Now we assume that (3.9). Next we consider the following equality

$$\begin{aligned}
&k\Big\| \sum_{j=0}^{n_h} \gamma_{k\cdot n_h - j_h - 1}(I - B^*B)^j \prod_{i=1}^{j}(1 - \gamma_{k\cdot n_h - i})e_0\Big\| \\
=&k\Big\| \sum_{j_h=0}^{n_h-1} \gamma_{k\cdot n_h - j_h - 1}(I - B^*B)^j \prod_{i=1}^{j}(1 - \gamma_{k\cdot n_h - i})e_0 \\
&+\gamma_{k\cdot n_h - j - 1}(I - B^*B)^{n_h} \prod_{i=1}^{n_h}(1 - \gamma_{k\cdot n_h - i})e_0\Big\| \\
\leq&\frac{1}{k}\Big\| \sum_{j_h=0}^{n_h-1} \gamma_{k\cdot n_h - j_h - 1}(I - B^*B)^{j_h} \prod_{i=1}^{j_h}(1 - \gamma_{k\cdot n_h - i})e_0\Big\| \\
&+\frac{1}{k}\Big\|\gamma_{k\cdot n_h - n_h - 1}(I - B^*B)^{j_h} \prod_{i=1}^{j_h}(1 - \gamma_{k\cdot n_h - i})e_0\Big\| \\
\leq&\frac{q_3}{k}\big((\ln(n_h + e))^{-\beta}\big)\big\|w\big\| + \frac{q_3}{k\cdot n_h}(t_0 + k\cdot n_h - n_h - 1)^{-\phi} \qquad (3.12) \\
&\cdot\big\|(I - B^*B)^{n_h} f(B^*B)w\big\| \\
\leq&\frac{q_3}{k}\big((\ln(n_h + e))^{-\beta}\big)\big\|w\big\| + \frac{q_3}{k\cdot n_h}(t_0 + k\cdot n_h - n_h - 1)^{-\phi} \\
&\cdot\big((\ln(n_h + e))^{-\beta}\big)\big\|w\big\|
\end{aligned}$$

From the above proof we can deduce that

$$\frac{q_3}{k}(\ln(k\cdot n_h - n_h - 1 + t_0))^{-\phi}(\ln(n_h+e))^{-\beta} \le \ln(n_h+1+e))^{-\beta}. \tag{3.13}$$

So,

$$\begin{aligned} &k\Big\|\sum_{j=0}^{n_h}\gamma_{k\cdot n_h - j_h - 1}(I-B^*B)^j\prod_{i=1}^{j}(1-\gamma_{k\cdot n_h - i})e_0\Big\| \\ \le & max\{\frac{q_3}{k},\frac{q_3}{k\cdot n_h}\}\ln(n_h+1+e))^{-\beta}\|w\|\left(\frac{\ln(n_h+e))^{-\beta}}{\ln(n_h+1+e))^{-\beta}}+1\right) \\ \le & \frac{q_3}{k}\ln(n_h+1+e))^{-\beta}\|w\|. \end{aligned} \tag{3.14}$$

□

4. Convergence Analysis

To investigate the convergence rate of the modified Landweber method under the logarithmic source condition, we choose the regularization parameter n according to the generalized discrepancy principle, i.e., the iteration is stopped after $N = N\big(y^\delta,\delta)\big)$ steps with

$$\big\|\big(y^\delta - F(x^\delta_{N_i})\big)\big\| \le \beta\delta < \big\|y^\delta - F(x^\delta_{n_i})\big\|, 0 \le n_i < N_i, i = 1,\dots,k. \tag{4.1}$$

Here $\beta \ge \frac{2-\eta}{1-\eta}$ positive number. In addition to the discrepancy principle, F satisfies the local property in the open ball $x \in B_\rho(x_0)$, of radius ρ around x_0,

$$\big\|F(x) - F(\hat{x}) - F'(x)(x-\hat{x})\big\| \le \eta\big\|F(x) - F(\hat{x})\big\|, \quad \eta < \frac{1}{2}, \tag{4.2}$$

with $x, \hat{x} \in B_\rho(x_0) \subset D(F)$. Utilizing the triangle inequality yields

$$\frac{1}{1+\eta}\big\|F'(x)(x-\hat{x})\big\| \le \big\|F(x) - F(\hat{x})\big\| \le \frac{1}{1-\eta}\big\|F(x)-F(\hat{x})\big\|, \eta < \frac{1}{2}, \tag{4.3}$$

to ensure at least local convergence to a solution x^+ of equation (3) in $B_{\frac{\rho}{2}}(x_0)$.

Theorem 4.1. *Assume that the problem in equation $F(x) = y$ has a solution $x^+ \in B_{\frac{\rho}{2}}(x_0)$, y^δ that satisfies the functional inequality*

$$\big\|y^\delta - y\big\| \le \delta, \tag{4.4}$$

and F satisfy the following functional equation and functional inequality

$$F'(x) = R_x F'(x^+) \quad and \quad \|R_x - I\| \leq c_L \|x - x^+\|. \tag{4.5}$$

Assume that the Fréchet derivative of F is narrow such that

$$\|F'(x)\| \leq 1, \forall x \in B_{\frac{\varrho}{2}}(x_0).$$

Furthermore, assume that the headspring condition in equations

$$f_\beta(h, g) := \begin{cases} \left(\ln \frac{e}{\lambda}\right)^{-k\beta} & for \quad 0 < \lambda \leq 1, \\ 0 & for \quad \lambda = e, \end{cases} \tag{4.6}$$

and

$$x^+ - x_0 = f\big(F'(x^+)^*\big)F(x^+)\big)w, x \in X, \tag{4.7}$$

is fulfilled and that the modified Landweber method is stopped according to equation

$$\frac{1}{1+\eta}\|F'(x)(x - \hat{x})\| \leq \|F(x) - F(\hat{x})\| \leq \frac{1}{1-\eta}\|F(x) - F(\hat{x})\|, \quad \eta < \frac{1}{2}, \tag{4.8}$$

If $\|w\|$ is sufficiently small, then there exists a constant E_2 depending only on β and $\|w\|$ with

$$\|e_n\| \leq E_2\big(\tag{4.9}$$

$$lnn\big)^{-k\beta} \tag{4.10}$$

and

$$\|y^\delta - F(x^\delta)\| \leq 4E_2 M\big(n+1\big)^{-\frac{1}{k}}\big(\ln n\big)^{-k\beta}. \tag{4.11}$$

Proof. First we consider $e_{n_i} = x^+ - x^\delta_{n_i}$ for the error of the $n_i th$ iteration $x^\delta_{n_i}$ of equation

$$\big(x^+ - x^\delta_{n_i+1}\big) = f\big(F'(x^+)^* F'(x^+)\big)w, \quad w \in X, \tag{4.12}$$

and $P = F'(x^+)$. So we represent the equation (4.12) in the following form

$$\begin{aligned} &\big(x^+ - x^\delta_{n_i+1}\big) \\ =&(1-\gamma_{n_i})\big(x^+ - x^\delta_{n_i+1}\big) + F'(x^\delta_{n_i})^*(y^\delta - F(x^\delta_{n_i}) - \gamma_{n_i}(x_0 - x^+). \end{aligned} \tag{4.13}$$

