ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **13** (2024), no.3, 455–461
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.13.3.11

FUNCTIONS WHOSE DERIVATIVE HAS POSITIVE REAL PART

M. Obradović and V. Ravichandran

ABSTRACT. It is well-known that a normalized analytic function for which the quantity $1 + zf''(z)/f'(z)$ lies in the half-plane $\operatorname{Re} w > -1/2$ is close-to-convex and hence univalent. In this paper, we show that the derivative of the function f has positive real part if the quantity $1+\alpha zf''(z)/f'(z)$ with $\alpha > 0$ lies in the sector $|\arg w| < \arctan(\alpha)$.

1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ be the class of functions f analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ and normalized by the condition $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ be its subclass consisting of univalent functions. A function $f \in \mathcal{A}$ is convex if $f(\mathbb{D})$ is convex and it is starlike if $f(\mathbb{D})$ is starlike with respect to the origin. Analytically, a function $f \in \mathcal{A}$ is convex if $1 + zf''(z)/f'(z)$ takes values in the right half-plane. The function $f \in \mathcal{A}$ is starlike if $zf'(z)/f(z)$ takes values in the right half-plane. If these quantities takes values in the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > \alpha\}$, the functions are respectively called convex functions of order α and starlike functions of order

[1]*corresponding author*
2020 *Mathematics Subject Classification.* 30C45, 30C80.
Key words and phrases. starlike functions, convex functions, close-to-convex functions, functions with positive real part.
Submitted: 26.07.2024; *Accepted:* 12.08.2024; *Published:* 27.09.2024.

α. A function $f \in \mathcal{A}$ is close-to-convex if there exists a convex function g such that

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0 \quad (z \in \mathbb{D}).$$

Close-to-convex functions were introduced and studied by Kaplan [4]. Functions in $\mathcal{A}$ that are convex, starlike and close-to-convex are univalent. Kaplan [4] (see Duren [1]) proved that a locally univalent analytic function f is close-to-convex if and only if

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) d\theta > -\pi \quad (z = re^{i\theta})$$

for each r $(0 < r < 1)$ and for each pair of real numbers θ_1 and θ_2 with $\theta_1 < \theta_2$. This characterization of close-to-convexity shows that a function $f \in \mathcal{A}$ satisfying the condition

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > -\frac{1}{2} \quad (z \in \mathbb{D}) \tag{1.1}$$

is close-to-convex function and hence univalent. We note that the condition given by (1.1) is equivalent to the condition

$$\operatorname{Re}\left(1 + \frac{2}{3}\frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathbb{D}). \tag{1.2}$$

A function that satisfy (1.1) is sometime called a convex function of order $-1/2$. Though a convex function is starlike of order 1/2, a function satisfying the condition (1.1), or equivalently (1.2) is not necessarily for the function to have derivative with positive real part in $\mathbb{D}$ as the next example shows.

Example 1. *Define the function* $f_1 : \mathbb{D} \to \mathbb{C}$ *by*

$$f_1(z) = \frac{z(1 - \frac{z}{2})}{(1 - z)^2}.$$

For this function f_1, *we have*

$$\operatorname{Re}\left(1 + \frac{zf_1''(z)}{f_1'(z)}\right) = \operatorname{Re}\left(1 + 3\frac{z}{1 - z}\right) > -\frac{1}{2} \quad (z \in \mathbb{D}).$$

This shows that f_1 *is convex of order* $-1/2$. *Since*

$$f_1'(z) = \frac{1}{(1 - z)^3},$$

we have, at $z = i$,

$$\operatorname{Re} f_1'(i) = -\frac{1}{4} < 0.$$

Therefore, it follows that

$$\operatorname{Re} f_1'(z) < 0$$

at the point $z = i$ and hence in a neighbourhood of it. Therefore, the function f_1 is not a function whose derivative has positive real part(see Fig. 1).

