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ON THE EXISTENCE AND UNIQUENESS OF CLASSICAL SOLUTION FOR AN INITIAL-BOUNDARY VALUE PROBLEM FOR A DISCRETE BOLTZMANN SYSTEM IN TWO SPACE DIMENSIONS

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ABSTRACT. The initial-boundary value problem for the two-dimensional regular four-velocity discrete boltzmann system is analyzed in a rectangle. The existence and uniqueness of classical global positive solution, bounded with its first partial derivatives are proved for a range of bounded data by the use of fixed points tools. A bound for the solution and its partial derivatives is provided.

1. INTRODUCTION

Discrete velocity models of gas are simplified models of the Boltzmann equation obtained by assuming that the velocities of the gas particles belong to a finite set of vectors. The nonlinear integro-partial derivative Boltzmann equation is replaced by a system of semilinear hyperbolic equations associated to the number densities of the particles having the given velocities. After the pionner works of Broadwell [1, 2] who introduced the first physically convenient models in the sense that they can model actual gas flows and the theory for the general discrete velocity model for binary collision given by Gatignol [3], the discrete kinetic theory of

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gas develops in two directions: the mathematical study of the kinetic equations encompassing the existence and the uniqueness theory as well as the construction of exact solutions and the modelling and the resolution of flow problems.

The existence theory for the discrete velocity models of the Boltzmann equation, not only supports the mathematical understanding of these models but also underpins the reliability of numerical methods used in engineering and physics. The good mathematical structure of the kinetic equations associated to these models lead to the rapid development of the mathematical theory of discrete velocity models. Many results concerning the proof of the global existence and the uniqueness of the solutions of the initial-boundary value problem have been obtained in the one-dimensional case [4, 5, 10, 12, 13]. Most of these studies concern the so-called three velocity and four velocity Broadwell models which are the symmetrical models obtained from the six and the eight velocity spatial models of Broadwell by a symmetry with respect to any coordinate plane and axis respectively. Exact solutions have been proposed for the three velocity Broadwell model [6].

The situation is quite different for multi-dimensional problems even in the steady case. In [19], using techniques based on the fractional steps method, the problem of existence and uniqueness of the solution of the initial boundary value problem is solved for the two velocity Carleman model. In the steady case, the boundary value problem for the general two-dimensional four velocity Broadwell model is investigated in [9, 15–18] the existence of a solution is proved and exact solutions are built. An extension to a fifteen velocity three speed discrete model is done in [16].

In this work, the initial-boundary value problem for a two-dimensional four velocity model of Broadwell is considered in a rectangle; we prove for a range of bounded initial and boundary data, the existence and uniqueness of the classical global positive solution which is bounded with its first partial derivatives and we provide a bound for the solution and its partial derivatives.

The paper is organized as follows. In section 2 we briefly describe the model, state the initial-boundary value problem and present the main result of the paper which is proved in section 4. In section 3 we establish the positivity of the solution of the initial-boundary value problem.

2. Statement of the problem

2.1. The discrete velocity model.

The general plane four velocity discrete models of Broadwell denoted by B_θ, $\theta \in \left[0, \frac{\pi}{2}\right[$ are among the simplest discrete velocity models and have been used to study initial and boundary value problems in one dimension [2,8,17] and to build exact solutions [9,15]. In the basis $(\vec{e_1}, \vec{e_2})$ of orthonormal reference $(O, \vec{e_1}, \vec{e_2})$ of the plane $\mathbb{R}^2$ its velocities are $\vec{u_1} = c(cos\theta, sin\theta)$, $\vec{u_2} = c(-sin\theta, cos\theta)$, $\vec{u_3} = -\vec{u_2}$, $\vec{u_4} = -\vec{u_1}$, where $\theta = angle(\vec{e_1}, \vec{u_1})$ accounts of the orientation of the discrete velocity model with respect to the reference.

Let $N_i(t, x, y)$ be the number density of the gas molecules with velocity $\vec{u}_i$, $i = 1, 2, 3, , 4$ at the time t and at the position $M(x, y)$ the kinetic equations of the model are:

(2.1)
$$\begin{cases} \frac{\partial N_1}{\partial t} + ccos\theta \frac{\partial N_1}{\partial x} + csin\theta \frac{\partial N_1}{\partial y} & = & Q \\ \frac{\partial N_2}{\partial t} - csin\theta \frac{\partial N_2}{\partial x} + ccos\theta \frac{\partial N_2}{\partial y} & = & -Q \\ \frac{\partial N_3}{\partial t} + csin\theta \frac{\partial N_3}{\partial x} - ccos\theta \frac{\partial N_3}{\partial y} & = & -Q \\ \frac{\partial N_4}{\partial t} - ccos\theta \frac{\partial N_4}{\partial x} - csin\theta \frac{\partial N_4}{\partial ye} & = & Q, \end{cases}$$

$$Q = 2cs\left(N_2 N_3 - N_1 N_4\right).$$

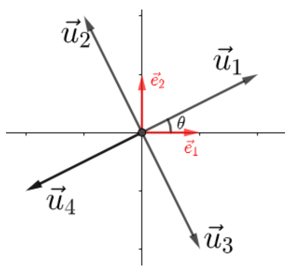

Figure 1. The model B_θ

The total density ρ and the macroscopic velocity $\vec{U}(U, V)$ of a gas described by the model are defined by:

$$
\begin{aligned}
\rho| &= N_1 + N_2 + N_3 + N_4, \\
\rho U &= \cos(\theta)[N_1 - N_4] - \sin(\theta)[N_2 - N_3], \\
\rho V &= \sin(\theta)[N_1 - N_4] + \cos(\theta)[N_2 - N_3].
\end{aligned} \tag{2.2}
$$

The Maxwellian densities of the model associated with the macroscopic variables ρ, U and V are given by the relations:

$$
\begin{cases}
N_{1M} = \frac{\rho}{4}\left[1 + \cos(2\theta)\,(u^2 - v^2) + 2uv\sin(2\theta) + 2u\cos(\theta) + 2v\sin(\theta)\right] \\
N_{2M} = \frac{\rho}{4}\left[1 - \cos(2\theta)\,(u^2 - v^2) - 2uv\sin(2\theta) + 2v\cos(\theta) - 2u\sin(\theta)\right] \\
N_{3M} = \frac{\rho}{4}\left[1 - \cos(2\theta)\,(u^2 - v^2) - 2uv\sin(2\theta) - 2v\cos(\theta) + 2u\sin(\theta)\right] \\
N_{4M} = \frac{\rho}{4}\left[1 + \cos(2\theta)\,(u^2 - v^2) + 2uv\sin(2\theta) - 2u\cos(\theta) - 2v\sin(\theta)\right]
\end{cases} \tag{2.3}
$$

The mild problem in consideration in the sequel results from the modelling of a gas flow in a rectangular box by the model B_0.

2.2. Initial-boundary value problem.

Given $\Omega = [a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2$, $I = [0; T] \subset \mathbb{R}$, we set $\mathscr{P} = [0; T] \times [a_1; b_1] \times [a_2; b_2]$ and consider the system Σ^0 defined by:

$$\frac{\partial N_1}{\partial t} + c\frac{\partial N_1}{\partial x} = Q(N),\ (t, x, y) \in \mathring{\mathscr{P}} \tag{2.4}$$

$$\frac{\partial N_2}{\partial t} + c\frac{\partial N_2}{\partial y} = -\,Q(N),\ (t, x, y) \in \mathring{\mathscr{P}} \tag{2.5}$$

$$\frac{\partial N_3}{\partial t} - c\frac{\partial N_3}{\partial y} = -\,Q(N),\ (t, x, y) \in \mathring{\mathscr{P}} \tag{2.6}$$

$$\frac{\partial N_4}{\partial t} - c\frac{\partial N_4}{\partial x} = Q(N),\ (t, x, y) \in \mathring{\mathscr{P}} \tag{2.7}$$

$$N_i\,(0, x, y) = N_i^0\,(x, y)\,,\ (x, y) \in [a_1; b_1] \times [a_2; b_2]\,, i = 1, \cdots, 4 \tag{2.8}$$

$$N_1\,(t, a_1, y) = N_1^-\,(t, y)\,,\ (t, y) \in [0; T] \times [a_2; b_2] \tag{2.9}$$

$$N_2\,(t, x, a_2) = N_2^-\,(t, x)\,,\ (t, x) \in [0; T] \times [a_1; b_1] \tag{2.10}$$

$$N_3\,(t, x, b_2) = N_3^+\,(t, x)\,,\ (t, x) \in [0; T] \times [a_1; b_1] \tag{2.11}$$

$$N_4\,(t, b_1, y) = N_4^+\,(t, y)\,,\ (t, y) \in [0; T] \times [a_2; b_2] \tag{2.12}$$

$$N_1^0\,(a_1, y) = N_1^-\,(0, y)\,,\ y \in [a_2; b_2] \tag{2.13}$$

$$N_2^0(x, a_2) = N_2^-(0, x), \; x \in [a_1; b_1] \tag{2.14}$$

$$N_3^0(x, b_2) = N_3^+(0, x), \; x \in [a_1; b_1] \tag{2.15}$$

$$N_4^0(b_1, y) = N_4^+(0, y), \; y \in [a_2; b_2] \tag{2.16}$$

where

$$Q(N) = 2cS(N_2N_3 - N_1N_4), \quad N_i^0, i = 1, \ldots, 4, \tag{2.17}$$

are the initial data and $N_1^-, N_2^-, N_3^+, N_4^+$ the boundary data. We assume in the sequel that $N_i^0, (i = 1, 2, 3, 4), N_1^-, N_2^-, N_3^+, N_4^+$ are non-negative and continuous, that they have bounded and continuous first order partial derivatives. In the sequel, $C(X, Y)$ denotes the set of continuous functions from the set X into the set Y. Our aim is to prove for the system Σ^0, the existence of non-negative solutions in $C\left(\mathscr{P}; \mathbb{R}^4\right)$ (thus bounded solutions) that have bounded first order partial derivatives.