Since $e_{n_i} = x^+ - x^\delta_{n_i}$ and $P := F'(x^+)$, we present e_{n_i} as follows:

$$\begin{aligned}
&e_{n_i+1}\\
&= (1-\gamma_{n_i})e_{n_i} + F'(x^\delta_{n_i})^*(y^\delta - F(x^\delta_{n_i}) - \gamma_{n_i}(x_0 - x^+)\\
&= (1-\gamma_{n_i})(I - P^*P)e_{n_i} + (1-\gamma_{n_i})P^*Pe_{n_i}\\
&+ F'(x^\delta_{n_i})^*(y^\delta - F(x^\delta_{n_i})\gamma_{n_i}(x_0 - x^+)\\
&= (1-\gamma_{n_i})(I - P^*P)e_{n_i} + (1-\gamma_{n_i})P^*\big[F(x^\delta_{n_i}) - F(x^+) - P(x_{n_i} - x^+)\big]\\
&+ \big[P^* - F'(x^\delta_{n_i})^*\big](y^\delta - F(x^\delta_{n_i})) - \gamma_{n_i}P^*(y^\delta - F(x^\delta_{n_i}))\\
&+ (1-\gamma_{n_i})P^*(y - y^\delta) - \gamma_{n_i}(x_0 - x^+)\\
&= (1-\gamma_{n_i})(I - P^*P)e_{n_i} + (1-\gamma_{n_i})P^*\big[F(x^\delta_{n_i}) - F(x^+) - P(x_{n_i} - x^+)\big]\\
&+ \big[P^* - P^*R^*_{x^\delta_{n_i}}\big](y^\delta - F(x^\delta_{n_i})) - \gamma_{n_i}P^*(y^\delta - F(x^\delta_{n_i}))\\
&+ (1-\gamma_{n_i})P^*(y - y^\delta) - \gamma_{n_i}(x_0 - x^+)\\
&= (1-\gamma_{n_i})(I - P^*P)e_{n_i} + (1-\gamma_{n_i})P^*\big[F(x^\delta_{n_i}) - F(x^+) - P(x_{n_i} - x^+)\big]\\
&+ P^*\big[(1-\gamma_{n_i})I - R^*_{x^\delta_{n_i}}\big](y^\delta - F(x^\delta_{n_i})) - \gamma_{n_i}P^*(y^\delta - F(x^\delta_{n_i}))\\
&+ (1-\gamma_{n_i})P^*(y - y^\delta) - \gamma_{n_i}(x_0 - x^+).
\end{aligned} \tag{4.14}$$

Next we put

$$v_{n_i} = (1-\gamma_{n_i})(F(x^\delta_{n_i}) - F(x^+)) - P(x^\delta_{n_i} - x^+) + \big[(1-\gamma_{n_i})I - R^*_{x^\delta_{n_i}}\big](y^\delta - F(x^\delta_{n_i})).$$

So equation (4.14) is written as:

$$\begin{aligned}
e_{n_i+1} &= (1-\gamma_{n_i})(I - P^*P)e_{n_i} + (1-\gamma_{n_i})P^*(y - y^\delta)\\
&-\gamma_{n_i}(x_0 - x^+) + P^*v_{n_i}.
\end{aligned} \tag{4.15}$$

By repetition equation (4.15), we obtain the closed expression for the errors

$$\begin{aligned}
&e_{n_1} + \ldots + e_{n_k}\\
&= \Big[\prod_{j_1=0}^{n_1-1}(1-\gamma_{j_1})(I - P^*P)^{n_1} + \sum_{j_1=0}^{n_1-1}\gamma_{n_1-j_1-1}(I - B^*B)^{j_1}\prod_{i=1}^{j_1}(1-\gamma_{n_1-i})\Big]e_0\\
&+ \Big[\sum_{j_1=1}^{n_1}(I - P^*P)^{j_1}\prod_{i=1}^{j_1}(1-\gamma_{n_1-i})\Big]P^*(y - y^\delta)
\end{aligned}$$

$$+\sum_{j_1=0}^{n_1-1}\prod_{i=n_l-j_1}^{n_1-1}(1-\gamma_i)(I-P^*P)^{j_1}P^*v_{n_1-j_1-1}+\cdots$$

$$+\Big[\prod_{j_k=0}^{n_k-1}(1-\gamma_{j_k})(I-P^*P)^{n_k}+\sum_{j_k=0}^{n_k-1}\gamma_{n_k-j_k-1}(I-B^*B)^{j_k}\prod_{i=1}^{j_k}(1-\gamma_{n_k-i})\Big]e_0$$

$$+\Big[\sum_{j_k=1}^{n_k}(I-P^*P)^{j_k}\prod_{i=1}^{j_k}(1-\gamma_{n_k-i})\Big]P^*(y-y^\delta)$$

$$+\sum_{j_k=0}^{n_k-1}\prod_{i=n_k-j_l}^{n_k-1}(1-\gamma_i)(I-P^*P)^{j_k}P^*v_{n_k-j_k-1}. \tag{4.16}$$

Furthermore, it holds

$$P(e_{n_1}+\ldots+e_{n_k})$$

$$=\Big[P\prod_{j_1=0}^{n_1-1}(1-\gamma_{j_1})(I-P^*P)^{n_1}+P\sum_{j_1=0}^{n_1-1}\gamma_{n_1-j_1-1}(I-B^*B)^{j_1}\prod_{i=1}^{j_1}(1-\gamma_{n_1-i})\Big]e_0$$

$$+\Big[P\sum_{j_1=1}^{n_1}(I-P^*P)^{j_1}\prod_{i=1}^{j_1}(1-\gamma_{n_1-i})\Big]P^*(y-y^\delta)$$

(4.17)

$$+P\sum_{j_1=0}^{n_1-1}\prod_{i=n_l-j_1}^{n_1-1}(1-\gamma_i)(I-P^*P)^{j_1}P^*v_{n_1-j_1-1}+\cdots$$

$$+\Big[P\prod_{j_k=0}^{n_k-1}(1-\gamma_{j_k})(I-P^*P)^{n_k}+P\sum_{j_k=0}^{n_k-1}\gamma_{n_k-j_k-1}(I-B^*B)^{j_k}\prod_{i=1}^{j_k}(1-\gamma_{n_k-i})\Big]e_0$$

$$+\Big[P\sum_{j_k=1}^{n_k}(I-P^*P)^{j_k}\prod_{i=1}^{j_k}(1-\gamma_{n_k-i})\Big]P^*(y-y^\delta)$$

$$+P\sum_{j_k=0}^{n_k-1}\prod_{i=n_k-j_l}^{n_k-1}(1-\gamma_i)(I-P^*P)^{j_k}P^*v_{n_k-j_k-1}.$$

So, $\big(e_{n_1},e_{n_2},\ldots,e_{n_k}\big)$ infer that $P\big(e_{n_1},e_{n_2},\ldots,e_{n_k}\big)$. Next, for $0<n_i<N_i$, using the discrepancy principle, triangle inequality, equation (4.3), and

(4.18) $\|y^\delta - F(x^\delta_{N_i})\| \le k\|y^\delta - F(x^\delta_{N_i})\| - \kappa\delta < \|y^\delta - F(x^\delta_{n_i})\|, 0 \le n_i < N_i,$

$i = 1, \ldots, k$, $\kappa > \frac{2-\eta_i}{1-\eta_i}, i = 1, \ldots, k$. Using Lemma 3.1, Lemma 3.2, and equation (4.18), we obtain

$$\begin{aligned}\|v_{n_i}\| &\le (1-\gamma_{n_i})\|(F(x^\delta_{n_i}) - F(x^+)) - P(x^\delta_{n_i} - x^+)\| \\ &+ \|(1-\gamma_{n_i})I - R^*_{x^\delta_{n_i}}\|\|(y^\delta - F(x^\delta_{n_i}))\| \\ &\le \frac{1}{2}(1-\gamma_{n_i})\|e_{n_i}\|\|Pe_{n_i}\|c_L + \frac{1}{2}P_R\|e_{n_i}\frac{2}{1-\eta_i}\|\|Pe_{n_i}\| \\ &\le \hat{c}\|e_{n_i}\|\|Pe_{n_i}\|, \qquad (4.19)\end{aligned}$$

$\hat{c} := \frac{c_L}{2} + \frac{P_R}{1-\eta_i}, i = 1, \ldots, k$, and $1 - \eta_i \le 1$.

