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SOME PROPERTIES OF TOTALLY ANTI-MAGIC GRAPHS

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ABSTRACT. We call an undirected graph with n edges *totally anti-magic* if in every labelling of edges by the numbers $1, 2, \ldots, n$ all nonzero vertex sums are distinct. We study totally anti-magic disjoint unions of stars and adding whiskers to any graph to achieve a totally anti-magic graph.

1. INTRODUCTION

For all the terminology the reader can consult [1].

A *graph* $\Gamma = (V(\Gamma), E(\Gamma))$ consists of a finite *vertex set* $V(\Gamma)$ and an *edge set*, where an edge is an unordered pair of distinct vertices of Γ. We will use the notation $x - y$ to denote an edge. When we have an edge $x - y$, we say that x and y are *adjacent*. The number of edges adjacent to a vertex v is called the *degree* of the vertx v. If Γ and Δ are two graphs with disjoint vertex sets, the *disjoint union* of Γ and Δ is the graph $\Gamma \cup \Delta$ whose set of vertices is $V(\Gamma \cup \Delta) = V(\Gamma) \cup V(\Delta)$ and the set of edges is $E(\Gamma \cup \Delta) = E(\Gamma) \cup E(\Delta)$.

An *edge labelling* (or shortly *labelling*) of a graph Γ is an assignement of a positive integer (i.e., *label*) to each edge of Γ. When we have a labelling on Γ, we say that Γ is a *labelled* graph. Let v be a vertex of a labelled graph Γ. The *vertex sum* of v, denoted by sum(v), is the sum of the labels assigned to all edges adjacent to

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v. If no edge is adjacent to v, then $\mathrm{sum}(v) = 0$. If v is a vertex of a graph Γ with n edgess and v is adjacent to k edges, then we define $\mathrm{sps}(v) = 1 + 2 + \cdots + k$ (*sps* stands for *"smallest possible sum"*) and $\mathrm{bps}(v) = n + (n-1) + \cdots + (n-k+1)$ (*bps* stands for *"biggest possible sum"*).

Definition 1.1. *A graph Γ with $|E(\Gamma)| = n$ is called totally anti-magic if for every labelling of Γ by the numbers $1, 2, \ldots, n$ all the vertices of nonzero degree have distinct vertex sums.*

On Figures 1-9 we give all totally anti-magic graphs that have up to four vertices.

Figure 1

Figure 2

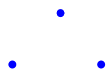

Figure 3

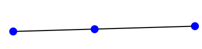

Figure 4

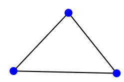

Figure 5

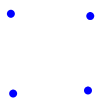

Figure 6

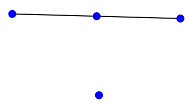

Figure 7

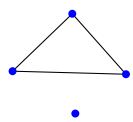

Figure 8

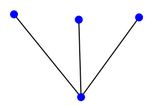

Figure 9

2. Totally anti-magic disjoint unions of stars

A *star* is a graph having a vertex (called *root*) to which all edges are adjacent. A star having zero edges consists of the root only. We denote a star with k vertices by $S(k)$. If $S(k_1), S(k_2), S(k_3), \ldots, S(k_n)$ are n stars with disjoint vertex sets, we denote $S(k_1, k_2, \ldots, k_n)$ the graph $S(k_1) \cup S(k_2) \cup \cdots \cup S(k_n)$ and call this graph a *disjoint union of n stars*.

Proposition 2.1.

(i) *For any $n \geq 2$ there are infinitely many n-tuples $(k_1, k_2, \ldots, k_n)$ such that $S(k_1, k_2, \ldots, k_n)$ is totally anti-magic.*

(ii) *For any fixed k_1 for which there exists a totally anti-magic $S(k_1, k_2, \ldots, k_n)$, the number of distinct totally anti-magic disjoint unions of n stars with the smallest star having k_1 elements is finite.*

Proof. (i) Consider the following conditions:

$$\text{sps}(k_1) > k_1 + k_2 + \cdots + k_n, \tag{1}$$

$$\text{bps}(k_1) < \text{sps}(k_2), \tag{2_1}$$

$$\text{bps}(k_2) < \text{sps}(k_3), \tag{2_2}$$

$$\dots$$

$$\text{bps}(k_{n-1}) < \text{sps}(k_n). \tag{2_{n-1}}$$

We claim that these conditions are sufficient for $S(k_1, k_2, \dots, k_n)$ to be totally anti-magic.