2.3. Main theorem.

Notation 2.1. *For every function $u : X \longrightarrow \mathbb{R}$ whose domain is $D_u \subset X$ such that u is bounded on D_u, let us denote $\|u\|_\infty = \sup_{x \in D_u} |u(x)|$ and for $U = (U_i)_{i=1}^4 : X \longrightarrow \mathbb{R}^4$, such that every $U_i : D_u \left(\subset X\right) \longrightarrow \mathbb{R}$ is bounded, $\|U\| = \max_{1 \le i \le 4} \|U_i\|_\infty$.*

Notation 2.2. *For $u : D_u \longrightarrow \mathbb{R}$ such that $u \equiv u(\alpha, \beta)$ is bounded on D_u and such that $\frac{\partial u}{\partial \alpha}, \frac{\partial u}{\partial \beta}$ are bounded on their domain, let us set*

$$\|u\|_1 = \max\left\{\|u\|_\infty, \left\|\frac{\partial u}{\partial \alpha}\right\|_\infty, \left\|\frac{\partial u}{\partial \beta}\right\|_\infty\right\}.$$

Let us consider the following parameters

$$p \equiv 4cS\left(1 + 2 \cdot \max\left\{4T; \frac{2}{c}(b_1 - a_1); \frac{1}{c}(b_2 - a_2)\right\}\right);$$

$$q \equiv \max\left\{\max\{1; 2c\}\left\|N_1^0\right\|_1; (1+c)\left\|N_1^-\right\|_1; \max\{1; 2c\}\left\|N_2^0\right\|_1;\right.$$

$$\max\{2;(1+c)\}\left\|N_2^-\right\|_1;\max\{1;2c\}\left\|N_3^0\right\|_1;\max\{2;(1+c)\}\left\|N_3^+\right\|_1;$$
$$\left.\max\{1;2c\}\left\|N_4^0\right\|_1;(2+c)\left\|N_4^+\right\|_1\right\}.$$

We prove in the sequel the following result.

Theorem 2.1. *Suppose* $pq \le \frac{1}{4}$. *Then the system* Σ^0 (2.4)-(2.12) *has an unique non-negative solution*

$$N=(N_1,N_2,N_3,N_4)\in C\left([0;T]\times[a_1,b_1]\times[a_2,b_2];\mathbb{R}^4\right)$$

such that

$$\|N\| \le \frac{1+\sqrt{1-4pq}}{2p}, \tag{2.18}$$

$\frac{\partial N_i}{\partial t}, \frac{\partial N_i}{\partial x}, \frac{\partial N_i}{\partial y}$ *are defined in* $]0;T[\times]a_1,b_1[\times]a_2,b_2[$ *except possibly on a finite number of planes including the four planes with respective equations*

$$-ct+x=a_1;-ct+y=a_2;ct+y=b_2;ct+x=b_1; \tag{2.19}$$

$\frac{\partial N_i}{\partial t}, \frac{\partial N_i}{\partial x}, \frac{\partial N_i}{\partial y}$ *are continuous and bounded, for* $i=1,2,3,4$, *and satisfy*

$$\begin{aligned}&\max_{1\le i\le 4}\left\{\|N_i\|_\infty,\left\|\frac{\partial N_i}{\partial t}\right\|_\infty,\left\|\frac{\partial N_i}{\partial x}\right\|_\infty,\left\|\frac{\partial N_i}{\partial y}\right\|_\infty\right\}\\ &\le \max\left\{1,\frac{2}{c}\right\}\frac{1+\sqrt{1-4pq}}{2p}.\end{aligned} \tag{2.20}$$

3. Non-negativity of the solution

Let's consider the change of variables $\mathscr{F}:(t,x,y)\longmapsto(\eta_1,\eta_2,\eta_3)$ such that $\eta_1=x/c$, $\eta_2=t/2-x/2c+y/2c$ and $\eta_3=t/2-x/2c-y/2c$. We have $\frac{D(\eta_1,\eta_2,\eta_3)}{D(t,x,y)}=\frac{1}{2c^2}\neq 0$ and the inverse of $\mathscr{F}$ is defined by $\mathscr{F}^{-1}:(\eta_1,\eta_2,\eta_3)\longmapsto(t,x,y)$ such that $t=\eta_1+\eta_2+\eta_3$, $x=c\eta_1$ and $y=c\eta_2-c\eta_3$. The transformed of the mixed problem Σ^0, through the change of variables, is the following problem Σ^1:

$$\frac{\partial \widetilde{N_1}}{\partial \eta_1} = Q(\widetilde{N}),\ (\eta_1, \eta_2, \eta_3) \in \overbrace{\mathscr{F}(\mathscr{P})}^{\circ} \equiv \mathring{\mathscr{P}}' \tag{3.1}$$

$$\frac{\partial \widetilde{N_2}}{\partial \eta_2} = -Q(\widetilde{N}),\ (\eta_1, \eta_2, \eta_3) \in \mathring{\mathscr{P}}' \tag{3.2}$$

$$\frac{\partial \widetilde{N_3}}{\partial \eta_3} = -Q(\widetilde{N}),\ (\eta_1, \eta_2, \eta_3) \in \mathring{\mathscr{P}}' \tag{3.3}$$

$$-\frac{\partial \widetilde{N_4}}{\partial \eta_1} + \frac{\partial \widetilde{N_4}}{\partial \eta_2} + \frac{\partial \widetilde{N_4}}{\partial \eta_3} = Q(\widetilde{N}),\ (\eta_1, \eta_2, \eta_3) \in \mathring{\mathscr{P}}' \tag{3.4}$$

$$\widetilde{N_1}\left(-\eta_2 - \eta_3, \eta_2, \eta_3\right) = N_1^0\left(-c\eta_2 - c\eta_3, c\eta_2 - c\eta_3\right) \tag{3.5}$$

$$\widetilde{N_2}\left(\eta_1, -\eta_1 - \eta_3, \eta_3\right) = N_2^0\left(c\eta_1, -c\eta_1 - 2c\eta_3\right) \tag{3.6}$$

$$\widetilde{N_3}\left(\eta_1, \eta_2, -\eta_1 - \eta_2\right) = N_3^0\left(c\eta_1, c\eta_1 + 2c\eta_2\right) \tag{3.7}$$

$$\widetilde{N_4}\left(-\eta_2 - \eta_3, \eta_2, \eta_3\right) = N_4^0\left(-c\eta_2 - c\eta_3, c\eta_2 - c\eta_3\right) \tag{3.8}$$

$$\widetilde{N_1}\left(\frac{1}{c}a_1, \eta_2, \eta_3\right) = N_1^-\left(\frac{1}{c}a_1 + \eta_2 + \eta_3, c\eta_2 - c\eta_3\right) \tag{3.9}$$

$$\widetilde{N_2}\left(\eta_1, \eta_3 + \frac{a_2}{c}, \eta_3\right) = N_2^-\left(\eta_1 + 2\eta_3 + \frac{a_2}{c}, c\eta_1\right) \tag{3.10}$$

$$\widetilde{N_3}\left(\eta_1, \eta_2, \eta_2 - \frac{b_2}{c}\right) = N_3^+\left(\eta_1 + 2\eta_2 - \frac{b_2}{c}, c\eta_1\right) \tag{3.11}$$

$$\widetilde{N_4}\left(\frac{1}{c}b_1, \eta_2, \eta_3\right) = N_4^+\left(\frac{1}{c}b_1 + \eta_2 + \eta_3, c\eta_2 - c\eta_3\right) \tag{3.12}$$