Next we need to show that

$$\|e_{n_h}\| \le \hat{P}_2(\ln(n_h + e))^{-\beta}. \qquad (4.20)$$

Inferred

$$k\|e_{n_h}\| \le \hat{P}_2(\ln(n_h + e))^{-k\beta} \qquad (4.21)$$

and

$$\|Pe_{n_h}\| \le \hat{P}_2(n_h + 1)^{\frac{-1}{2}}(\ln(n_h + e))^{-\beta}. \qquad (4.22)$$

Inferred

$$k\|Pe_{n_h}\| \le \hat{P}_2(n_h + 1)^{\frac{-k}{2}}(\ln(n_h + e))^{-k\beta}. \qquad (4.23)$$

Using the inductive method.

In case 1 with n=0, equations (4.22) and (4.23) are satisfied. Next we assume that equations (4.22) and (4.23) hold for $k = n - 1$. We only need to prove that equations (4.22) and (4.23) hold for n=k. Indeed. We rewrite equation (4.16) as follows: in order not to lose generality, we assume that $n_1 = n_2 = \cdots = n_k = n_h$ then according to (4.16) we have

$$
\begin{aligned}
\|e_{n_h}\| \le & \Big\| \prod_{j_1=0}^{n_h-1}(1-\gamma_{j_h})(I-P^*P)^{n_h}e_0\Big\| \\
& + \Big\| \sum_{j_h=0}^{n_h-1}\gamma_{n_h-j_h-1}(I-B^*B)^{j_h}\prod_{i=1}^{j_h}(1-\gamma_{n_h-i})e_0\Big\| \\
& + \Big\| \sum_{j_h=1}^{n_h}(I-P^*P)^{j_h-1}\prod_{i=1}^{j_h}(1-\gamma_{n_h-i})P^*(y-y^\delta)\Big\| \\
& + \Big\| \sum_{j_h=0}^{n_h-1}\prod_{i=n_h-j_h}^{n_h-1}(1-\gamma_i)(I-P^*P)^{j_h}P^*v_{n_h-j_h-1}\Big\| \qquad (4.24)
\end{aligned}
$$

Form $\|P\| \le 1$. So, we have

$$
\Big\| \sum_{j_s=1}^{n_h-1}(I-P^*P)^s P^*\Big\| \le \sqrt{n_s}
$$

and

$$
\Big\|(I-P^*P)^{j_k}P^*\Big\| \le (j_k+1)^{-k/2}, \quad j_k \ge 1.
$$

Therefore we have

$$
\begin{aligned}
& \Big\| \sum_{j_h=1}^{n_h}(I-P^*P)^{j_h-1}\prod_{i=1}^{j_h}(1-\gamma_{n_h-i})P^*(y-y^\delta)\Big\| \\
\le & \Big\| \sum_{j_h=1}^{n_h}(I-P^*P)^{j_h-1}P^*\Big\| \|(y-y^\delta)\| \le \sqrt{n_h}\delta. \qquad (4.25)
\end{aligned}
$$

Next, we consider

$$
\begin{aligned}
& \Big\| \sum_{j_h=0}^{n_h-1}\prod_{i=n_h-j_h}^{n_h-1}(1-\gamma_i)(I-P^*P)^{j_h}P^*v_{n_h-j_h-1}\Big\| \\
\le & \Big\| \sum_{j_h=0}^{n_h-1}(I-P^*P)^{j_h}P^*\Big\| \|v_{n_h-j_h-1}\| \le \sum_{j_h=0}^{n_h-1}(j_h+1)^{-k/2}\|v_{n_h-j_h-1}\| \qquad (4.26)
\end{aligned}
$$

So, using Lemma 3.1, Lemma 3.2, and equations (4.25) and (4.26) to equation (4.24), we obtain

$$\begin{aligned}\|e_{n_h}\| \leq& \frac{q_1}{k}(\ln(n_h+e))^{-k\beta}\|w\| + \frac{\hat{q}_1}{k}(\ln(n_h+e))^{-k\beta}\|w\| + \frac{\sqrt{n_h}}{k}\delta \\ &+ \frac{1}{k}\sum_{j_h=0}^{n_h-1}(j_h+1)^{-k/2}\|v_{n_h-j_h-1}\|. \end{aligned} \tag{4.27}$$

Then, using equation (4.19) to estimate the last term of equation (4.27), we obtain

$$\sum_{j_h=0}^{n_h-1}(j_h+1)^{-k/2}\|v_{n_h-j_h-1}\| \leq \hat{q}_1 \sum_{j_h=0}^{n_h-1}(j_h+1)^{-k/2}\|Pe_{n_h-j_h-1}\|\|e_{n_h-j_h-1}\|. \tag{4.28}$$

We apply the assumption of the induction in equations (4.21) and (4.23) into equation (4.28):

$$\begin{aligned}&\sum_{j_k=0}^{n_h-1}(j_h+1)^{-k/2}\|v_{n_h-j_h-1}\| \\ &\leq \hat{q}_1 \sum_{j_h=0}^{n_h-1}(j_h+1)^{-k/2}\|Pe_{n_h-j_h-1}\|\|e_{n_h-j_h-1}\| \\ &\leq \hat{q}_1\hat{P_2^2}\sum_{j_k=0}^{n_h-1}(\frac{j_h+1}{n_h+1})^{-k/2}(\frac{n_h-j_h}{n_h+1})^{-k/2}(\ln(n_h-j_k-1+e))^{-k\beta}(\frac{1}{n_h+1}).\end{aligned} \tag{4.29}$$

Rewriting equation (4.29), we have

$$\begin{aligned}\sum_{j_h=0}^{n_h-1}(j_h+1)^{-k/2}\|v_{n_h-j_h-1}\| =& \hat{q}_1\hat{P_2^2}(\ln(n_h+e))^{-k\beta}\sum_{j_h=0}^{n_h-1}(\frac{j_h+1}{n_k+1})^{-k/2} \\ &\cdot(\frac{n_h-j_h}{n_h+1})^{-k/2}\left[\frac{\ln(n_h+e)}{(\ln(n_h-j_h-1+e))}\right]^{k\beta}(\frac{1}{n_h+1})\end{aligned} \tag{4.30}$$

$$\begin{aligned}\leq& \hat{q}_1\hat{P_2^2}(\ln(n_h+e))^{-k\beta}\sum_{j_h=0}^{n_h-1}(\frac{j_h+1}{n_h+1})^{-k/2} \\ &\cdot(\frac{n_h-j_h}{n_h+1})^{-k/2}\left[\frac{\ln(n_h+e)}{(\ln(n_h-j_h-1+e))}\right]^{k\beta}(\frac{1}{n_h+1}),\end{aligned} \tag{4.31}$$

$k \in \mathbb{N}, k \geq 2$. Next we prove similar to the proof in the Assumption 3. Firstly, $n - j \geq 1$ provides

$$\ln\left(\frac{n_h + e}{n_h - j_h - 1 + e}\right) \ln(n_h - j_h - 1 + e) \geq \ln\left(\frac{n_h + e}{n_h - j_h - 1 + e}\right). \tag{4.32}$$