Indeed, suppose the conditions are satisfied. Let the edges of $S(k_1, k_2, \dots, k_n)$ be labelled by the numbers $1, 2, 3, \dots, n$ in any way. The condition (1) implies

$$\text{sps}(k_i) > k_1 + k_2 + \cdots + k_n \tag{3}$$

for any $i = 1, 2, \dots, n$. The condition (3) implies that

$$\text{sum}(u_i) > \text{sum}(v_{p,q}) \tag{4}$$

for any $p = 1, 2, \dots, n$, $q = 1, 2, \dots, k_p$, $i = 1, 2, \dots, n$. The conditions $(2_1) - (2_{n-1})$ imply that

$$\text{sum}(u_1) < \text{sum}(u_2) < \cdots < \text{sum}(u_n). \tag{5}$$

Because of (4) and (5) and the obvious fact that the sums at two different leaves are different, we can conclude that $S(k_1, k_2, \dots, k_n)$ is totally anti-magic.

The condition (1) can be written as

$$k_1 + 2k_2 + 2k_3 + \cdots + 2k_n < k_1^2. \tag{6}$$

The condition (2_i) can be written as

$$m + (m-1) + \cdots + (m - (k_i - 1)) < \frac{k_{i+1}(k_{i+1}+1)}{2},$$

where $m = k_1 + k_2 + \cdots + k_n$. This in turn can be written as

$$k_i^2 + \sum_{j=1,\, j\neq i}^{n} 2k_ik_j + k_i < k_{i+1}^2 + k_{i+1}. \tag{7_i}$$

The following condition is stronger than (7_i):

$$k_i^2 + \sum_{j=1,\, j\neq i}^{n} 2k_ik_j < k_{i+1}^2.$$

It can be written as

$$1 + 2\sum_{j=1,\, j\neq i}^{n} \frac{k_j}{k_i} < \left(\frac{k_{i+1}}{k_i}\right)^2,$$

i.e.,

$$(8_i) \qquad 1 + 2\sum_{j=1}^{i-1} \frac{k_j}{k_i} + 2\sum_{j=i+1}^{n} \frac{k_j}{k_i} < \left(\frac{k_{i+1}}{k_i}\right)^2.$$

The following condition is stronger than $(8.i)$:

$$(9_i) \qquad 2i - 1 + 2\sum_{j=i+1}^{n} \frac{k_j}{k_i} < \left(\frac{k_{i+1}}{k_i}\right)^2.$$

The condition (6) can be written as

$$(10) \qquad 1 + 2\frac{k_2}{k_1} + 2\frac{k_3}{k_1} + \cdots + 2\frac{k_n}{k_1} < k_1.$$

Thus the conditions (10) and (9_i) for $i = 1, 2, \ldots, n-1$, are sufficient for $S(k_1, k_2, \ldots, k_n)$ to be totally anti-magic.

If we now introduce the following notation:

$$A_1 = \frac{k_2}{k_1},\ A_2 = \frac{k_3}{k_2},\ \ldots,\ A_{n-1} = \frac{k_n}{k_{n-1}},$$

we can write these conditions in the following way:

$$(9'_{n-1}) \qquad 2n - 3 + 2A_{n-1} < A_{n-1}^2,$$

$$(9'_{n-2}) \qquad 2n - 5 + 2A_{n-2} + 2A_{n-2}A_{n-1} < A_{n-2}^2,$$

$$(9'_{n-3}) \qquad 2n - 7 + 2A_{n-3} + 2A_{n-3}A_{n-2} + 2A_{n-3}A_{n-2}A_{n-1} < A_{n-3}^2,$$

$$\ldots$$

$$(9'_1) \qquad 1 + 2A_1 + 2A_1A_2 + 2A_1A_2A_3 + \cdots + 2A_1A_2\ldots A_{n-1} < A_1^2,$$

$$(10') \qquad 1 + 2A_1 + 2A_1A_2 + 2A_1A_2A_3 + \cdots + 2A_1A_2\ldots A_{n-1} < k_1.$$

For $S(k_1, k_2, \ldots, k_n)$ to be totally anti-magic it is enough to find positive integers $A_1, A_2, \ldots, A_{n-1}, k_1$ satisfying these conditions. Clearly there are infinitely many positive integers A_{n-1} satisfying $(9'_{n-1})$. If we fix such an A_{n-1}, then there are infinitely many positive integers A_{n-2} satisfying $(9'_{n-2})$. And so on. There are infinitely many positive integers A_1 satisfying $(9'_1)$ when $A_{n-1}, A_{n-2}, \dots, A_2$ are fixed. (All this is true because of the form of the quadratic inequalities that we get.) Finally, if we fix one such integer A_1, we can select k_1 in infinitely many ways so that $(10')$ holds. Selection of $A_1, A_2, \dots, A_{n-1}$ and k_1 uniquely determines $k_2, k_3, \dots, k_n$. Thus (i) is proved.