3.1. **Non-negative operator.**

Proposition 3.1. *Let $\sigma > 0$. The problem (3.1-3.12) is equivalent to:*

$$\frac{\partial \widetilde{N_1}}{\partial \eta_1} + \sigma\rho\left(\widetilde{N}\right)\widetilde{N_1} = \sigma\rho\left(\widetilde{N}\right)\widetilde{N_1} + Q\left(\widetilde{N}\right),\ (\eta_1, \eta_2, \eta_3) \in \overbrace{\mathscr{F}(\mathscr{P})}^{\circ} \equiv \mathring{\mathscr{P}}' \tag{3.13}$$

$$\frac{\partial \widetilde{N_2}}{\partial \eta_2} + \sigma\rho\left(\widetilde{N}\right)\widetilde{N_2} = \sigma\rho\left(\widetilde{N}\right)\widetilde{N_2} - Q\left(\widetilde{N}\right),\ (\eta_1, \eta_2, \eta_3) \in \mathring{\mathscr{P}}' \tag{3.14}$$

$$\frac{\partial \widetilde{N_3}}{\partial \eta_3} + \sigma\rho\left(\widetilde{N}\right)\widetilde{N_3} = \sigma\rho\left(\widetilde{N}\right)\widetilde{N_3} - Q\left(\widetilde{N}\right),\ (\eta_1, \eta_2, \eta_3) \in \mathring{\mathscr{P}}' \tag{3.15}$$

$$(3.16)\qquad -\frac{\partial\widetilde{N_4}}{\partial\eta_1}+\frac{\partial\widetilde{N_4}}{\partial\eta_2}+\frac{\partial\widetilde{N_4}}{\partial\eta_3}+\sigma\rho\left(\widetilde{N}\right)\widetilde{N_4}=\sigma\rho\left(\widetilde{N}\right)\widetilde{N_4}+Q\left(\widetilde{N}\right),\quad (\eta_1,\eta_2,\eta_3)\in\mathring{\mathscr{P}}'$$

with the conditions (3.5)-(3.12) *and* $\rho\left(\widetilde{N}\right)=\sum_{i=1}^{4}\widetilde{N_i}$.

Proof. The proof is obvious so that the equations (3.13)-(3.16) are obtained by adding $\sigma\rho\left(\widetilde{N}\right)\widetilde{N_i},\ i=1,2,3,4$ to the members of equations (3.1)-(3.4). □

In the sequel we denote $Q_i^\sigma\left(\widetilde{N}\right)=\sigma\rho\left(\widetilde{N}\right)\widetilde{N_i}+Q\left(\widetilde{N}\right),\ i=1,4$ and $Q_i^\sigma\left(\widetilde{N}\right)=\sigma\rho\left(\widetilde{N}\right)\widetilde{N_i}-Q\left(\widetilde{N}\right),\ i=2,3$.

Proposition 3.2. *Let* $\sigma>0$. *Let* $\widetilde{M}=\left(\widetilde{M_1},\widetilde{M_2},\widetilde{M_3},\widetilde{M_4}\right)$ *be a fixed 4-tuple of continuous functions defined from* $\mathscr{P}'$ *to* $\mathbb{R}$. *Let's put* $\left|\widetilde{M}\right|=\left(\left|\widetilde{M_1}\right|,\left|\widetilde{M_2}\right|,\left|\widetilde{M_3}\right|,\left|\widetilde{M_4}\right|\right)$. *Let's consider the decoupled system* $\left(\Sigma^1_{\sigma,\widetilde{M}}\right)$ *of the following equations:* $\widetilde{N}=\left(\widetilde{N_1},\widetilde{N_2},\widetilde{N_3},\widetilde{N_4}\right)$, (3.17)-(3.20)

$$(3.17)\qquad l\frac{\partial\widetilde{N_1}}{\partial\eta_1}+\sigma\rho\left(\left|\widetilde{M}\right|\right)\widetilde{N_1}=Q_1^\sigma\left(\left|\widetilde{M}\right|\right),\ (\eta_1,\eta_2,\eta_3)\in\overbrace{\mathscr{F}\left(\mathscr{P}\right)}^{\circ}\equiv\mathring{\mathscr{P}}'$$

$$(3.18)\qquad \frac{\partial\widetilde{N_2}}{\partial\eta_2}+\sigma\rho\left(\left|\widetilde{M}\right|\right)\widetilde{N_2}=Q_2^\sigma\left(\left|\widetilde{M}\right|\right),\ (\eta_1,\eta_2,\eta_3)\in\mathring{\mathscr{P}}'$$

$$(3.19)\qquad \frac{\partial\widetilde{N_3}}{\partial\eta_3}+\sigma\rho\left(\left|\widetilde{M}\right|\right)\widetilde{N_3}=Q_3^\sigma\left(\left|\widetilde{M}\right|\right),\ (\eta_1,\eta_2,\eta_3)\in\mathring{\mathscr{P}}'$$

$$(3.20)\qquad -\frac{\partial\widetilde{N_4}}{\partial\eta_1}+\frac{\partial\widetilde{N_4}}{\partial\eta_2}+\frac{\partial\widetilde{N_4}}{\partial\eta_3}+\sigma\rho\left(\left|\widetilde{M}\right|\right)\widetilde{N_4}=Q_4^\sigma\left(\left|\widetilde{M}\right|\right),\ (\eta_1,\eta_2,\eta_3)\in\mathring{\mathscr{P}}'$$

with the conditions (3.5)-(3.12). *Then for sufficiently large* σ, *the problem* $\left(\Sigma^1_{\sigma,\widetilde{M}}\right)$ *has an unique continuous and non-negative solution.*

Proof. The problem (3.17)-(3.20) is a linear problem associated with (3.13)-(3.16). Using the conditions (3.5)-(3.12), it's unique solution is:

$$\begin{aligned}
&\widetilde{N_1}(\eta_1,\eta_2,\eta_3)\\
&=\left(\int_{-\eta_2-\eta_3}^{\eta_1} e^{\sigma\int_{-\eta_2-\eta_3}^{s}\rho\left(\left|\widetilde{M}\right|\right)(r,\eta_2,\eta_3)dr} Q_1^\sigma\left(\left|\widetilde{M}\right|\right)(s,\eta_2,\eta_3)\,ds+\overline{N_1^0}(\eta_2,\eta_3)\right)\\
&\times e^{-\sigma\int_{-\eta_2-\eta_3}^{\eta_1}\rho\left(\left|\widetilde{M}\right|\right)(s,\eta_2,\eta_3)ds}\cdot\mathbb{I}_{-c\eta_2-c\eta_3\geq a_1}(\eta_1,\eta_2,\eta_3)\\
&+\left(\int_{\frac{1}{c}a_1}^{\eta_1} e^{\sigma\int_{\frac{1}{c}a_1}^{s}\rho\left(\left|\widetilde{M}\right|\right)(r,\eta_2,\eta_3)dr} Q_1^\sigma\left(\left|\widetilde{M}\right|\right)(s,\eta_2,\eta_3)\,ds+\overline{N_1^-}(\eta_2,\eta_3)\right)\\
&\times e^{-\sigma\int_{\frac{1}{c}a_1}^{\eta_1}\rho\left(\left|\widetilde{M}\right|\right)(s,\eta_2,\eta_3)ds}\cdot\mathbb{I}_{-c\eta_2-c\eta_3\leq a_1}(\eta_1,\eta_2,\eta_3)
\end{aligned}\tag{3.21}$$

$$\begin{aligned}
&\widetilde{N_2}(\eta_1,\eta_2,\eta_3)\\
&=\left(\int_{-\eta_1-\eta_3}^{\eta_2} e^{\sigma\int_{-\eta_1-\eta_3}^{s}\rho\left(\left|\widetilde{M}\right|\right)(\eta_1,r,\eta_3)dr} Q_2^\sigma\left(\left|\widetilde{M}\right|\right)(\eta_1,s,\eta_3)\,ds+\overline{N_2^0}(\eta_1,\eta_3)\right)\\
&\times e^{-\sigma\int_{-\eta_1-\eta_3}^{\eta_2}\rho\left(\left|\widetilde{M}\right|\right)(\eta_1,s,\eta_3)}\cdot\mathbb{I}_{-c\eta_1-2c\eta_3\geq a_2}(\eta_1,\eta_2,\eta_3)\\
&+\left(\int_{\eta_3+\frac{a_2}{c}}^{\eta_2} e^{\sigma\int_{\eta_3+\frac{a_2}{c}}^{s}\rho\left(\left|\widetilde{M}\right|\right)(\eta_1,r,\eta_3)dr} Q_2^\sigma\left(\left|\widetilde{M}\right|\right)(\eta_1,s,\eta_3)\,ds+\overline{N_2^-}(\eta_1,\eta_3)\right)\\
&\times e^{-\sigma\int_{\eta_3+\frac{a_2}{c}}^{\eta_2}\rho\left(\left|\widetilde{M}\right|\right)(\eta_1,s,\eta_3)ds}\cdot\mathbb{I}_{-c\eta_1-2c\eta_3\leq a_2}(\eta_1,\eta_2,\eta_3)
\end{aligned}\tag{3.22}$$