With $n_h - 1 \geq j_h \geq 0$, the properties of the logarithm provide

$$\begin{aligned} \frac{\ln(n_h + e)}{(\ln(n_h - j_h - 1 + e))} &= \frac{\ln(n_h + e)}{\ln(n_h + 1)}\left(1 + \frac{\ln(\frac{n_h+1}{n_h - j_h - 1 + e})}{\ln(n_h - j_h - 1 + e)}\right) \\ &\leq M_0\left(1 + \ln\left(\frac{n_h + 1}{n_k - j_h - 1 + e}\right)\right), \end{aligned} \tag{4.33}$$

with constant $M_0 < 2$ and $n_h \in \mathbb{N}^*$. From equation (4.32) can be estimated as follows:

$$\begin{aligned} \sum_{j_h=0}^{n_h-1} (j_h + 1)^{-k/2} \|v_{n_h - j_h - 1}\| &= \hat{c_1}\hat{P_2^2}(\ln(n_h + e))^{-k\beta} \sum_{j_h=0}^{n_h-1} \left(\frac{j_h + 1}{n_k + 1}\right)^{-k/2} \\ &\cdot \left(\frac{n_h - j_h}{n_h + 1}\right)^{-k/2}\left[\frac{\ln(n_h + e)}{(\ln(n_h - j_h - 1 + e))}\right]^{k\beta}\left(\frac{1}{n_h + 1}\right) \\ &\leq \hat{c_1}\hat{P_2^2}(\ln(n_h + e))^{-p} \sum_{j_h=0}^{n_h-1} \left(\frac{j_h + 1}{n_h + 1}\right)^{-k/2} \\ &\cdot \left(\frac{n_h - j_h}{n_k + 1}\right)^{-k/2}\left[\frac{\ln(n_h + e)}{(\ln(n_h - j_h - 1 + e))}\right]^{k\beta}\left(\frac{1}{n_h + 1}\right). \end{aligned} \tag{4.34}$$

The last summation is bounded since, put $r := \frac{1}{2(n_h+1)}$, the integral

$$\int_r^{1-r} x^{\frac{-k}{2}}(1 - x)^{\frac{-k}{2}}(1 - \ln(1 - x))^{k\beta} dx, \tag{4.35}$$

with a constant M that depends on n, we substitute the above information into the equation (4.27)

$$\begin{aligned} \|e_{n_h}\| &\leq c_h(\ln(n_h + e))^{-k\beta}\|w\| + \hat{c_h}(\ln(n_h + e))^{-k\beta}\|w\| \\ &\quad + \sqrt{n_h}\delta + c_{p_h}\hat{P_2^2}(\ln(n_h + 2))^{-\beta} \\ &\leq (c_h + \hat{c_h})\|w\| + \hat{c_h}M_0^{k\beta}M_p\hat{P_2^2}(\ln(k \cdot n_h + 2))^{-k\beta} + \sqrt{n_h}\delta. \end{aligned} \tag{4.36}$$

So, in general we have

$$\|e_{n_1+n_2+\ldots+n_k}\| \leq [\sum_{j=1}^{k} \frac{c_j+\hat{c_j}}{k}\|w\| + \hat{c_m} M_0^{k\beta} M_p \hat{P_2^2}](\ln(\sum_{j=1}^{k} n_j + e))^{-k\beta} + \frac{\sqrt{n_1+n_2+\ldots+n_k}}{k}\delta. \tag{4.37}$$

Similarly, equation (4.17) we have

$$\begin{aligned} \|Pe_{n_h}\| = & \left\|P\prod_{i=0}^{n_h-1}(1-\gamma_{j_h})(I-P^*P)^{n_h}e_0\right\| \\ & + \left\|P\sum_{j_h=0}^{n_h-1}\gamma_{n_h-j_h-1}(I-B^*B)^{j_h}\prod_{i=1}^{j_h}(1-\gamma_{n_h-i})e_0\right\| \\ & + \left\|P\sum_{j_h=1}^{n_h}(I-P^*P)^{j_h}\prod_{i=1}^{j_h}(1-\gamma_{n_h-i})P^*(y-y^\delta)\right\| \\ & + \left\|P\sum_{j_h=0}^{n_h-1}\prod_{i=n_h-j_h}^{n_h-1}(1-\gamma_i)(I-P^*P)^{j_h}P^*v_{n_h-j_h-1}\right\|. \end{aligned} \tag{4.38}$$

From the hypothesis $\|P\| \leq 1$, we have

$$\|(I-P*P)^{j_r}PP^*\| \leq (j_r+1)^{-1} \tag{4.39}$$

and

$$\|P\sum_{j_h=0}^{n_h-1}(I-P*P)^{j_h}PP^*\| \leq \|I-(I-P^*P)^{n_h}\| \leq 1. \tag{4.40}$$

Consequently,

$$\left\|P\sum_{j_k=1}^{n_h}(I-P^*P)^{j_h-1}\prod_{i=1}^{j_h}(1-\gamma_{n_h-i})P^*(y-y^\delta)\right\| \leq \|I-(I-P^*P)^{n_h}\| \leq 1 \tag{4.41}$$

and

$$\begin{aligned} & \left\|P\sum_{j_h=0}^{n_h-1}\prod_{i=n_h-j_h}^{n_h-1}(1-\gamma_i)(I-P^*P)^{j_h}P^*v_{n_h-j_h-1}\right\| \\ & \leq \sum_{j_h=0}^{n_h-1}(j_h+1)^{-1}\|v_{n_h-j_h-1}\|. \end{aligned} \tag{4.42}$$

From the Lemma 3.1 with $n_h > 1$, Lemma 3.2 and equations (4.41) and (4.42) for the equation (4.40) we have

$$\|Pe_{n_h}\| \le c_2(n_h+1)^{-1/2}(\ln(n_h+e))^{-\beta}\|w\| + \hat{c_2}(1+n_k)^{-1/2}(\ln(n_h+e))^{-\beta}\|w\|$$
$$+\delta + \sum_{j_h=0}^{n_h-1}(j_h+1)^{-1}\|v_{n_h-j_h-1}\|. \tag{4.43}$$

We estimate the last term of equation (4.43) by using equations (4.19) and (4.33) and the fact that $(\ln(n+e))^{-kp} \le 1$ as follows:

$$\sum_{j_h=0}^{n_h-1}(j_h+1)^{-1}\|v_{n_h-j_h-1}\|.$$
$$\le \hat{c_1}\sum_{j_h=0}^{n_h-1}(j_h+1)^{-1/2}\|Pe_{n_h-j_h-1}\|\|e_{n_h-j_h-1}\|$$
$$\le \hat{c_1}\hat{P_2^2}\sum_{j_h=0}^{n_h-1}\Big(\frac{j_h+1}{n_h+1}\Big)^{-1/2}\Big(\frac{n_h-j_h}{n_h+1}\Big)^{-1/2}(\ln(n_h-j_h-1+e))^{-2p}$$
$$= \hat{c_1}\hat{P_2^2}(n_h+1)^{-1/2}(\ln(n_h+e))^{-kp}$$
$$\times\sum_{j_h=0}^{n_h-1}\Big(\frac{j_h+1}{n_h+1}\Big)^{-1/2}\left(\frac{\ln(n_h+e)}{\ln(n_h-j_h-1+e)}\right)^{2p}(\ln(n_h+e))^{-p}\frac{1}{n_k+1}$$
$$\le \hat{c_1}\hat{P_2^2}(n_h+1)^{-1/2}(\ln(n_h+e))^{-kp}$$
$$\sum_{j_h=0}^{n_h-1}\Big(\frac{j_h+1}{n_h+1}\Big)^{-1}\Big(\frac{n_h-j_h}{n_h+1}\Big)^{-1/2}\left(1-\frac{\ln(n_h-j_h)}{\ln(n_h+1)}\right)^{2p}\frac{1}{n_h+1} \tag{4.44}$$