(ii) Let k_1 be fixed. Note that the condition (1), i.e., (6), is necessary for $S(k_1, k_2, \dots, k_n)$ to be totally anti-magic. [If it is not satisfied, we can label $S(u_1)$ by $1, 2, \dots, k$, and then the label $\frac{k_1(k_1+1)}{2}$ will appear in some of $S(u_2), S(u_3), \dots, S(u_n)$. So some leaf of $S(u_2), S(u_3), \dots, S(u_n)$ will have the same sum as u_1.] It follows from (6) that $k_1^2 > 2k_i$ $(i = 2, 3, \dots, n)$, i.e., $k_i < \frac{k_1^2}{2}$. So there are only finitely many options for $k_2, k_3, \dots, k_n$. The statement (ii) is proved. □

Example 1. *Let* $n = 3$. *Let* $A_2 = 4$, $A_1 = 2A_2 + 3 = 11$, $k_1 = A_1^2 - A_1 + 2 = 112$. *Then* $k_2 = A_1 \cdot k_1 = 1232$, $k_3 = A_2 \cdot k_2 = 4928$. *So we have an example of a totally anti-magic disjoint union of three stars:* $S(112, 1232, 4928)$.

Remark 2.1. *There are infinitely many* $(k_1, k_2, \dots, k_n)$ *with* $1 \le k_1 \le k_2 \le \cdots \le k_n$ *such that* $S(k_1, k_2, \dots, k_n)$ *is totally anti-magic. This is true because in the proof of the previous proposition we can choose* k_1 *in infinitely many ways and each choice results in a totally anti-magic* $S(k_1, k_2, \dots, k_n)$.

Question 2.1. *We can raise the following questions:*

(i) *For any* $n \ge 3$, *what is the smallest* $k_1 = k_1(n)$ *for which there exists a totally anti-magic* $S(k_1, k_2, \dots, k_n)$?

(ii) *What is the smallest* k_1 *for which there exists a totally anti-magic* $S(k_1, k_2, k_3)$ *with* $k_1 \le k_2 \le k_3$? *What is the smallest* k_2 *for that* k_1? *What is the smallest* k_3 *for those* k_1 *and* k_2?

If $S(k_1)$ and $S(k_2)$ are two disjoint stars with roots u_1 and u_2 respectively, then the graph $\Gamma = (V(S(k_1)) \cup V(S(k_2)), E(S(k_1)) \cup E(S(k_2)) \cup \{u_1 - u_2\})$ is called the *root-connected two stars*. We denote Γ by $\overline{S}(k_1, k_2)$.

Proposition 2.2. *There are infinitely many pairs* (k_1, k_2) *of positive integers such that* $\overline{S}(k_1, k_2)$ *is totally anti-magic.*

Proof. We reason similarly as in Proposition 2.1. It is easy to see that the following two conditions

$$\text{sps}(u_1) > k_1 + k_2 + 1, \tag{11}$$

$$\text{bps}(u_1) < \text{sps}(u_2) \tag{12}$$

are sufficient for $\overline{S}(k_1, k_2)$ to be totally anti-magic. These two conditions have the following form:

$$2k_2 < k_1^2 + k_1 + 1, \tag{13}$$

$$k_1^2 + 2k_1k_2 + 3k_1 < k_2^2 + k_2. \tag{14}$$

We will now replace these two conditions by stronger conditions that are sufficient for $\overline{S}(k_1, k_2)$ to be totally anti-magic. We replace (3) by

$$2k_2 < k_1^2 + k_1. \tag{15}$$

We replace (4) by

$$k_1^2 + 2k_1k_2 + 3k_2 < k_2^2 + k_2,$$

i.e., with

$$k_1^2 + 2k_1k_2 + 2k_2 < k_2^2,$$

and then we replace this condition by an even stronger condition

$$k_1^2 + 4k_1k_2 < k_2^2. \tag{16}$$

Let $A = \frac{k_2}{k_1}$. Dividing (15) by k_1 and (16) by k_1^2 we obtain

$$2A < k_1 + 1, \tag{17}$$

$$1 + 4A < A^2. \tag{18}$$

The equation (18) has infinitely many solutions A. For any A which is a solution of (18) we can find infinitely many k_1 which are solutions of (17). Hence there are infinitely (k_1, k_2) such that $\overline{S}(k_1, k_2)$ is totally anti-magic. □