$$\begin{aligned}
&\widetilde{N_3}(\eta_1,\eta_2,\eta_3)\\
&=\left(\int_{-\eta_1-\eta_2}^{\eta_3} e^{\sigma\int_{-\eta_1-\eta_2}^{s}\rho\left(\left|\widetilde{M}\right|\right)(\eta_1,\eta_2,r)dr} Q_3^\sigma\left(\left|\widetilde{M}\right|\right)(\eta_1,\eta_2,s)\,ds+\overline{N_3^0}(\eta_1,\eta_2)\right)\\
&\times e^{-\sigma\int_{-\eta_1-\eta_2}^{\eta_3}\rho\left(\left|\widetilde{M}\right|\right)(\eta_1,\eta_2,s)ds}\cdot\mathbb{I}_{c\eta_1+2c\eta_2\leq b_2}(\eta_1,\eta_2,\eta_3)\\
&+\left(\int_{\eta_2-\frac{b_2}{c}}^{\eta_3} e^{\sigma\int_{\eta_2-\frac{b_2}{c}}^{s}\rho\left(\left|\widetilde{M}\right|\right)(\eta_1,\eta_2,r)dr} Q_3^\sigma\left(\left|\widetilde{M}\right|\right)(\eta_1,\eta_2,s)\,ds+\overline{N_3^+}(\eta_1,\eta_2)\right)\\
&\qquad\times e^{-\sigma\displaystyle\int_{\eta_2-\frac{b_2}{c}}^{\eta_3}\rho\left(\left|\widetilde{M}\right|\right)(\eta_1,\eta_2,s)ds}\cdot\mathbb{I}_{c\eta_1+2c\eta_2\geq b_2}(\eta_1,\eta_2,\eta_3)
\end{aligned}\tag{3.23}$$

(3.24)
$$
\begin{aligned}
&\widetilde{N_4}(\eta_1,\eta_2,\eta_3) = \mathbb{I}_{2c\eta_1+c\eta_2+c\eta_3\le b_1}(\eta_1,\eta_2,\eta_3) \\
&\cdot\left[\int_0^{\eta_1+\eta_2+\eta_3}\left(e^{\sigma\int_0^s \rho\left(\left|\widetilde{M}\right|\right)(-r+2\eta_1+\eta_2+\eta_3;r-\eta_1-\eta_3;r-\eta_1-\eta_2)dr}\right)\right. \\
&\times Q_4^\sigma\left(\left|\widetilde{M}\right|\right)(-s+2\eta_1+\eta_2+\eta_3;s-\eta_1-\eta_3;s-\eta_1-\eta_2)\,ds \\
&\left.+\overline{N_4^0}(\eta_1,\eta_2,\eta_3)\right]e^{-\sigma\int_0^{\eta_1+\eta_2+\eta_3}\rho\left(\left|\widetilde{M}\right|\right)(-s+2\eta_1+\eta_2+\eta_3;s-\eta_1-\eta_3;s-\eta_1-\eta_2)ds} \\
&+\mathbb{I}_{2c\eta_1+c\eta_2+c\eta_3\ge b_1}(\eta_1,\eta_2,\eta_3) \\
&\cdot\left[\int_0^{\left(-\eta_1+\frac{1}{c}b_1\right)}\left(e^{\sigma\int_0^s\rho\left(\left|\widetilde{M}\right|\right)\left(-r+\frac{1}{c}b_1;r+\eta_1+\eta_2-\frac{1}{c}b_1;r+\eta_1+\eta_3-\frac{1}{c}b_1\right)dr}\right)\right. \\
&\times Q_4^\sigma\left(\left|\widetilde{M}\right|\right)\left(-s+\frac{1}{c}b_1,s+\eta_1+\eta_2-\frac{1}{c}b_1,s+\eta_1+\eta_3-\frac{1}{c}b_1\right)ds \\
&\left.+\overline{N_4^+}(\eta_1,\eta_2,\eta_3)\right]e^{-\sigma\int_0^{\left(-\eta_1+\frac{1}{c}b_1\right)}\rho\left(\left|\widetilde{M}\right|\right)\left(-s+\frac{1}{c}b_1;s+\eta_1+\eta_2-\frac{1}{c}b_1;s+\eta_1+\eta_3-\frac{1}{c}b_1\right)ds}
\end{aligned}
$$

where

(3.25)
$$
\begin{cases}
\overline{N_1^0}(\eta_2,\eta_3)\equiv N_1^0(-c\eta_2-c\eta_3,c\eta_2-c\eta_3) \\
\text{and } \overline{N_1^-}(\eta_2,\eta_3)\equiv N_1^-\left(\frac{1}{c}a_1+\eta_2+\eta_3,c\eta_2-c\eta_3\right)
\end{cases}
$$

(3.26)
$$
\begin{cases}
\overline{N_2^0}(\eta_1,\eta_3)\equiv N_2^0(c\eta_1,-c\eta_1-2c\eta_3) \\
\text{and } \overline{N_2^-}(\eta_1,\eta_3)\equiv N_2^-\left(\eta_1+2\eta_3+\frac{a_2}{c},c\eta_1\right)
\end{cases}
$$

(3.27)
$$
\begin{cases}
\overline{N_3^0}(\eta_1,\eta_2)\equiv N_3^0(c\eta_1,c\eta_1+2c\eta_2) \\
\text{and } \overline{N_3^+}(\eta_1,\eta_2)\equiv N_3^+\left(\eta_1+2\eta_2-\frac{b_2}{c},c\eta_1\right)
\end{cases}
$$

(3.28)
$$
\begin{cases}
\overline{N_4^0}(\eta_1,\eta_2,\eta_3)\equiv N_4^0(2c\eta_1+c\eta_2+c\eta_3,c\eta_2-c\eta_3) \\
\text{and } \overline{N_4^+}(\eta_1,\eta_2,\eta_3)\equiv N_4^+\left(2\eta_1+\eta_2+\eta_3-\frac{1}{c}b_1,c\eta_2-c\eta_3\right).
\end{cases}
$$

Let us show that if σ is sufficiently large, for all $\widetilde{M}$, the solution $\widetilde{N}=(\widetilde{N_1},\widetilde{N_2},\widetilde{N_3},\widetilde{N_4})$ of $(\Sigma^1_{\sigma,\widetilde{M}})$ where $\widetilde{N_1},\widetilde{N_2},\widetilde{N_3},\widetilde{N_4}$ are defined by (3.21)-(3.24), is non-negative.

As the data $N_1^-,\ N_1^0,\ N_2^0,\ N_2^-,\ N_3^0,\ N_3^+,\ N_4^0,\ N_4^+$ are non-negative, it is sufficient that $Q_i^\sigma\left(\left|\widetilde{M}\right|\right) \geq 0,\ i = 1,2,3,4$. One has:

$$
\begin{cases}
Q_1^\sigma\left(\left|\widetilde{M}\right|\right) = \sigma\left(\left|\widetilde{M_1}\right| + \left|\widetilde{M_2}\right| + \left|\widetilde{M_3}\right|\right)\left|\widetilde{M_1}\right| + 2cS\left|\widetilde{M_2}\right|\left|\widetilde{M_3}\right| \\
\qquad\qquad + (\sigma - 2cS)\left|\widetilde{M_1}\right|\left|\widetilde{M_4}\right| \\
Q_2^\sigma\left(\left|\widetilde{M}\right|\right) = \sigma\left(\left|\widetilde{M_1}\right| + \left|\widetilde{M_2}\right| + \left|\widetilde{M_4}\right|\right)\left|\widetilde{M_2}\right| + 2cS\left|\widetilde{M_1}\right|\left|\widetilde{M_4}\right| \\
\qquad\qquad + (\sigma - 2cS)\left|\widetilde{M_2}\right|\left|\widetilde{M_3}\right| \\
Q_3^\sigma\left(\left|\widetilde{M}\right|\right) = \sigma\left(\left|\widetilde{M_1}\right| + \left|\widetilde{M_3}\right| + \left|\widetilde{M_4}\right|\right)\left|\widetilde{M_3}\right| + 2cS\left|\widetilde{M_1}\right|\left|\widetilde{M_4}\right| \\
\qquad\qquad + (\sigma - 2cS)\left|\widetilde{M_2}\right|\left|\widetilde{M_3}\right| \\
Q_4^\sigma\left(\left|\widetilde{M}\right|\right) = \sigma\left(\left|\widetilde{M_2}\right| + \left|\widetilde{M_3}\right| + \left|\widetilde{M_4}\right|\right)\left|\widetilde{M_4}\right| + 2cS\left|\widetilde{M_2}\right|\left|\widetilde{M_3}\right| \\
\qquad\qquad + (\sigma - 2cS)\left|\widetilde{M_1}\right|\left|\widetilde{M_4}\right|
\end{cases} . \tag{3.29}
$$

From which we conclude that for $\sigma \geq 2cS$, the solution $\widetilde{N} = \left(\widetilde{N_1}, \widetilde{N_2}, \widetilde{N_3}, \widetilde{N_4}\right)$ of $\left(\Sigma^1_{\sigma,\widetilde{M}}\right)$ is non-negative. □

We can thus consider the non-negative operator

$$
\begin{aligned}
\mathcal{T}^\sigma : C\left(\ \mathscr{P}'\ ;\mathbb{R}^4\right) &\longrightarrow C\left(\ \mathscr{P}'\ ;\mathbb{R}^4\right) \\
\widetilde{M} &\longmapsto \widetilde{N}_{\widetilde{M}},
\end{aligned} \tag{3.30}
$$

where $\widetilde{N}_{\widetilde{M}}$ is the unique non-negative solution of the problem $\left(\Sigma^1_{\sigma,\widetilde{M}}\right)$ for sufficiently large σ.