The last summation is bounded since, with $r := \frac{1}{2(n_h+1)}$, the integral

$$\int_r^{1-r} x^{\frac{-1}{2}}(1-x)^{\frac{-1}{2}}(1-\ln(1-x))^{2p}dx \le \widehat{M_p} \tag{4.45}$$

with a positive constant $\widehat{M_p}$ independently of n. Substituting above information into (4.41) yields

$$\|Pe_{n_h}\| \le c_2(n_h+1)^{-1/2}(\ln(n_h+e))^{-kp}\|w\|$$
$$+\hat{c_2}(1+n_h)^{-1/2}(\ln(n_h+e))^{-kp}\|w\| \tag{4.46}$$

$$+\delta+\hat{c_p}\hat{P_2^2}(n_h+1)^{-1/2}(\ln(n_h+e))^{-kp}$$
$$\leq\big[(c_2+\hat{c_2})\|w\|+\hat{c_p}\hat{P_2^2}\big](1+n_h)^{-1/2}(\ln(n_h+e))^{-kp}+\delta.$$

So,

$$\begin{aligned}k\|Pe_{n_h}\| &\leq c_2(n_h+1)^{-k/2}(\ln(k\cdot n_h+e))^{-k\beta}\|w\| \\ &+\hat{c_2}(1+n_h)^{-k/2}(\ln(k\cdot n_h+e))^{-k\beta}\|w\| \\ &+\delta+\hat{c_p}\hat{P_2^2}(n_h+1)^{-k/2}(\ln(k\cdot n_h+e))^{-k\beta} \\ &\leq\big[(c_2+\hat{c_2})\|w\|+\hat{c_p}\hat{P_2^2}\big](1+n_h)^{-k/2}(\ln(k\cdot n_h+e))^{-k\beta}+\delta \qquad (4.47)\end{aligned}$$

Setting $h_*:=max\{c_1+\hat{c_1},c_2+\hat{c_2},\dots,c_k+\hat{c_k}\}$ equations (4.47) and (4.44) become

$$\|e_n\|\leq\big[\frac{h_*}{k}\|w\|+\hat{c_m}M_0^{2p}M_p\hat{P_2^2}\big](\ln(\sum_{j=1}^{k}n_j+e))^{-p}+\frac{\sqrt{n_1+n_2+\ldots+n_k}}{k}\delta \qquad (4.48)$$

and

$$\begin{aligned}\|Pe_{n_k}\| &=\leq\big[\frac{h_*}{k}\|w\|+\hat{c_m}M_0^{2p}M_p\hat{P_2^2}\big]\big(\sum_{j=1}^{k}n_j+1\big)^{-1/2} \\ &\cdot(\ln(\sum_{j=1}^{k}n_j+e))^{-p}+\delta. \qquad (4.49)\end{aligned}$$

Because of equations (4.1) and (4.3) we have

$$t\delta\leq\|y^\delta-F(x_{n_k}^\delta)\|\leq\delta+\big(\frac{1}{1+\eta}\big)\|Pe_{n_k}\|. \qquad (4.50)$$

Moreover,

$$(1-\eta)(t-1)\delta\|Pe_{n_k}\|\leq\big[h_*\|w\|+\hat{c_p}\hat{P_2^2}\big](n_k+1)^{\frac{-1}{2}}(\ln(n_{k+e}))^{-p}+\delta. \qquad (4.51)$$

Due to $t>\frac{2-\eta}{1-\eta}$, we have $\Gamma=(1-\eta)(t-1)-1$ We can rewrite equation (4.46) as follows:

$$\delta\leq\frac{1}{\Gamma}\big[h_*\|w\|+\hat{c_p}\hat{P_2^2}\big](n_k+1)^{\frac{-1}{2}}(\ln(n_{k+e}))^{-kp}. \qquad (4.52)$$

Applying equation (4.47) to equation (4.44), we get

$$\|e_{n_k}\| \le \big(1 + \frac{1}{\Gamma}\big)\big[h_* \|w\| + \hat{c_p}\hat{P_2^2}\big](\ln(n_{k+e}))^{-kp}. \tag{4.53}$$

For $\hat{c_p} = max\{c_p, \hat{c_p}\}$, in a similar manner, equation (4.45) can be written as

$$\|Pe_{n_k}\| \le \big(1 + \frac{1}{\Gamma}\big)\big[h_* \|w\| + \hat{c_p}\hat{P_2^2}\big](n_k+1)^{-1/2}(\ln(n_{k+e}))^{-kp}. \tag{4.54}$$

Finally, we select $\|w\|$ such that $\big(1 + \frac{1}{\Gamma}\big)\big[h_* \|w\| + \hat{c_p}\hat{P_2^2}\big] \le M_2$. This is always possible for sufficiently small $\|w\|$. Therefore, the induction is completed. Using equation (4.20), we have

$$\|e_{n_k}\| \le \hat{P}_2\big(\frac{\ln n}{\ln(n_k+e)}\big)^{kp}(\ln(n))^{-kp} \le \hat{P}_2(\ln(n))^{-kp}. \tag{4.55}$$

and similarly, by using equation (4.18), we get

$$\begin{aligned}\|y^\delta - F(x^\delta_{n_k})\| &\le \frac{2}{1-\eta}P_2(n_k+1)^{-1/2}\big(\frac{\ln n}{\ln(n_k+e)}\big)^{kp}(\ln(n))^{-kp}\\ &\le P_2(n_k+1)^{-1/2}(\ln(n))^{-kp}.\end{aligned} \tag{4.56}$$

Thus, the assertion is obtained. □

Theorem 4.2. *1. Assume that the problem in equation $F(x) = y$ has a solution $x^+ \in B_{\frac{\varrho}{2}}(x_0)$, y^δ satisfy the functional inequality satisfy*

$$\|y^\delta - y\| \le \delta, \tag{4.57}$$

and F satisfy the following functional equation and functional inequality

$$F'(x) = R_x F'(x^+) \quad and \quad \|R_x - I\| \le c_L\|x - x^+\|. \tag{4.58}$$

Assume that the Fréchet derivative of F is narrow such that

$$\|F'(x)\| \le 1, \forall x \in B_{\frac{\varrho}{2}}(x_0).$$

Furthermore, assume that the headspring condition in equations

$$f_\beta(h,g) := \begin{cases} \big(\ln \frac{e}{\lambda}\big)^{-k\beta} & for \quad 0 < \lambda \le 1 \\ ,0 & for \quad \lambda = e, \end{cases} \tag{4.59}$$

and

$$x^+ - x_0 = f\big(F'(x^+)^*)F(x^+)\big)w, x \in X \tag{4.60}$$

is fulfilled and that the modified Landweber method is stopped according to equation

$$\| \sum_{i=1}^{k} \big(y^{\delta} - F(x^{\delta}_{N_i})\big) \| \leq \beta\delta < \sum_{i=1}^{k} \|y^{\delta} - F(x^{\delta}_{n_i})\|, 0 \leq n_i < N_i, i = 1, \ldots, k. \tag{4.61}$$

If $\|w\|$ *is sufficiently small and* $1 \leq \beta \leq 2$*, then*

$$\| \sum_{i=1}^{k} N_i \ln(N_i)^{k\beta} \| \leq \frac{q^*}{\delta} \tag{4.62}$$

and

$$\sum_{i=1}^{k} \|x^+ - x^{\delta}_{N_i}\| \leq M^* \big(-\frac{1}{\ln \delta}\big)^{-k\beta}, \tag{4.63}$$

where $q^*, M^* > 0$ *are positive constants.*

Proof. Let put $e_0 := x^+ - x_0 = f(P^*P)w$. Then,

$$\begin{aligned} e_{n_1} &= \Big[\prod_{i_1=0}^{n_1-1} (1-\gamma_{i_1})(I - P^*P)^{n_1} + \sum_{j_1=0}^{n_1-1} \gamma_{n_1-j_1-1}(I - P^*P)^{j_1} \prod_{i=1}^{j_1} (1-\gamma_{n_1-i}) \Big] \\ &\quad \cdot f(P^*P)w + \Big[\sum_{j_1=1}^{n_1} (I - P^*P)^{j_1} \prod_{i=1}^{j_1} (1-\gamma_{n_1-i}) \Big] P^*(y - y^{\delta}) \\ &\quad + \sum_{j_1=0}^{n_1-1} \prod_{i=n_l-j_1}^{n_1-1} (1-\gamma_i)(I - P^*P)^{j_1} P^* v_{n_1-j_1-1} \end{aligned} \tag{4.64}$$