Example 2. *The smallest positive integer solution of (8) is $A = 5$ and then for that A the smallest solution of (7) is $k_1 = 10$. hence $k_2 = A \cdot k_1 = 50$. Thus we have a totally anti-magic graph $\overline{S}(10, 50)$.*

In the next theorem we will characterize all totally anti-magic graphs $\Gamma = S(k_1, k_2)$ with $1 \le k_1 \le k_2$.

Theorem 2.1. *The disjoint union $S(k_1, k_2)$ (with $1 \le k_1 \le k_2$) of two stars $S(k_1)$ and $S(k_2)$ is totally anti-magic if and only if (k_1, k_2) is one of the pairs from the following three sets:*

(i) $\{(4,5),(5,5),(5,8),(5,9),(6,7),(6,8),(6,11),(6,12),(6,14)\}$*;*

(ii) *the set of all (k_1, k_2) such that $k_1 \ge 7$ and*

$$\left\lceil \frac{2k_1 - 1 + \sqrt{8k_1^2 + 1}}{2} \right\rceil \le k_2 \le \frac{k_1(k_1-1)}{2} - 1;$$

(iii) *the set of all (k_1, k_2) such that $k_1 \ge 7$,*

$$k_1 \le k_2 \le \left\lceil \frac{2k_1 - 1 + \sqrt{8k_1^2 + 1}}{2} \right\rceil - 1$$

and $k_1 + k_2 \equiv 1$ or $2 \mod 4$.

Proof. Consider the following conditions:

$$\text{sps}(k_1) > m, \tag{19}$$

$$k_1 + k_2 \equiv 1 \text{ or } 2 \mod 4, \tag{20}$$

$$\text{bps}(k_1) < \text{sps}(k_2). \tag{21}$$

It is easy to prove that:

(a) the condition (19) is necessary for Γ to be totally anti-magic;

(b) the condition “(19) and ((20) or (21))" is sufficient for Γ to be totally anti-magic.

Also it is easy to see that all Γ with $k_1 = 1, 2,$ or 3 are not totally anti-magic, so we assume from now on that $k_1 \ge 4$.

The condition (19) can be written as

$$k_2 \le \frac{k_1(k_1-1)}{2} - 1. \tag{22}$$

Hence, due to (a) above, in order to find all totally anti-magic Γ’s, we need to consider only the possibilities:

$$k_2 = k_1,\ k_1 + 1,\ k_1 + 2,\ \ldots,\ \frac{k_1(k_1-1)}{2} - 1. \tag{23}$$

The condition (21) is equivalent with

(24) $$k_2 > \frac{2k_1 - 1 - \sqrt{8k_1^2+1}}{2}.$$

Note that

$$\frac{2k_1 - 1 - \sqrt{8k_1^2+1}}{2} > k_1.$$

The condition

$$\frac{2k_1 - 1 - \sqrt{8k_1^2+1}}{2} < \frac{k_1(k_1-1)}{2} - 1$$

is equivalent with

$$3k_1 + 1 + \sqrt{8k_1^2+1} < k_1^2.$$

If $k_1 > 6$, we have

$$1 + \sqrt{8k_1^2+1} < 3k_1,$$

hence

$$3k_1 + 1 + \sqrt{8k_1^2+1} < 6k_1 < k_1^2.$$

Thus if $k_1 > 6$, then in the sequence (23) all pairs (k_1, k_2) with

(25) $$\frac{2k_1 - 1 - \sqrt{8k_1^2+1}}{2} \leq k_2 \leq \frac{k_1(k_1-1)}{2} - 1$$

are totally anti-magic. It is easy to see that for $k_1 = 4, 5, 6$ the following are totally anti-magic pairs (k_1, k_2):

(26) $$\{(4,5),(5,5),(5,8),(5,9),(6,7),(6,8),(6,11),(6,12),(6,14)\}.$$

In order to find all totally anti-magic pairs (k_1, k_2), in addition to (25) and (26), we have to consider (for $k_1 \geq 7$) the pairs (k_1, k_2) such that

(27) $$k_1 \leq k_2 \leq \left\lceil \frac{2k_1 - 1 + \sqrt{8k_1^2+1}}{2} \right\rceil - 1.$$

Claim. Assume $k_1 \geq 7$. A pair (k_1, k_2), such that k_2 is from the sequence (27), is totally anti-magic if and only if $k_1 + k_2 \equiv 1$ or $2 \mod 4$.