3.2. **Non-negativity theorem.**

Theorem 3.1. *The solutions of the problem Σ^1 (eq.3.1-3.12) are non-negative.*

Proof. Let us verify that $\widetilde{N} = \left(\widetilde{N_1}, \widetilde{N_2}, \widetilde{N_3}, \widetilde{N_4}\right)$ is a solution of Σ^1 if $\widetilde{N}$ is a fixed point of the operator $\mathcal{T}^\sigma$ (3.30) for sufficiently large σ.

We have $\widetilde{M} = \left(\widetilde{M_1}, \widetilde{M_2}, \widetilde{M_3}, \widetilde{M_4}\right) \in C\left(\mathscr{P}' ; \mathbb{R}^4\right)$ is a fixed point of $\mathcal{T}^\sigma$ if $\widetilde{N}_{\widetilde{M}} = \widetilde{M}$, i.e., $\widetilde{M}$ is a solution of $\left(\Sigma^1_{\sigma,\widetilde{M}}\right)$, i.e.,

$$\frac{\partial \widetilde{M_1}}{\partial \eta_1} + \sigma\rho\left(\left|\widetilde{M}\right|\right)\widetilde{M_1} = Q_1^\sigma\left(\left|\widetilde{M}\right|\right),\ (\eta_1,\eta_2,\eta_3) \in \overbrace{\mathscr{F}(\mathscr{P})}^{\circ} \equiv \mathring{\mathscr{P}}', \tag{3.31}$$

$$\frac{\partial \widetilde{M_2}}{\partial \eta_2} + \sigma\rho\left(\left|\widetilde{M}\right|\right)\widetilde{M_2} = Q_2^\sigma\left(\left|\widetilde{M}\right|\right),\ (\eta_1,\eta_2,\eta_3) \in \mathring{\mathscr{P}}', \tag{3.32}$$

$$\frac{\partial \widetilde{M_3}}{\partial \eta_3} + \sigma\rho\left(\left|\widetilde{M}\right|\right)\widetilde{M_3} = Q_3^\sigma\left(\left|\widetilde{M}\right|\right),\ (\eta_1,\eta_2,\eta_3) \in \mathring{\mathscr{P}}', \tag{3.33}$$

$$-\frac{\partial \widetilde{M_4}}{\partial \eta_1} + \frac{\partial \widetilde{M_4}}{\partial \eta_2} + \frac{\partial \widetilde{M_4}}{\partial \eta_3} + \sigma\rho\left(\left|\widetilde{M}\right|\right)\widetilde{M_4} = Q_4^\sigma\left(\left|\widetilde{M}\right|\right),\ (\eta_1,\eta_2,\eta_3) \in \mathring{\mathscr{P}}'. \tag{3.34}$$

with the conditions (3.5)-(3.12); as $\widetilde{N}_{\widetilde{M}} = \widetilde{M}$ is positive i.e. $\left|\widetilde{M}\right| = \widetilde{M}$ for sufficiently large σ, (3.31)-(3.34) means $\widetilde{M}$ is a solution of (Σ^1_σ) which is equivalent to Σ^1. As $\mathcal{T}^\sigma$ is non-negative, so are its fixed points. □

4. Existence and uniqueness of bounded solution

We shall define an operator, the fixed points of which, are the solutions of the problem Σ^1 (3.1-3.12) and establish the existence of the fixed points by using the following Schauder's theorem ([14], p.25, Theorem 4.1.1).

Theorem 4.1. *(Schauder [14]) Let $\mathcal{M}$ be a non-empty convex subset of a normed space $\mathscr{B}$. Let $\mathcal{T}$ be a continuous compact mapping from $\mathcal{M}$ into $\mathscr{M}$. Then $\mathcal{T}$ has a fixed point.*

4.1. Fixed point problem.

Let $\widetilde{M} = \left(\widetilde{M_1}, \widetilde{M_2}, \widetilde{M_3}, \widetilde{M_4}\right)$ be a fixed 4-tuple of continuous functions from $\mathscr{P}'$ into $\mathbb{R}$. Let us consider, the following decoupled system $\left(\Sigma^1_{\widetilde{M}}\right)$ defined by (4.1)-(4.4).

$$\frac{\partial \widetilde{N_1}}{\partial \eta_1} = Q(\widetilde{M}),\ (\eta_1,\eta_2,\eta_3) \in \overbrace{\mathscr{F}(\mathscr{P})}^{\circ} \equiv \mathring{\mathscr{P}}', \tag{4.1}$$

$$\frac{\partial \widetilde{N_2}}{\partial \eta_2} = -Q(\widetilde{M}),\ (\eta_1,\eta_2,\eta_3) \in \mathring{\mathscr{P}}', \tag{4.2}$$

$$\frac{\partial \widetilde{N_3}}{\partial \eta_3} = -Q(\widetilde{M}),\ (\eta_1,\eta_2,\eta_3) \in \mathring{\mathscr{P}}', \tag{4.3}$$

$$-\frac{\partial \widetilde{N_4}}{\partial \eta_1} + \frac{\partial \widetilde{N_4}}{\partial \eta_2} + \frac{\partial \widetilde{N_4}}{\partial \eta_3} = Q(\widetilde{M}),\ (\eta_1,\eta_2,\eta_3) \in \mathring{\mathscr{P}}', \tag{4.4}$$

with the conditions (3.5)-(3.12). Then it follows from the resolution made in the proof of proposition (3.2) that the problem $\left(\Sigma^1_{\widetilde{M}}\right)$ has an unique continuous solution $\widetilde{N} = \left(\widetilde{N_1}, \widetilde{N_2}, \widetilde{N_3}, \widetilde{N_4}\right)$ defined by:

$$\begin{aligned} &\widetilde{N_1}(\eta_1,\eta_2,\eta_3) \\ &= \left(\int_{-\eta_2-\eta_3}^{\eta_1} Q\left(\widetilde{M}\right)(s,\eta_2,\eta_3)\,ds + \overline{N_1^0}(\eta_2,\eta_3)\right) \cdot \mathbb{I}_{-c\eta_2-c\eta_3 \geq a_1}(\eta_1,\eta_2,\eta_3) \\ &+ \left(\int_{\frac{1}{c}a_1}^{\eta_1} Q\left(\widetilde{M}\right)(s,\eta_2,\eta_3)\,ds + \overline{N_1^-}(\eta_2,\eta_3)\right) \cdot \mathbb{I}_{-c\eta_2-c\eta_3 \leq a_1}(\eta_1,\eta_2,\eta_3) \end{aligned} \tag{4.5}$$

$$\begin{aligned} &\widetilde{N_2}(\eta_1,\eta_2,\eta_3) \\ &= \left(\int_{-\eta_1-\eta_3}^{\eta_2} -Q\left(\widetilde{M}\right)(\eta_1,s,\eta_3)\,ds + \overline{N_2^0}(\eta_1,\eta_3)\right) \cdot \mathbb{I}_{-c\eta_1-2c\eta_3 \geq a_2}(\eta_1,\eta_2,\eta_3) \\ &+ \left(\int_{\eta_3+\frac{a_2}{c}}^{\eta_2} - Q\left(\widetilde{M}\right)(\eta_1,s,\eta_3)\,ds + \overline{N_2^-}(\eta_1,\eta_3)\right) \cdot \mathbb{I}_{-c\eta_1-2c\eta_3 \leq a_2}(\eta_1,\eta_2,\eta_3) \end{aligned} \tag{4.6}$$

$$\begin{aligned} &\widetilde{N_3}(\eta_1,\eta_2,\eta_3) \\ &= \left(\int_{-\eta_1-\eta_2}^{\eta_3} - Q\left(\widetilde{M}\right)(\eta_1,\eta_2,s)\,ds + \overline{N_3^0}(\eta_1,\eta_2)\right) \cdot \mathbb{I}_{c\eta_1+2c\eta_2 \leq b_2}(\eta_1,\eta_2,\eta_3) \\ &+ \left(\int_{\eta_2-\frac{b_2}{c}}^{\eta_3} - Q\left(\widetilde{M}\right)(\eta_1,\eta_2,s)\,ds + \overline{N_3^+}(\eta_1,\eta_2)\right) \cdot \mathbb{I}_{c\eta_1+2c\eta_2 \geq b_2}(\eta_1,\eta_2,\eta_3) \end{aligned} \tag{4.7}$$

$$\begin{aligned} &\widetilde{N_4}(\eta_1,\eta_2,\eta_3) = \mathbb{I}_{2c\eta_1+c\eta_2+c\eta_3 \leq b_1}(\eta_1,\eta_2,\eta_3) \\ &\cdot \left[\int_0^{\eta_1+\eta_2+\eta_3} Q\left(\widetilde{M}\right)(-s+2\eta_1+\eta_2+\eta_3; s-\eta_1-\eta_3; s-\eta_1-\eta_2)\,ds \right. \\ &\left. + \overline{N_4^0}(\eta_1,\eta_2,\eta_3)\right] + \mathbb{I}_{2c\eta_1+c\eta_2+c\eta_3 \geq b_1}(\eta_1,\eta_2,\eta_3) \end{aligned} \tag{4.8}$$

$$\cdot\left[\int_0^{\left(-\eta_1+\frac{1}{c}b_1\right)} Q\left(\widetilde{M}\right)\left(-s+\frac{1}{c}b_1, s+\eta_1+\eta_2-\frac{1}{c}b_1, s+\eta_1+\eta_3-\frac{1}{c}b_1\right)ds\right.$$
$$\left.+\overline{N_4^+}\left(\eta_1,\eta_2,\eta_3\right)\right].$$