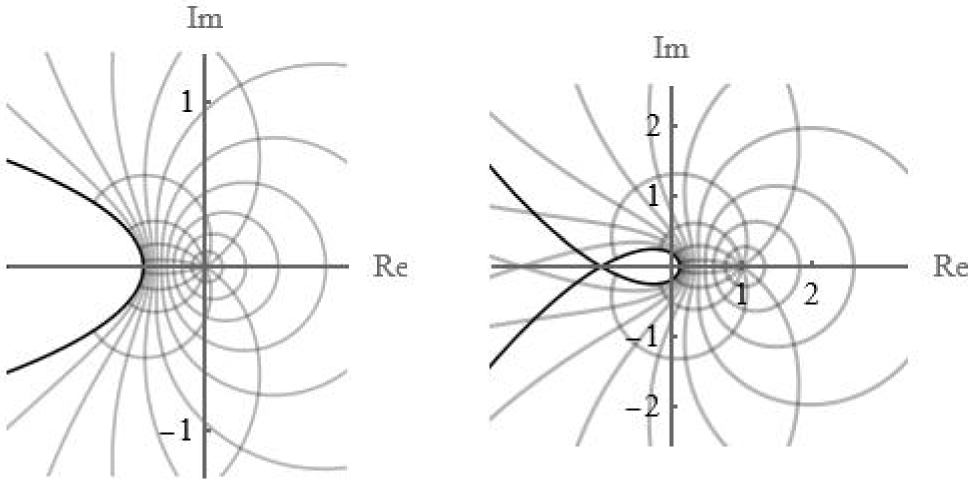

FIGURE 1. The image of $\mathbb{D}$ under $f_1(z)$ and $f_1'(z)$

We note that under some restriction on the coefficients, we can prove starlikeness of the convex functions of order $-1/2$. For example, Miller and Mocanu [3, p. 68] have shown a convex function f of order $-1/2$ is starlike of order $1/2$ if its second coefficient vanishes, that is, $f''(0) = 0$. In this paper, we investigate bounded turningness of the function f when the quantity $1 + \alpha z f''(z)/f'(z)$ takes values in certain sector in the right half-plane where $\alpha \geq 1$. In particular, we show that a function $f \in \mathcal{A}$ is has derivative f' with positive real part if

$$\left| \arg \left(1 + \frac{2}{3} \frac{z f''(z)}{f'(z)} \right) \right| < \arctan \frac{2}{3} \approx 0.588 \quad (z \in \mathbb{D}).$$

Our proof uses the following result of Nunokawa in [5] in the theory of first order differential subordination:

Lemma 1.1. *Let $p : \mathbb{D} \to \mathbb{C}$ be analytic, $p(0) = 1$ and $p(z) \neq 0$ for $z \in \mathbb{D}$. Suppose that there exists a point $z_0 \in \mathbb{D}$ such that $\operatorname{Re} p(z) > 0$ for $|z| < |z_0|$ and $\operatorname{Re} p(z_0) =$*

0, *where* $p(z_0) \neq 0$, *i.e.* $p(z_0) = ia$, a *is real and* $a \neq 0$. *Then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where $k \geq \frac{1}{2}\left(a + \frac{1}{a}\right)$, *when* $a > 0$, *and* $k \leq \frac{1}{2}\left(a + \frac{1}{a}\right)$, *when* $a < 0$.

2. Maine results

Theorem 2.1. *Let* $\alpha > 0$. *A function* $f \in \mathcal{A}$ *has derivative* f' *with positive real part if it satisfies the inequality*

$$\left|\arg\left(1 + \alpha \frac{zf''(z)}{f'(z)}\right)\right| < \arctan(\alpha) \quad (z \in \mathbb{D}). \tag{2.1}$$

Proof. Define the function $p : \mathbb{D} \to \mathbb{C}$ by

$$p(z) = f'(z).$$

Then, we have $p(0) = 1$ and

$$\frac{zp'(z)}{p(z)} = \frac{zf''(z)}{f'(z)}.$$

Using this, we see that the inequality (2.1) is equivalent to

$$\left|\arg\left(1 + \alpha \frac{zp'(z)}{p(z)}\right)\right| < \arctan(\alpha) \quad (z \in \mathbb{D}). \tag{2.2}$$

By applying Lemma 1.1, we show that $\operatorname{Re} p(z) > 0$ for all $z \in \mathbb{D}$ for any analytic function p satisfying the inequality (2.2).

First, we show that $p(z) \neq 0$ for all $z \in \mathbb{D}$. On a contrary, suppose that there exists $z_1 \in \mathbb{D}$ such that z_1 is the zero of order m of the function p. Then, it follows that $p(z) = (z - z_1)^m p_1(z)$, where m is positive integer, p_1 is analytic in $\mathbb{D}$ with $p_1(z_1) \neq 0$, and further

$$\frac{zp'(z)}{p(z)} = \frac{mz}{z - z_1} + \frac{zp_1'(z)}{p_1(z)}.$$

This means that the real part of the right hand side can tend to $-\infty$ when $z \to z_1$, which is a contradiction to the assumption of the theorem regarding the argument. Thus, we have $p(z) \neq 0$ for all $z \in \mathbb{D}$.

Now, suppose that the inequality (2.2) holds but p does not satisfy $\operatorname{Re} p(z) > 0$ for some $z \in \mathbb{D}$. Since $p(0) = 1$, there exists a point $z_0 \in \mathbb{D}$ such that $\operatorname{Re} p(z) > 0$ for $|z| < |z_0|$ and $\operatorname{Re} p(z_0) = 0$, where $p(z_0) \neq 0$. If we put $p(z_0) = ia$, a is real and

$a \neq 0$, then by Lemma 1.1 we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik,$$

where $k \geq \frac{1}{2}(a + 1/a)$, when $a > 0$, and $k \leq \frac{1}{2}(a + 1/a)$, when $a < 0$.