$$\begin{aligned} e_{n_k} &= \Big[\prod_{i_k=0}^{n_k-1} (1-\gamma_{i_k})(I - P^*P)^{n_k} + \sum_{j_k=0}^{n_k-1} \gamma_{n_k-j_k-1}(I - P^*P)^{j_k} \prod_{i=1}^{j_k} (1-\gamma_{n_k-i}) \Big] \\ &\quad \cdot f(P^*P)w + \Big[\sum_{j_k=1}^{n_k} (I - P^*P)^{j_k} \prod_{i=1}^{j_k} (1-\gamma_{n_k-i}) \Big] P^*(y - y^{\delta}) \\ &\quad + \sum_{j_k=0}^{n_k-1} \prod_{i=n_k-j_l}^{n_k-1} (1-\gamma_i)(I - P^*P)^{j_k} P^* v_{n_k-j_k-1}. \end{aligned}$$

then

$$
\begin{aligned}
e_{N_1} &= \Big[\prod_{i_1=0}^{N_1-1} (1-\gamma_{i_1})(I-P^*P)^{n_1} + \sum_{j_1=0}^{N_1-1} \gamma_{n_1-j_1-1} (I-P^*P)^{j_1} \prod_{i=1}^{j_1} (1-\gamma_{N_1-i}) \Big] \\
&\quad \cdot f(P^*P)w + \Big[\sum_{j_1=1}^{N_1} (I-P^*P)^{j_1} \prod_{i=1}^{j_1} (1-\gamma_{N_1-i}) \Big] P^*(y-y^\delta) \\
&\quad + \sum_{j_1=0}^{N_1-1} \prod_{i=N_1-j_1}^{N_1-1} (1-\gamma_i)(I-P^*P)^{j_1} \hat{f}(P^*P) \hat{v}_{N_1-j_1-1} \qquad (4.65)
\end{aligned}
$$

$$
\begin{aligned}
e_{N_k} &= \Big[\prod_{i_k=0}^{N_k-1} (1-\gamma_{i_k})(I-P^*P)^{N_k} + \sum_{j_k=0}^{N_k-1} \gamma_{N_k-j_k-1} (I-P^*P)^{j_k} \prod_{i=1}^{j_k} (1-\gamma_{N_k-i}) \Big] \\
&\quad \cdot f(P^*P)w + \Big[\sum_{j_k=1}^{N_k} (I-P^*P)^{j_k} \prod_{i=1}^{j_k} (1-\gamma_{N_k-i}) \Big] P^*(y-y^\delta) \\
&\quad + \sum_{j_k=0}^{N_k-1} \prod_{i=N_k-j_k}^{N_k-1} (1-\gamma_i)(I-P^*P)^{j_k} \hat{f}(P^*P) \hat{v}_{N_k-j_k-1}. \qquad (4.66)
\end{aligned}
$$

For $v_{N_l-j_m-1} = \hat{v}_{N_l-j_m-1}, l = 1,\dots,k, m = 0,\dots,N_l-1$ and $\hat{f}(P^*P) := \int_0^1 \lambda^{1/2}(1-\ln\lambda)^{k\beta} dE_\lambda$ from Assumption 1 we put $r = -k\beta$ and we get

$$
\big\|(I-P^*P)^{j_i}\hat{f}(P^*P)\big\| \le q_i\big(j_i+1\big)^{-1/2}(\ln(j_i+1))^p, i = 1,\dots,k, q_i \in \mathbb{R}^+. \qquad (4.67)
$$

Next, according to Assumption 3 we put $h_i = N_i - 1, i = 1,\dots,k$, and

$$
\begin{aligned}
&\sum_{j_i=0}^{h_i-1} (j_i+1)^{-1/2} \big(\ln(j_i+1)\big)^p \big(h_i-j_i+1\big)^{-1/2} \big(\ln\big(h_i-j_i+1\big)\big)^{-2k\beta} \\
&= \sum_{j_i=0}^{h_i-1} (j_i+1)^{-1/2} \big(\ln(j_i+1)\big)^p \big(h_i-j_i+1\big)^{-1/2} \big(\ln\big(h_i-j_i+1\big)\big)^{-2k\beta} \qquad (4.68) \\
&\quad + (h_i+1)^{-1/2}(\ln(h_i+1))^p \le M + (N)_i^{-1/2}(\ln(N_i)^p, i = 1,\dots,k.
\end{aligned}
$$

For $i \in \{1,2,\dots,k\}$ we consider the following equality

$$
\begin{aligned}
\|e_{N_k}\| = \| & \prod_{i_k=0}^{N_k-1}(1-\gamma_{i_k})(I-P^*P)^{N_k}w\| \\
& + \sum_{j_k=0}^{N_k-1}\gamma_{N_k-j_k-1}\prod_{i=1}^{j_k}(1-\gamma_{N_k-i})\|(I-P^*P)^{j_k}w\|f(P^*P) \\
& + \Big[\sum_{j_k=1}^{N_k}(I-P^*P)^{j_k}\prod_{i=1}^{j_k}(1-\gamma_{N_k-i})\Big]P^*(y-y^\delta) \\
& + \sum_{j_k=0}^{N_k-1}\prod_{i=N_k-j_k}^{N_k-1}(1-\gamma_i)(I-P^*P)^{j_k}\hat{f}(P^*P)\hat{v}_{N_k-j_k-1}. \qquad (4.69)
\end{aligned}
$$

Now we consider the sub-equation of e_{N_i} as follows:

$$
\begin{aligned}
& \| \prod_{i_k=0}^{N_k-1}(1-\gamma_{i_k})(I-P^*P)^{N_k}w\| \\
& + \sum_{j_k=0}^{N_k-1}\gamma_{N_k-j_k-1}\prod_{i=1}^{j_k}(1-\gamma_{N_k-i})\|(I-P^*P)^{j_k}w\| \qquad (4.70) \\
& + \prod_{i=N_k-j_k}^{N_k-1}(1-\gamma_i)(I-P^*P)^{j_k}\hat{f}(P^*P)\hat{v}_{N_k-j_k-1}.
\end{aligned}
$$

From equations (4.19), (4.20), (4.19), (4.67), and (4.68), we put

$$
\begin{aligned}
D_{N_i} = \| & \prod_{i_k=0}^{N_k-1}(1-\gamma_{i_k})(I-P^*P)^{N_k}w\| \\
& + \sum_{j_k=0}^{N_k-1}\gamma_{N_k-j_k-1}\prod_{i=1}^{j_k}(1-\gamma_{N_k-i})\|(I-P^*P)^{j_k}w\|f(P^*P) \\
& + \prod_{i=N_k-j_k}^{N_k-1}(1-\gamma_i)(I-P^*P)^{j_k}\hat{f}(P^*P)\hat{v}_{N_k-j_k-1}. \qquad (4.71) \\
& \le (N_i+1)\|w\| + c_2\sum_{j_i=0}^{N_i-1}(j_i+1)^{-1/2}(\ln(j_i+1))^p\|\hat{v}_{N_k-j_k-1}\|
\end{aligned}
$$

$$\leq (N_i+1)\|w\| + c_2\hat{c}_1 \sum_{j_i=0}^{N_i-1} (j_i+1)^{-1/2}(\ln(j_i+1))^p \|Pe_{N_i-j_1-1}\| \|e_{N_i-j_i-1}\|$$
$$\leq (N_i+1)\|w\| + c_2\hat{c}_1\hat{P}_2^2 \sum_{j_i=0}^{N_i-1} (j_i+1)^{-1/2}(\ln(j_i+1))^p (N_i-j_i)^{-1/2}$$
$$\cdot \big(\ln(N_i-j_i-1+e)\big)^{-2p}$$
$$\leq (N_i+1)\|w\| + M + (N_i)^{-1/2}(\ln(N_i)^p, i=1,\dots,k.$$