Proof of the claim. $(\Rightarrow)$ Follws from (b) above.

$(\Leftarrow)$ Suppose that $k_1 \geq 7$, k_2 is from the sequence (27) and $k_1 = k_2 \equiv 0$ or $3 \mod 4$. We will show that then Γ is not totally anti-magic.

We first consider the case $k_1 \equiv 0 \mod 4$. Let $k_1 = 4l$ ($l \geq 2$). We have the following table:

TABLE 1.

k_1	k_2	bps(u_1) − sps(u_2)	
$4l$	$4l$	$16l^2$	
$4l$	$4l+1$	$16l^2-1$	tam
$4l$	$4l+2$	$16l^2-3$	tam
$4l$	$4l+3$	$16l^2-6$	
$\cdots$	$\cdots$	$\cdots$	
$4l$	$4l+t$	$16l^2-\frac{t(t+1)}{2}$	
$\cdots$	$\cdots$	$\cdots$	

We need to show that in each in which $k_1+k_2 \equiv 0$ or $3 \mod 4$, the pair (k_1,k_2) is not totally anti-magic. Since

$$\left\lceil \frac{2k_1-1+\sqrt{8k_1^2+1}}{2} \right\rceil - 1 = 4l + \left\lceil \frac{\sqrt{128l^2+1}-1}{2} \right\rceil - 1,$$

every t that we have in Table 1 satisfies

$$t \le \lceil \frac{\sqrt{128l^2+1}-1}{2} \rceil - 1. \tag{28}$$

Note that we consider only $t \equiv 0$ or $3 \mod 4$. For such t's the sum $1+2+\cdots+t$ is even. Now fix a t from $\{0,1,2,\ldots,\lceil \frac{\sqrt{128l^2+1}-1}{2} \rceil - 1\}$, such that $t \equiv 0$ or $3 \mod 4$. We want to show that $S(k_1,k_2)$ is not totally anti-magic. We will find a lebelling such that sum(u_1) = sum(u_2). Initially, put the biggest possible sum on the first star and the smallest possible sum on the second star. So the labels on the first star are:

$$4l+t+1,\ 4l+t+2,\ \ldots,\ 8l+t. \tag{29}$$

The labels on the second star are:

$$1,\ 2,\ \ldots,\ t, \tag{30}$$

$$t+1,\ t+2,\ \ldots,\ 4l+t. \tag{31}$$

Consider the following possible switches ($4l$ of them)

$4l+t+1$	$\leftrightarrow$	$t+1$
$4l+t+2$	$\leftrightarrow$	$t+2$
...	...	...
$8l+t$	$\leftrightarrow$	$4l+t$

We call them "$4l$-switches" since the difference in each of them is $4l$. Initially,

$$\text{sum}(u_1) - \text{sum}(u_2) = 16l^2 - \frac{t(t+1)}{2} > 0. \tag{32}$$

If we perform $2l$ $4l$-switches, then

$$\text{sum}(u_1) - \text{sum}(u_2) = -\frac{t(t+1)}{2} < 0. \tag{33}$$

Hence, there is an integer $x \geq 0$ and $< 2l$ such that

$$x \cdot 4l \leq 8l^2 - \frac{t(t+1)}{4}, \tag{34}$$

but

$$(x+1) \cdot 4l > 8l^2 - \frac{t(t+1)}{4}, \tag{35}$$

For this x we have

$$0 \leq 8l^2 - \frac{t(t+1)}{4} - 4xl < 4l. \tag{36}$$

Denote $d = 8l^2 - \frac{t(t+1)}{4} - 4xl$. Since $0 \leq d < 4l$ (that is (36)), we can select a label a on the first star and the label b on the second star such that $a - b = d$. (The differences between the labels on the first star and the labels on the second star go from 1 to $8l + t - 1$.) We first switch these two labels. This switch eliminates the possibilities of two $4l$-switches. But we still have at our disposition $4l - 2$ $4l$-switches. So we can now do x $4l$-switches.The sum of differences in all these switches is precisely $8l^2 - \frac{k(k+1)}{4}$. Hence after these switches we have $\text{sum}(u_1) = \text{sum}(u_2)$.