We can thus define the following operator

$$\begin{aligned}\mathcal{T}: C\left(\ \mathscr{P}'\ ;\mathbb{R}^4\right) &\longrightarrow C\left(\ \mathscr{P}'\ ;\mathbb{R}^4\right)\\ \widetilde{M} &\longmapsto \mathcal{T}\left(\widetilde{M}\right)=\left(\mathcal{T}_i\left(\widetilde{M}\right)\right)_{i=1}^4\end{aligned}\tag{4.9}$$

where $\mathcal{T}\left(\widetilde{M}\right)=\left(\mathcal{T}_i\left(\widetilde{M}\right)\right)_{i=1}^4$ is the unique solution of the problem $\left(\Sigma^1_{\widetilde{M}}\right)$.

It immediately follows from (4.1)-(4.4) and (3.1)-(3.12) that the solutions of Σ^1 are the fixed points of $\mathcal{T}$.

4.2. **Continuity of the operator of the fixed point problem.**

Proposition 4.1. *The operator $\mathcal{T}$ (4.9) is continuous.*

Proof. Relation (4.5) gives for $\widetilde{M},\widetilde{N}\in C\left(\mathscr{P}',\mathbb{R}^4\right)$:

$$\begin{aligned}&\left\|\mathcal{T}_1\left(\widetilde{M}\right)-\mathcal{T}_1\left(\widetilde{N}\right)\right\|_\infty\\ &\leq \max\left\{\sup_{(\eta_1,\eta_2,\eta_3)\in\mathscr{P}'}\left|\int_{-\eta_2-\eta_3}^{\eta_1}\left[Q\left(\widetilde{M}\right)-Q\left(\widetilde{N}\right)\right](s,\eta_2,\eta_3)\,ds\right|;\right.\\ &\left.\sup_{(\eta_1,\eta_2,\eta_3)\in\mathscr{P}'}\left|\int_{\frac{1}{c}a_1}^{\eta_1}\left[Q\left(\widetilde{M}\right)-Q\left(\widetilde{N}\right)\right](s,\eta_2,\eta_3)\,ds\right|\right\}.\end{aligned}\tag{4.10}$$

But (2.17) yields

$$\begin{aligned}Q\left(\widetilde{M}\right)-Q\left(\widetilde{N}\right)&=2cS\left(\widetilde{M_2}-\widetilde{N_2}\right)\widetilde{M_3}+2cS\widetilde{N_2}\left(\widetilde{M_3}-\widetilde{N_3}\right)\\ &-2cS\left(\widetilde{M_1}-\widetilde{N_1}\right)\widetilde{M_4}-2cS\widetilde{N_1}\left(\widetilde{M_4}-\widetilde{N_4}\right),\end{aligned}\tag{4.11}$$

hence

$$\begin{aligned}&\left\|Q\left(\widetilde{M}\right)-Q\left(\widetilde{N}\right)\right\|_{\infty}\\&\leq 2cS\left\|\widetilde{M_2}-\widetilde{N_2}\right\|_{\infty}\left\|\widetilde{M_3}\right\|_{\infty}+2cS\left\|\widetilde{N_2}\right\|_{\infty}\left\|\widetilde{M_3}-\widetilde{N_3}\right\|_{\infty}\\&+2cS\left\|\widetilde{M_1}-\widetilde{N_1}\right\|_{\infty}\left\|\widetilde{M_4}\right\|_{\infty}+2cS\left\|\widetilde{N_1}\right\|_{\infty}\left\|\widetilde{M_4}-\widetilde{N_4}\right\|_{\infty},\end{aligned}$$

and

$$\left\|Q\left(\widetilde{M}\right)-Q\left(\widetilde{N}\right)\right\|_{\infty}\leq 4cS\left\|\widetilde{M}-\widetilde{N}\right\|\left\|\widetilde{M}\right\|+4cS\left\|\widetilde{N}\right\|\left\|\widetilde{M}-\widetilde{N}\right\|$$

$$\left\|Q\left(\widetilde{M}\right)-Q\left(\widetilde{N}\right)\right\|_{\infty}\leq 4cS\left(\left\|\widetilde{M}\right\|+\left\|\widetilde{N}\right\|\right)\left\|\widetilde{M}-\widetilde{N}\right\|;$$

hence (4.10) implies

$$\begin{aligned}&\left\|\mathcal{T}_1\left(\widetilde{M}\right)-\mathcal{T}_1\left(\widetilde{N}\right)\right\|_{\infty}\\&\leq\max\Bigg\{\left(\sup_{(\eta_1,\eta_2,\eta_3)\in\mathscr{P}'}(\eta_1+\eta_2+\eta_3)\right)\cdot 4cS\left(\left\|\widetilde{M}\right\|+\left\|\widetilde{N}\right\|\right)\left\|\widetilde{M}-\widetilde{N}\right\|;\\&\left(\sup_{(\eta_1,\eta_2,\eta_3)\in\mathscr{P}'}\left(\eta_1-\frac{1}{c}a_1\right)\right)\cdot 4cS\left(\left\|\widetilde{M}\right\|+\left\|\widetilde{N}\right\|\right)\left\|\widetilde{M}-\widetilde{N}\right\|\Bigg\}.\end{aligned}\tag{4.12}$$

Now $0\leq\eta_1+\eta_2+\eta_3\leq T$ and $a_1\leq c\eta_1\leq b_1$; hence $0\leq\eta_1-\frac{1}{c}a_1\leq\frac{1}{c}(b_1-a_1)$; from which

$$\begin{aligned}&l\left\|\mathcal{T}_1\left(\widetilde{M}\right)-\mathcal{T}_1\left(\widetilde{N}\right)\right\|_{\infty}\\&\leq\max\left\{T,\frac{1}{c}(b_1-a_1)\right\}\cdot 4cS\left(\left\|\widetilde{M}\right\|+\left\|\widetilde{N}\right\|\right)\left\|\widetilde{M}-\widetilde{N}\right\|.\end{aligned}\tag{4.13}$$

Similarly from (4.6) we have

$$\begin{aligned}&\left\|\mathcal{T}_2\left(\widetilde{M}\right)-\mathcal{T}_2\left(\widetilde{N}\right)\right\|_{\infty}\\&\leq\max\Bigg\{\sup_{(\eta_1,\eta_2,\eta_3)\in\mathscr{P}'}\left|\int_{-\eta_2-\eta_3}^{\eta_1}\left[Q\left(\widetilde{M}\right)-Q\left(\widetilde{N}\right)\right](s,\eta_2,\eta_3)\,ds\right|;\\&\sup_{(\eta_1,\eta_2,\eta_3)\in\mathscr{P}'}\left|\int_{\eta_3+\frac{a_2}{c}}^{\eta_2}\left[Q\left(\widetilde{M}\right)-Q\left(\widetilde{N}\right)\right](s,\eta_2,\eta_3)\,ds\right|\Bigg\}\end{aligned}\tag{4.14}$$

and

$$\left\|\mathcal{T}_2\left(\widetilde{M}\right)-\mathcal{T}_2\left(\widetilde{N}\right)\right\|_\infty$$

$$\leq \max\left\{\left(\sup_{(\eta_1,\eta_2,\eta_3)\in\mathscr{P}'}(\eta_1+\eta_2+\eta_3)\right)\cdot 4cS\left(\left\|\widetilde{M}\right\|+\left\|\widetilde{N}\right\|\right)\left\|\widetilde{M}-\widetilde{N}\right\|;\right. \tag{4.15}$$

$$\left.\left(\sup_{(\eta_1,\eta_2,\eta_3)\in\mathscr{P}'}\left(\eta_2-\eta_3-\frac{a_2}{c}\right)\right)\cdot 4cS\left(\left\|\widetilde{M}\right\|+\left\|\widetilde{N}\right\|\right)\left\|\widetilde{M}-\widetilde{N}\right\|\right\}.$$