So, from equation (4.65) we conclude that

(4.72)
$$\|e_{N_i}\| \leq \|D_{N_i}f(P^*P)\| + \|\sum_{j_i=0}^{N_i-1}(I-P^*P)^{j_i}P^*\|\delta \leq \|D_{N_i}f(P^*P)\| + \sqrt{N_i}\delta,$$

$i=1,\dots,k$. From Assumption 2 and equation (4.9) for some $c_4>0$, we have

$$\|D_{N_i}f(P^*P)\| \leq c_4(-\ln\delta)^{-p}\big[(N_i+1)\|w\| + M + (N_i)^{-1/2}(\ln(N_i)^p\big], \tag{4.73}$$

$i=1,\dots,k$. So,

$$\|e_{N_i}\| \leq c_4(-\ln\delta)^{-p}\big[(N_i+1)\|w\| + M + (N_i)^{-1/2}(\ln(N_i)^p\big] + \sqrt{N_i}\delta, \tag{4.74}$$

$i=1,\dots,k$. We apply equation (4.52); then,

$$(N_i+1)^{1/2}(\ln(N_i+e))^p \leq \frac{1}{\Gamma\delta}\big[c_*\|w\| + \hat{c}\hat{P}_2^2\big] = \frac{c_5}{\delta}, \tag{4.75}$$

$i=1,\dots,k$. for some positive c5. By the fact that

$$N_i(\ln(N_i))^{2p} \leq (N_i+1)(\ln(N_i+e))^{2p} \leq \big(\frac{c_5}{\delta}\big)^2, \tag{4.76}$$

$i=1,\dots,k$. By Lemma 4, we have

$$N_i = \frac{c_6(-\ln\delta)^{-2p}}{\delta^2}, i=1,\dots,k. \tag{4.77}$$

Applying equation (68) to equation (66), we get

$$\begin{aligned}\|e_{N_i}\| \leq{}& c_4(-\ln\delta)^{-p}\big[(N_i+1)\|w\| + M + (N_i)^{-1/2}(\ln(N_i)^p\big]\\ &+ c_6(-\ln\delta)^{-p},\end{aligned} \tag{4.78}$$

$i = 1, \dots, k$, or

$$\|e_{N_i}\| \le (-\ln \delta)^{-p}\big(c_4\big[(N_i+1)\|w\| + M + (N_i)^{-1/2}(\ln(N_i)^p\big] + c_6\big), \tag{4.79}$$

$i = 1, \dots, k$. So,

$$\sum_{i=1}^{k} \|e_{N_i}\| \le (-\ln \delta)^{-p}\big(c_4\big[\sum_{i=1}^{k}(N_i+1)\|w\| + kM + \sum_{i=1}^{k}(N_i)^{-1/2}(\ln(N_i)^p\big] + c_6\big). \tag{4.80}$$

or

$$\sum_{i=1}^{k} \|e_{N_i}\| \le (-\ln \delta)^{-p} c_4 \sum_{i=1}^{k}(N_i+1)\|w\| + (-\ln \delta)^{-p} kM + (-\ln \delta)^{-p}\sum_{i=1}^{k}(N_i)^{-1/2}(\ln(N_i)^p + kc_6(-\ln \delta)^{-p}. \tag{4.81}$$

□

5. Conclusion

In this paper, we give lemmas such as Lemma 3.1 and Lemma 3.2 to analyze the convergence of the Inverse Math problem using Landweber's Algorithm. That is the main result in this paper.

References

[1] L. Landweber: *An iteration formula for Fredholm integral equations of the first kind*, Amer. J. Math **73** (1951), 615 – 624.

[2] P. L. Combettes and J.-C. Pesquet.: *"Proximal splitting methods in signal processing," in: Fixed-Point Algorithms for Inverse Problems in Science and Engineering, (H. H. Bauschke, R. S. Burachik, P. L. Combettes, V. Elser, D. R. Luke, and H. Wolkowicz, Editors, pp.* (2011), 185 – 212.

[3] P.Pornsawad, N. Sapsakul, C. Böckmann: *A modified asymptotical regularization of nonlinear ill-posed problems.* , Mathematics **7** (2019), 419.

[4] U.Tautenhahn: *On the asymptotical regularization of nonlinear ill-posed problems.* Inverse Probl **10** (1994), 1405-1418.

[5] P. PORNSAWAD, C. BÖCKMANN: *Modified iterative Runge-Kutta-type methods for nonlinear ill-posed problems* ,Numer. Funct. Anal. Optim. **37** (2016), 1562–11589.

[6] T. HOHAGE: *Logarithmic convergence rates of the iteratively regularized Gauss-Newton method for an inverse potential and an inverse scattering problem*, Inverse Probl. **13** (1997), 1279.

[7] P. PORNSAWAD, C. BÖCKMANN: *Modified iterative Runge-Kutta-type methods for nonlinear ill-posed problems.*, Numer. Funct. Anal. Optim. **37** (2016), 1562–1589.

[8] P. MAHALE, NAIR,M: *Tikhonov regularization of nonlinear ill-posed equations under general source condition.*, J. Inv. Ill-Posed Probl. **15** (2007), 813–829.

[9] O. SCHERZER: *A modified Landweber iteration for solving parameter estimation problems.*,Appl. Math. Optim. **38** (1998), 45–68.

[10] B. KALTENBACHER, NEUBAUER, A. SCHERZER, O: *Iterative Regularization Methods for Nonlinear Ill-Posed Problems*, De Gruyter: Berlin, Germany; Boston, MA, USA, () (2008).

[11] M. HANKE: *Acelerated Landweber iterations for the solution of ill-posed problems*, Numer. Math. **60**() (1991), 341–373.

[12] C.W. GROETSH: *The Theory of Tikhonov Regularization for Fredholm equations of the First Kind*, Pitman, Boston. () (1984).

[13] H.W. ENGL, K. KUNISH, AND A. NEUBAUER: *Neubauer. Convergene rates for Tikhonov regularization of nonlinear ill- posed problems*, Inverse Probl. **5**() (1989), 532–540.

[14] H. W. ENGL AND O. SHERZER: *Convergence rates results for iterative methods for solving nonlinear ill-posed problems*, Can. J. Math. **13**() (2000), 7–34.

[15] A.K. LOUIS: *Inverse und schlecht gestellte Probleme*, Stuttgart, Teubner. () (1989).

[16] G.M. VAINIKKO, A.VERETENNIKOV,Y: *UnivIteration Procedures in Ill-Posed Problems*, Moscow, Nauka (in Russian). () (1986).

[17] B. EICKE: *Iteration methods for convexly constrained ill-posed problems in Hilbert space*, Numer. Funct. Anal. OptimC. **13**() (1992), 413–429.

[18] B. JOHANSSON, ELFVING, T. KOZLOVC, V. CENSOR, Y. FORSSEN, P.E. GRANLUND, G: *The application of an oblique-projected Landweber method to a model of supervised learning*, Math. Comput. Modelling. **43** (2006), 892–909.

[19] H.J. TRUSSELL, M. CIVANLAR, R: *The Landweber iteration and projection onto convex sets*, IEEE Trans. Acoust., Speech, Signal Process. **33** (1985), 1632–1634.

[20] A. ANASTASIOS KYRILLIDIS VOLKAN CEVHER: *Recipes on hard thresholding methodss*, Recipes for hard thresholding methods. (2011), 353–356.