Next we consider the case $k_1 \equiv 1 \mod 4$. Let $k_1 = 4l + 1$ ($l \geq 2$). We have Table 2.

We need to show that in each row in which $k_1 + k_2 \equiv 0$ or $3 \mod 4$, the pair (k_1, k_2) is not totally anti-magic. Since

$$\left\lceil \frac{2k_1 - 1 + \sqrt{8k_1^2 + 1}}{2} \right\rceil - 1 = 4l + 1 + \left\lceil \frac{\sqrt{8(4l+1)l^2 + 1} - 1}{2} \right\rceil - 1,$$

every t that we have in the table (30) satisfies

$$t \leq \lceil \frac{8(4l+1)^2 + 1}{2} \rceil - 1.$$

TABLE 2. Table

k_1	k_2	bps$(u_1)-$sps(u_2)	
$4l+1$	$4l+1$	$(4l+1)^2$	tam
$4l+1$	$4l+2$	$(4l+1)^2-1$	
$4l+1$	$4l+3$	$(4l+1)^2-3$	
$4l+1$	$4l+4$	$(4l+1)^2-6$	tam
...	...	...	
$4l+1$	$4l+t$	$(4l+1)^2-\frac{(t+1)}{2}$	
...	...	...	

Note that we consider only $t \equiv 2$ or $3 \mod 4$. For such t's the sum $\frac{(t-1)t}{2}$ is odd (so $(4l+1)^2-\frac{(t-1)t}{2}$ is even).

Now fix a t from $\{0,1,2,\ldots,\lceil\frac{8(4l+1)^2+1}{2}\rceil-1\}$, such that $t \equiv 2$ or $3 \mod 4$. We want to show that $S(k_1,k_2)$ is not totally anti-magic. We will find a lebelling such that sum$(u_1)=$ sum(u_2). Initially, put the biggest possible sum on the first star and the smallest possible sum on the second star. So the labels on the first star are:

$$4l+t+1,\ 4l+t+2,\ \ldots,\ 8l+t+1. \tag{37}$$

The labels on the seond star are

$$1,\ 2,\ \ldots,\ t-1,\ t,\ t+1,\ \ldots,\ 4l+t. \tag{38}$$

Note that if $t=1$, the graph $\Gamma=S(4l+1,4l+1)$ is totally anti-magic since the sum $1+2+\cdots+8l+2=\frac{(8l+2)(8l+3)}{2}$ is odd. Let $t>1$ (i.e., $t-1\geq 0$). Consider the following switches ($4l+1$ of them):

$4l+t+1$	$\leftrightarrow$	$t-1$
$4l+t+2$	$\leftrightarrow$	t
...	...	...
$8l+t+1$	$\leftrightarrow$	$4l+t-1$

We call them "$4l+2$-switches" since the difference in each of them is $4l+2$. The difference bps$(u_1)-$sps(u_2) from Table B can be written as:

$$(4l+1)^2-\frac{(t-1)t}{2}=4l(4l+2)-\frac{t^2-t-2}{2},$$

where both $4l(4l+2)$ and $\frac{t^2-t-2}{2}$ are even. The half of that difference is $2l(4l+2) - \frac{t^2-t-2}{4}$. Initially

$$\text{sum}(u_1) - \text{sum}(u_2) = 4l(4l+2) - \frac{t^2-t-2}{2} > 0. \tag{39}$$

If we perform $2l$ $4l+2$-switches, then

$$\text{sum}(u_1) - \text{sum}(u_2) = -\frac{t^2-t-2}{2} < 0. \tag{40}$$

Hence there is an integer $x \geq 0$ and $< 2l$ such that

$$x \cdot (4l+2) \leq 2l(4l+2) - \frac{t^2-t-2}{4}, \tag{41}$$

but

$$(x+1) \cdot (4l+2) > 2l(4l+2) - \frac{t^2-t-2}{4}. \tag{42}$$

For this x we have

$$0 \leq 2l(4l+2) - \frac{t^2-t-2}{4} - x \cdot (4l+2) < 4l+2. \tag{43}$$

Denote $d = 2l(4l+2) - \frac{t^2-t-2}{4} - x \cdot (4l+2)$. Since $0 \leq d < 4l+2$ (that is (34)), we can select a label a on the first star and a label b on the second star such that $a - b = d$. (The difference between the labels on the first star and the labels on the second star go from 1 to $8l+t$.) We first switch these two labels. This switch eliminates the possibilities of two $4l+2$-switches. But we still have at our disposition $4l-1$ $4l+2$-switches. So we can now do x $4l+2$-switches. The sum of differences in all these switches is precisely $2l(4l+2) - \frac{t^2-t-2}{4}$. Hence after these switches we have $\text{sum}(u_1) = \text{sum}(u_2)$. □