But $0 \leq \eta_1+\eta_2+\eta_3 \leq T$ and $a_2 \leq c\eta_2 - c\eta_3 \leq b_2$; hence $0 \leq \eta_2-\eta_3-\frac{a_2}{c} \leq \frac{1}{c}(b_2-a_2)$; thus

$$\left\|\mathcal{T}_2\left(\widetilde{M}\right)-\mathcal{T}_2\left(\widetilde{N}\right)\right\|_\infty$$

$$\leq \max\left\{T,\frac{1}{c}(b_2-a_2)\right\}\cdot 4cS\left(\left\|\widetilde{M}\right\|+\left\|\widetilde{N}\right\|\right)\left\|\widetilde{M}-\widetilde{N}\right\|. \tag{4.16}$$

Similarly (4.7) and (4.8) yield

$$\left\|\mathcal{T}_3\left(\widetilde{M}\right)-\mathcal{T}_3\left(\widetilde{N}\right)\right\|_\infty$$

$$\leq \max\left\{T,\frac{1}{c}(b_2-a_2)\right\}\cdot 4cS\left(\left\|\widetilde{M}\right\|+\left\|\widetilde{N}\right\|\right)\left\|\widetilde{M}-\widetilde{N}\right\|. \tag{4.17}$$

and

$$\left\|\mathcal{T}_4\left(\widetilde{M}\right)-\mathcal{T}_4\left(\widetilde{N}\right)\right\|_\infty$$

$$\leq \max\left\{T,\frac{1}{c}(b_1-a_1)\right\}\cdot 4cS\left(\left\|\widetilde{M}\right\|+\left\|\widetilde{N}\right\|\right)\left\|\widetilde{M}-\widetilde{N}\right\|. \tag{4.18}$$

Now from (4.13)-(4.16)- (4.17)-(4.18) and

$$\left\|\mathcal{T}\left(\widetilde{M}\right)-\mathcal{T}\left(\widetilde{N}\right)\right\| = \max_{1\leq i\leq 4}\left\|\mathcal{T}_i\left(\widetilde{M}\right)-\mathcal{T}_i\left(\widetilde{N}\right)\right\|_\infty$$

we have

$$\left\|\mathcal{T}\left(\widetilde{M}\right)-\mathcal{T}\left(\widetilde{N}\right)\right\|$$

$$\leq \underbrace{\max\left\{T,\frac{1}{c}(b_1-a_1),\frac{1}{c}(b_2-a_2)\right\}\cdot 4cS}_{\equiv p'}\left(\left\|\widetilde{M}\right\|+\left\|\widetilde{N}\right\|\right)\left\|\widetilde{M}-\widetilde{N}\right\|. \tag{4.19}$$

For a fixed $\widetilde{N}$ and a fixed $R>0$, $\left\|\widetilde{M}-\widetilde{N}\right\| \leq R \implies \left\|\widetilde{M}\right\| + \left\|\widetilde{N}\right\| \leq 2\left\|\widetilde{N}\right\| + R$ thus forall $\varepsilon > 0$,

$$\left\|\widetilde{M}-\widetilde{N}\right\| \leq \min\left\{R; \frac{\varepsilon}{p'\left(2\left\|\widetilde{N}\right\|+R\right)}\right\} \implies \left\|\mathcal{T}\left(\widetilde{M}\right)-\mathcal{T}\left(\widetilde{N}\right)\right\| \leq \varepsilon.$$

We deduce that $\mathcal{T}$ is continuous. □

4.3. **Convex set on which the operator is compact.**

Proposition 4.2. *Suppose* $\widetilde{M} = \left(\widetilde{M_1}, \widetilde{M_2}, \widetilde{M_3}, \widetilde{M_4}\right) \in C\left(\mathscr{P}' ; \mathbb{R}^4\right)$ *such that* $\frac{\partial \widetilde{M_i}}{\partial \eta_1}, \frac{\partial \widetilde{M_i}}{\partial \eta_2}, \frac{\partial \widetilde{M_i}}{\partial \eta_3}$ *are defined in* $\mathring{\mathscr{P}}'$*, except possibly on a finite number of planes, and are continuous and bounded for all* $i = 1,2,3,4$*. Then all the derivatives* $\frac{\partial \mathcal{T}_i(\widetilde{M})}{\partial \eta_j}$*,*$(j = 1,2,3)$*,* $(i = 1,2,3,4)$ *are defined in* $\mathring{\mathscr{P}}'$*, except possibly on a finite number of planes, and are continuous and bounded.*

Proof. It follows immediately from the formula (4.5)-(4.8) as the derivatives of both the integrand and data are defined, continuous and bounded. □

Let E the sub-space of $C\left(\mathscr{P}' ; \mathbb{R}\right)$ consisting of functions u that are continuous on $\mathscr{P}'$ such that $\frac{\partial u}{\partial \eta_j}$, $j = 1,2,3$ are defined in $\mathring{\mathscr{P}}'$ except possibly on a finite number of planes, and are continuous and bounded. The above proposition states that $\forall \widetilde{M} \in E^4 \subset C\left(\mathscr{P}'; \mathbb{R}^4\right), \mathcal{T}\left(\widetilde{M}\right) \in E^4$.

Proposition 4.3. *Let us set for all* $R > 0$,

$$\mathscr{M}_R \equiv \left\{\widetilde{N} \in E^4 : \mathscr{N}\left(\widetilde{N}\right) \equiv \max\left\{\left\|\widetilde{N}\right\|, \left\|\frac{\partial \widetilde{N}}{\partial \eta_1}\right\|, \left\|\frac{\partial \widetilde{N}}{\partial \eta_2}\right\|, \left\|\frac{\partial \widetilde{N}}{\partial \eta_3}\right\|\right\} \leq R\right\}. \tag{4.20}$$

Then, $\mathscr{M}_R$ *is a non-empty convex subset of* $C\left(\mathscr{P}'; \mathbb{R}^4\right)$.

Proof. $\mathscr{M}_R$ is non-empty for it contains the zero function. For $\widetilde{M}, \widetilde{N} \in \mathscr{M}_R$, and $\lambda \in \mathbb{R}$ such that $0 \leq \lambda \leq 1$ we have $\lambda\widetilde{M} + (1-\lambda)\widetilde{N} \in E^4$. Moreover $\mathscr{N}\left(\lambda\widetilde{M} + (1-\lambda)\widetilde{N}\right) \leq R$ follows from triangular inequality. □

Proposition 4.4. *The operator* $\mathcal{T}$ *is compact on* $\mathscr{M}_R$ *forall* $R > 0$.

Proof. First, we prove that $\mathcal{T}(\mathscr{M}_R)$ is bounded in $C(\mathscr{P}'; \mathbb{R}^4)$. From

$$Q\left(\widetilde{M}\right) = 2cS\left(\widetilde{M_2}\widetilde{M_3} - \widetilde{M_1}\widetilde{M_4}\right) \tag{4.21}$$

we infer

$$\left\|Q\left(\widetilde{M}\right)\right\|_\infty = \left\|2cS\left(\widetilde{M_2}\widetilde{M_3} - \widetilde{M_1}\widetilde{M_4}\right)\right\|_\infty \leq 4cS\left(\mathscr{N}\left(\widetilde{M}\right)\right)^2. \tag{4.22}$$

So, (4.5)-(4.8) yields for all $\widetilde{M} \in \mathscr{M}_R$:

$$\begin{aligned} \left\|\mathcal{T}_1\left(\widetilde{M}\right)\right\|_\infty &\leq 4cS \cdot \max\left\{T; \frac{1}{c}(b_1 - a_1)\right\}\left(\mathscr{N}\left(\widetilde{M}\right)\right)^2 \\ &+ \max\left\{\left\|\overline{N_1^0}\right\|_1; \left\|\overline{N_1^-}\right\|_1\right\}. \end{aligned} \tag{4.23}$$

$$\begin{aligned} \left\|\mathcal{T}_2\left(\widetilde{M}\right)\right\|_\infty &\leq 4cS \cdot \max\left\{T; \frac{1}{c}(b_2 - a_2)\right\}\left(\mathscr{N}\left(\widetilde{M}\right)\right)^2 \\ &+ \max\left\{\left\|\overline{N_2^0}\right\|_1; \left\|\overline{N_2^-}\right\|_1\right\} \end{aligned} \tag{4.24}$$

$$\begin{aligned} \left\|\mathcal{T}_3\left(\widetilde{M}\right)\right\|_\infty &\leq 4cS \cdot \max\left\{T; \frac{1}{c}(b_2 - a_2)\right\}\left(\mathscr{N}\left(\widetilde{M}\right)\right)^2 \\ &+ \max\left\{\left\|\overline{N_3^0}\right\|_1; \left\|\overline{N_3^+}\right\|_1\right\} \end{aligned} \tag{4.25}$$

$$\begin{aligned} \left\|\mathcal{T}_4\left(\widetilde{M}\right)\right\|_\infty &\leq 4cS \cdot \max\left\{T; \frac{1}{c}(b_1 - a_1)\right\}\left(\mathscr{N}\left(\widetilde{M}\right)\right)^2 \\ &+ \max\left\{\left\|\overline{N_4^0}\right\|_1; \left\|\overline{N_4^+}\right\|_1\right\}. \end{aligned} \tag{4.26}$$