[21] H.W. ENGL, HANKE, M. NEUBAUER, A: *Regularization of Inverse Problems*, Kluwer, Dordrecht. (1996).

[22] B. BLASCHKE, NEUBAUER, A. SCHERZER, O: *On convergence rates for the iteratively regularized Gauss-Newton method*, IMA J. Numer. Anal. Can. J. Math. **17** (1997), 421–436.

[23] M. HANKE, NEUBAUER, A. SCHERZER, O: *A convergence analysis of the Landweber iteration for nonlinear ill-posed problems*, Numer. Math. **72** (1995), 21–37.

[24] J. QI-NIAN: *On the iteratively regularized Gauss-Newton method for solving nonlinear ill-posed problems*, Math. Comput. **69** (2000), 1603–1623.

[25] D. PRADEEP, M.P. RAJAN: *A regularized iterative scheme for solving non-linear ill-posed problems. Numer*, Numer. Func. Anal. Opt. **37**(5) (2016), 342–362.

[26] A. NEUBAUER: *On Landweber iteration for nonlinear ill-posed problems in Hilbert scales*, Numer. Math **85** (2000), 309–328.

[27] O. SCHERZER: *Convergence criteria of iterative methods based on Landweber iteration for solving nonlinear problems*, J. Math. Anal. Appl. **194** (1995), 911–933.

[28] M. HANKE: *Accelerated Landweber iterations for the solution of ill-posed problems*, Numer. Math. **60** (1991), 341–373.

[29] C.W. GROETSCH: *The Theory of Tikhonov Regularization for Fredholm equations of the First Kind*,Pitman, Boston. (1984).

[30] H.W. ENGL, K. KUNISCH, AND A. NEUBAUER: *Convergence rates for Tikhonov regularization of nonlinear ill-posed problems*, Inverse Probl. **5** (1989), 523–540.

[31] H. W. ENGL AND O. SCHERZER : *Convergence rates results for iterative methods for solving nonlinear ill-posed problems*, Can. J. Math. **13**() (2000), 7–34.

[32] A.B. BAKUSHINSKII: *The problem of the convergence of the iteratively regularized Gau-Newton method*, Comput. Maths. Math. Phys. **32**() (1992), 1353–1359.

[33] A.B. BAKUSHINSKII: *Iterative methods for solving non-linear operator equations without condition of regularity*, a new approach. Russian Math. Dokl. **330**() (1993), 282–284.

[34] A.B. BAKUSHINSKII: *Universal linear approximations of solutions to non-linear operator equations and their applications*, J. Inv. Il l-Posed Problems. **5**() (1997), 501–521.

[35] SA.B. BAKUSHINSKII AND A.V: *Goncharskii. Iterative Methods for the Solution of Incorret Problems*, Nauka, Moscow. () (1989).

[36] A.B. BAKUSHINSKII AND A.V: *Goncharskii. Ill-Posed Problems: Theory and Applications*, Kluwer Academic Publishers, Dordreht, Boston, London. () (1994).

[37] BO. ¨CKMANN, C. KAMMANEE, A. BRAUNSS, A: *ULogarithmic convergence rate of Levenberg–Marquardt method with application to an inverse potential problems*, J. Inv. Ill-Posed Probl. **19**() (2011), 345–367.

[38] F. HETTLICH, W. RUNDELL: *Iterative methods for the reconstruction of an inverse potential problems*, Inverse Probl. **12**() (1996), 251–266.

[39] F. HETTLICH, W.RUNDELL: *A second degree method for nonlinear inverse problems*, SIAM J. Numer. Anal. **37**() (1999), 587–620.

[40] K. VAN DEN DOEL, ASCHER, U: *On level set regularization for highly ill-posed distributed parameter estimation problems*, J. Comput. Phys. **216**() (2006), 707–723.

[41] T. HOHAGE: *Logarithmic convergence rates of the iteratively regularized Gauss—Newton method for an inverse potential and an inverse scattering problems*, CInverse Probl. **13**(5) (1997), 1279.

[42] P. PORNSAWAD, N. SAPSAKUL, C. BÖCKMANN: *A modified asymptotical regularization of nonlinear ill-posed problems*, Mathematics. **7**() (2019), 419.

[43] U. TAUTENHAHN: *On the asymptotical regularization of nonlinear ill-posed problems*, Inverse Probl.**10**() (1994), 1405–1418.

[44] Y. ZHANG, B. HOFMANN: *On the second order asymptotical regularization of linear ill-posed inverse problems*, Appl. Anal. () (2018).

[45] P. SPORNSAWAD, C. BÖCKMANN: *Modified iterative Runge-Kutta-type methods for nonlinear ill-posed problems*, Numer. Funct. Anal. Optim. **37**(5) (2016), 1562–1589.

[46] P. MAHALE, M. NAIR: *Tikhonov regularization of nonlinear ill-posed equations under general source condition*, J. Inv. Ill-Posed Probl. **15**() (2007), 813–829.

[47] V. ROMANOV, KABANIKHIN, S. ANIKONOV, Y. BUKHGEIM, A: *Ill-Posed and Inverse Problems*, Dedicated to Academician Mikhail Mikhailovich Lavrentiev on the Occasion of his 70th Birthday, De Gruyter: Berlin, Germany,() (2018).

[48] O. SCHERZER: *A modified Landweber iteration for solving parameter estimation problems*, Appl. Math. Optim. **38**() (1998), 45–68.

[49] P. DEUFLHARD, ENGL, W. SCHERZER, O: *A convergence analysis of iterative methods for the solution of nonlinear ill-posed problems under affinely invariant conditions* , Mathematics Inverse Probl **14** (1998), 1081–1106.

[50] P. PORNSAWAD, SAPSAKUL, N. BÖCKMANN, C.: *A modified asymptotical regularization of nonlinear ill-posed problems.* , Mathematics **7** (2019), 419.

[51] T. SCHUSTER, KALTENBACHER, B.HOFMANN, B. KAZIMIERSKI, K: *Regularization Methods in Banach Spaces*, Radon Series on Computational and Applied Mathematics; De Gruyter: Berlin, Germany (2012).

[52] V. ALBANI, ELBAU, P. DE HOOP, M.V. SCHERZER, O: *Optimal convergence rates results for linear inverseproblems in Hilbert spaces* , Numer. Funct. Anal. Optim **37** (2016), 521–540.

[53] B. KALTENBACHER, NEUBAUER, A. SCHERZER, O: *Iterative Regularization Methods for Nonlinear Ill-Posed Problems* , De Gruyter: Berlin, Germany; Boston, MA, USA (2008).

[54] A. K. LOUIS: *Inverse und Schlecht Gestellte Probleme* , Teubner Studienbücher Mathematik, B. G. Teubner. Stuttgart, Germany, (1989), 1989.

[55] G.VAINIKKO, VETERENNIKOV, A.Y: *Iteration Procedures in Ill-Posed Problems* , Nauka: Moscow, Russia, (1986).

[56] M. HANKE, NEUBAUER, A. SCHERZER, O: *A convergence analysis of the Landweber iteration for nonlinear ill-posed problems* , MNumer. Math **72** (1995), 21–37.

[57] C.BO ¨CKMANN, PORNSAWAD, P: *Iterative Runge-Kutta-type methods for nonlinear ill-posed problems* , Inverse Probl. **24** (2008), 025002.

[58] L. LY VAN AN: *Extending convergence analysis of a landweber method for solving nonlinear ill-posed problems* , ADV MATH SCI JOURNA Advances in Mathematics: Scientific Journal s **13** (2024), 1857–8434.

FACULTY OF MATHEMATICS TEACHER EDUCATION
TAY NINH UNIVERSITY
TAY NINH, VIETNAM.
Email address: lyvanan145@gmail.com, lyvananvietnam@gmail.com