3. Transforming a graph into a totally anti-magic graph by adding whiskers

Let Γ be a graph, $x \in V(\Gamma)$, $y \notin V(\Gamma)$. Then the edge $x - y$ is called a *whisker* for Γ. We say that the graph $\Gamma' = (V(\Gamma) \cup \{y\}, E(\Gamma) \cup \{x - y\})$ is obtained from Γ by adding the whisker $x - y$ to it.

Conjecture. *To any graph Γ one can add finitely many whiskers so that the new graph obtained in that way is totally anti-magic.*

Proposition 3.1. *Let* Γ *be a connected graph with* ≤ 4 *vertices. Then the above conjecture holds for* Γ.

Proof. Figures 10-12 represent all connected graphs having up to four vertices that are not totally anti-magic. The proof for graphs on Figures 10 and 11 follows from Proposition 2.2. We proceed with a proof for the graph on Figure 12. The proof is similar to the proof of Proposition 2.1. It is easy to see that we need to add whiskers to each of the four vertices , otherwise the new graph would not be totally anti-magic. We will show that we can add k_i whiskers to the root u_i $(i = 1, 2, 3, 4)$, as on Figure 13, with $1 \leq k_1 \leq k_2 \leq k_3 \leq k_4$. Denote the resulting graph $\overline{\Gamma}$.

The following conditions are sufficient for $\overline{\Gamma}$ to be totally anti-magic.

$$\text{sps}(u_1) > k_1 + k_2 + k_3 + k_4 + 4, \tag{44}$$

$$\text{bps}(u_1) < \text{sps}(u_2), \tag{45}$$

$$\text{bps}(u_2) < \text{sps}(u_3), \tag{46}$$

$$\text{bps}(u_3) < \text{sps}(u_4), \tag{47}$$

Figure 10

Figure 11

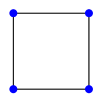

Figure 12

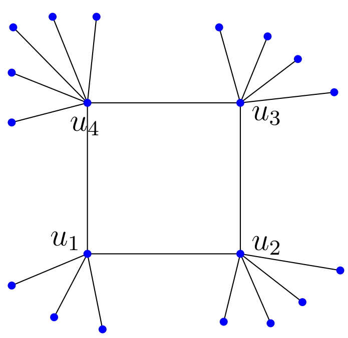

Figure 13

The conditon (44) can be written as

$$2k_2 + 2k_3 + 2k_4 + 2 < k_1^2 + 3k_1.$$

Since it is easy to see that $k_1 \geq 2$ the following condition is stronger than this one:

$$2k_2 + 2k_3 + 2k_4 < k_1^2 + 2k_1,$$

i.e.,

$$2\,\frac{k_2}{k_1} + 2\,\frac{k_3}{k_1} + 2\,\frac{k_4}{k_1} < k_1. \tag{48}$$

For any n, we will denote the sum $1 + 2 + 3 + \cdots + n$ by $\Sigma(n)$. The conditions (45)-(47) can be written as

$$\begin{aligned}
\Sigma(k_1 + k_2 + k_3 + k_4 + 4) &< \Sigma(k_2 + k_3 + k_4 + 2) + \Sigma(k_2 + 2),\\
\Sigma(k_1 + k_2 + k_3 + k_4 + 4) &< \Sigma(k_1 + k_3 + k_4 + 2) + \Sigma(k_3 + 2),\\
\Sigma(k_1 + k_2 + k_3 + k_4 + 4) &< \Sigma(k_1 + k_2 + k_4 + 2) + \Sigma(k_4 + 2).
\end{aligned}$$

After some calculations these conditions can be written as

$$\begin{aligned}
k_1^2 + 2k_1k_2 + 2k_1k_3 + 2k_1k_4 + 9k_1 + 4k_3 + 4k_4 + 12 &< k_2^2 + k_2,\\
k_2^2 + 2k_1k_2 + 2k_2k_3 + 2k_2k_4 + 4k_1 + 9k_2 + 4k_4 + 12 &< k_3^2 + k_3,\\
k_3^2 + 2k_1k_3 + 2k_2k_3 + 2k_3k_4 + 4k_1 + 4k_2 + 9k_3 + 12 &< k_4^2 + k_4.
\end{aligned}$$