Next, define the function $\Phi : \mathbb{D} \to \mathbb{C}$ by

$$\Phi(z) = 1 + \alpha \frac{zp'(z)}{p(z)}.$$

Then, we have

$$\operatorname{Re} \Phi(z_0) = 1, \qquad \operatorname{Im} \Phi(z_0) = \alpha k.$$

For $a > 0$, this implies that

$$\arg \Phi(z_0) = \arctan(\alpha k) \geq \arctan\left(\alpha \frac{1}{2}(a + \frac{1}{a})\right) \geq \arctan(\alpha).$$

Similarly, for $a < 0$:

$$\arg \Phi(z_0) = \arctan(\alpha k) \leq \arctan\left(\alpha \frac{1}{2}(a + \frac{1}{a})\right) \leq \arctan(\alpha).$$

Combining the cases $a > 0$ and $a < 0$, we receive

$$|\arg \Phi(z_0)| \geq \arctan(\alpha),$$

which is a contradiction to the relation (2.2). This show that $\operatorname{Re} p(z) > 0$ for all $z \in \mathbb{D}$ proving that f' has positive real part in $\mathbb{D}$. □

For a convex function f, we know that $f'(z) \prec 1/(1-z)^2$ and therefore f' does not have positive real part. Also, we have $|\arg(1 + zf''(z)/f'(z))| < \pi/2 \approx 1.5708$ for a convex function f. If we restrict the argument to smaller number, we have the following result (obtained by taking $\alpha = 1$ in Theorem 2.1).

Corollary 2.1. *A function $f \in \mathcal{A}$ has derivative f' with positive real part if it satisfies the inequality*

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \arctan 1 = \pi/4 \quad (z \in \mathbb{D}).$$

Theorem 2.1 for the case $\alpha = 2/3$ gives the following result.

Corollary 2.2. *A function $f \in \mathcal{A}$ has derivative f' with positive real part if it satisfies the inequality*

$$\left|\arg\left(1+\frac{2}{3}\frac{zf''(z)}{f'(z)}\right)\right| < \arctan\frac{2}{3} \approx 0.588 \quad (z \in \mathbb{D}).$$

Corollary 2.3. *A function $f \in \mathcal{A}$ has derivative f' with positive real part if it satisfies the inequality*

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1 \quad (z \in \mathbb{D}).$$

Proof. It is known [2] that the largest disc centered at $z = a$ inside the sector $\{w : |\arg w| < \gamma\pi/2\}$ is of radius $R_a = a\sin(\pi\gamma/2)$. It then follows that the disc with center 1 and radius

$$\frac{\alpha}{\sqrt{\alpha^2+1}}$$

is contained in the sector $\{w : |\arg w| < \arctan(\alpha)\}$. By Theorem 2.1, it follows that the function f has derivative f' with positive real part if

$$\left|\alpha\frac{zf''(z)}{f'(z)}\right| = \left|\left(\alpha\frac{zf''(z)}{f'(z)}+1\right)-1\right| < \frac{\alpha}{\sqrt{\alpha^2+1}} \quad (z \in \mathbb{D}).$$

Thus, the function f has derivative f' with positive real part if

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{1}{\sqrt{\alpha^2+1}} \quad (z \in \mathbb{D}).$$

The desired result follows if we let $\alpha \to 0$. □

References

[1] P.L. Duren: *Univalent function*, Springer-Verlag, New York, 1983.

[2] A. Gangadharan, V. Ravichandran, T. N. Shanmugam: *Radii of convexity and strong starlikeness for some classes of analytic functions*, J. Math. Anal. Appl. **211**(1) (1997), 301–313.

[3] S.S. Miller, P.T. Mocanu: *Differential subordinations*, Marcel Dekker, Inc., New York, Bazel, 2000.

[4] W. Kaplan: *Close-to-convex schlicht functions*, Michigan Math. J., **1** (1952), 169–185.

[5] M. Nunokawa: *On properties of non-Caratheodory functions*, Proc. Japan Acad. **68**, Ser.A (1992), 152-153.

[6] M. Obradović, N. Tuneski: *New criteria for starlikeness in the unit disc, arXiv 2405.07997v1-29 Apr. 2024*

Department of Mathematics,
Faculty of Civil Engineering, University of Belgrade,
Bulevar Kralja Aleksandra 73,
11000, Belgrade, Serbia.
Email address: obrad@grf.bg.ac.rs

Department of Mathematics,
National Institute of Technology,
Tiruchirappalli–620 015, India
Email address: ravic@nitt.edu; vravi68@gmail.com