We will replace these conditions by the following stronger ones:

$$\begin{aligned}
k_1^2 + 2k_1k_2 + 2k_1k_3 + 2k_1k_4 + 9k_1 + 4k_3 + 4k_4 + 12 &< k_2^2 + k_2,\\
k_2^2 + 2k_1k_2 + 2k_2k_3 + 2k_2k_4 + 4k_1 + 9k_2 + 4k_4 + 12 &< k_3^2 + k_3,\\
k_3^2 + 2k_1k_3 + 2k_2k_3 + 2k_3k_4 + 4k_1 + 4k_3 + 9k_3 + 12 &< k_4^2 + k_4.
\end{aligned}$$

We will replace these conditions by the following stronger ones:

$$\begin{aligned}
k_1^2 + 2k_1k_2 + 2k_1k_3 + 2k_1k_4 + 8k_1 + 4k_3 + 4k_4 + 12 &< k_2^2,\\
k_2^2 + 2k_1k_2 + 2k_2k_3 + 2k_2k_4 + 4k_1 + 8k_2 + 4k_4 + 12 &< k_3^2,\\
k_3^2 + 2k_1k_3 + 2k_2k_3 + 2k_3k_4 + 4k_1 + 4k_2 + 8k_3 + 12 &< k_4^2.
\end{aligned}$$

The following are stronger conditions than these:

$$k_1^2 + 10k_1k_2 + 6k_1k_3 + 6k_1k_4 + 12 < k_2^2,$$
$$k_2^2 + 6k_1k_2 + 10k_2k_3 + 6k_2k_4 + 12 < k_3^2,$$
$$k_3^2 + 6k_1k_3 + 6k_2k_3 + 10k_3k_4 + 12 < k_4^2.$$

and these are stronger than these:

$$k_1^2 + 14k_1k_2 + 10k_1k_3 + 10k_1k_4 < k_2^2,$$
$$k_2^2 + 10k_1k_2 + 14k_2k_3 + 10k_2k_4 < k_3^2,$$
$$k_3^2 + 10k_1k_3 + 10k_2k_3 + 14k_3k_4 < k_4^2.$$

These conditions in turn can be written as

$$1 + 14\,\frac{k_2}{k_1} + 10\,\frac{k_3}{k_1} + 10\,\frac{k_4}{k_1} < \left(\frac{k_2}{k_1}\right)^2,$$
$$1 + 10\,\frac{k_2}{k_1} + 14\,\frac{k_3}{k_1} + 10\,\frac{k_4}{k_1} < \left(\frac{k_3}{k_2}\right)^2,$$
$$1 + 10\,\frac{k_1}{k_3} + 10\,\frac{k_2}{k_3} + 14\,\frac{k_4}{k_3} < \left(\frac{k_4}{k_3}\right)^2.$$

Finally we replace these by the following stronger conditions:

$$1 + 14\,\frac{k_2}{k_1} + 10\,\frac{k_3}{k_1} + 10\,\frac{k_4}{k_1} < \left(\frac{k_2}{k_1}\right)^2, \tag{49}$$

$$11 + 14\,\frac{k_3}{k_2} + 10\,\frac{k_4}{k_2} < \left(\frac{k_3}{k_2}\right)^2, \tag{50}$$

$$21 + 14\,\frac{k_4}{k_3} < \left(\frac{k_4}{k_3}\right)^2. \tag{51}$$

We now introduce the following notation:

$$A_1 = \frac{k_2}{k_1}, A_2 = \frac{k_3}{k_2}, A_3 = \frac{k_4}{k_3}.$$

Then the conditions (48)-(51) can be written as

$$21 + 14A_3 < A_3^2, \tag{52}$$

$$11 + 14A_2 + 10A_2A_3 < A_2^2, \tag{53}$$

$$1 + 14A_1 + 10A_1A_2 + 10A_1A_2A_3 < A_1^2, \tag{54}$$

$$2A_1 + 2A_1A_2 + 2A_1A_2A_3 < k_1. \tag{55}$$

We can solve (52) for A_3, then fixing one solution A_3 for (52) we can solve (53) for A_2, then fixing one solution A_2 for (53) we can solve (54) for A_3, finally fixing one solution A_3 for (54) we can solve (55). From these solutions we get k_1, k_2, k_3, k_4. If we add that many whiskers to the vertices on Figure 13 the resulting graph is totally anti-magic. □

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