Then for all $\widetilde{M} \in \mathscr{M}_R$:

$$\begin{aligned} \left\|\mathcal{T}\left(\widetilde{M}\right)\right\| &\leq 4cS \cdot \max\left\{T; \frac{1}{c}(b_1 - a_1); \frac{1}{c}(b_2 - a_2)\right\}R^2 \\ &+ \max_{1\leq i\leq 4}\left\{\left\|\overline{N_i^0}\right\|_1; \left\|\overline{N_1^-}\right\|_1; \left\|\overline{N_2^-}\right\|_1; \left\|\overline{N_3^+}\right\|_1; \left\|\overline{N_4^+}\right\|_1\right\} \equiv R_1. \end{aligned} \tag{4.27}$$

Second, we prove that $\mathcal{T}(\mathscr{M}_R)$ is equicontinuous in $C(\mathscr{P}';\mathbb{R}^4)$. From

$$\frac{\partial Q\left(\widetilde{M}\right)}{\partial\eta_j}=2cS\left(\frac{\partial\widetilde{M}_2}{\partial\eta_j}\widetilde{M}_3+\widetilde{M}_2\frac{\partial\widetilde{M}_3}{\partial\eta_j}-\frac{\partial\widetilde{M}_1}{\partial\eta_j}\widetilde{M}_4-\widetilde{M}_1\frac{\partial\widetilde{M}_4}{\partial\eta_j}\right) \tag{4.28}$$

we infer

$$\left\|\frac{\partial Q\left(\widetilde{M}\right)}{\partial\eta_j}\right\|_\infty\le 2cS\cdot4\left(\mathscr{N}\left(\widetilde{M}\right)\right)^2\le 8cS\left(\mathscr{N}\left(\widetilde{M}\right)\right)^2. \tag{4.29}$$

On one hand we have

$$\left\|\frac{\partial\mathcal{T}_i\left(\widetilde{M}\right)}{\partial\eta_i}\right\|_\infty\le 4cS\left(\mathscr{N}\left(\widetilde{M}\right)\right)^2,\ i=1,2,3. \tag{4.30}$$

On other hand, by derivation in the equations (4.5)-(4.8), taking the norm and taking into account (4.29), we have

$$\begin{aligned}\left\|\frac{\partial\mathcal{T}_1\left(\widetilde{M}\right)}{\partial\eta_2}\right\|_\infty&\le 4cs\left(1+2\cdot\max\left\{T;\frac{1}{c}\left(b_1-a_1\right)\right\}\right)\left(\mathscr{N}\left(\widetilde{M}\right)\right)^2\\&+\max\left\{\left\|\overline{N_1^0}\right\|_1;\left\|\overline{N_1^-}\right\|_1\right\}.\end{aligned} \tag{4.31}$$

$$\begin{aligned}\left\|\frac{\partial\mathcal{T}_1\left(\widetilde{M}\right)}{\partial\eta_3}\right\|_\infty&\le 4cs\left(1+2\cdot\max\left\{T;\frac{1}{c}\left(b_1-a_1\right)\right\}\right)\left(\mathscr{N}\left(\widetilde{M}\right)\right)^2\\&+\max\left\{\left\|\overline{N_1^0}\right\|_1;\left\|\overline{N_1^-}\right\|_1\right\};\end{aligned} \tag{4.32}$$

$$\begin{aligned}\left\|\frac{\partial\mathcal{T}_2\left(\widetilde{M}\right)}{\partial\eta_1}\right\|_\infty&\le 4cs\left(1+2\cdot\max\left\{T;\frac{1}{c}\left(b_2-a_2\right)\right\}\right)\left(\mathscr{N}\left(\widetilde{M}\right)\right)^2\\&+\max\left\{\left\|\overline{N_2^0}\right\|_1;\left\|\overline{N_2^-}\right\|_1\right\};\end{aligned} \tag{4.33}$$

$$\left\|\frac{\partial \mathcal{T}_2\left(\widetilde{M}\right)}{\partial \eta_3}\right\|_\infty \leq 4cs\left(1+2\cdot\max\left\{T;\frac{1}{c}\left(b_2-a_2\right)\right\}\right)\left(\mathcal{N}\left(\widetilde{M}\right)\right)^2$$

$$+\max\left\{\left\|\overline{N_2^0}\right\|_1;\left\|\overline{N_2^-}\right\|_1\right\}; \tag{4.34}$$

$$\left\|\frac{\partial \mathcal{T}_3\left(\widetilde{M}\right)}{\partial \eta_1}\right\|_\infty \leq 4cs\left(1+2\cdot\max\left\{T;\frac{1}{c}\left(b_2-a_2\right)\right\}\right)\left(\mathcal{N}\left(\widetilde{M}\right)\right)^2$$

$$+\max\left\{\left\|\overline{N_3^0}\right\|_1;\left\|\overline{N_3^+}\right\|_1\right\}; \tag{4.35}$$

$$\left\|\frac{\partial \mathcal{T}_3\left(\widetilde{M}\right)}{\partial \eta_2}\right\|_\infty \leq 4cs\left(1+2\cdot\max\left\{T;\frac{1}{c}\left(b_2-a_2\right)\right\}\right)\left(\mathcal{N}\left(\widetilde{M}\right)\right)^2$$

$$+\max\left\{\left\|\overline{N_3^0}\right\|_1;\left\|\overline{N_3^+}\right\|_1\right\}; \tag{4.36}$$

$$\left\|\frac{\partial \mathcal{T}_4\left(\widetilde{M}\right)}{\partial \eta_1}\right\|_\infty \leq 4cs\left(1+2\cdot\max\left\{4T;\frac{2}{c}\left(b_1-a_1\right)\right\}\right)\left(\mathcal{N}\left(\widetilde{M}\right)\right)^2$$

$$+\max\left\{\left\|\overline{N_4^0}\right\|_1;\left\|\overline{N_4^+}\right\|_1\right\}; \tag{4.37}$$

$$\left\|\frac{\partial \mathcal{T}_4\left(\widetilde{M}\right)}{\partial \eta_2}\right\|_\infty \leq 4cs\left(1+2\cdot\max\left\{2T;\frac{1}{c}\left(b_1-a_1\right)\right\}\right)\left(\mathcal{N}\left(\widetilde{M}\right)\right)^2$$

$$+\max\left\{\left\|\overline{N_4^0}\right\|_1;\left\|\overline{N_4^+}\right\|_1\right\}; \tag{4.38}$$

$$\left\|\frac{\partial \mathcal{T}_4\left(\widetilde{M}\right)}{\partial \eta_3}\right\|_\infty \leq 4cs\left(1+2\cdot\max\left\{2T;\frac{1}{c}\left(b_1-a_1\right)\right\}\right)\left(\mathscr{N}\left(\widetilde{M}\right)\right)^2$$

$$+\max\left\{\left\|\overline{N_4^0}\right\|_1;\left\|\overline{N_4^+}\right\|_1\right\}. \tag{4.39}$$

Equations (4.30), (4.31)-(4.39) and (4.20) imply that $\frac{\partial \mathcal{T}_i(\widetilde{M})}{\partial \eta_j}$, $(j=1,2,3)$, $(i=1,2,3,4)$ are uniformly bounded on their domain when $\widetilde{M}$ varies within $\mathscr{M}_R$. As the $\frac{\partial \mathcal{T}_i(\widetilde{M})}{\partial \eta_j}$ are continuous, $\forall \widetilde{M}\in\mathscr{M}_R$, for $i=1,2,3,4$, $\mathcal{T}_i\left(\widetilde{M}\right)\in C\left(\ \mathscr{P}'\ ;\mathbb{R}\right)$ is differentiable on the domain of $\left(\frac{\partial \mathcal{T}_i(\widetilde{M})}{\partial \eta_j}\right)_{i,j}$. If $\mathcal{T}_i\left(\widetilde{M}\right)$ is differentiable at (η_1,η_2,η_3), let $d\left(\mathcal{T}_i\left(\widetilde{M}\right)\right)(\eta_1,\eta_2,\eta_3)$ denote the differential of $\mathcal{T}_i\left(\widetilde{M}\right)$ at (η_1,η_2,η_3). $d\left(\mathcal{T}_i\left(\widetilde{M}\right)\right)(\eta_1,\eta_2,\eta_3)\in\mathscr{L}\left(\mathbb{R}^3,\mathbb{R}\right)$, space of linear continuous functional on $\mathbb{R}^3$. As As $\frac{\partial \mathcal{T}_i(\widetilde{M})}{\partial \eta_j}$, $(j=1,2,3)$ are uniformly bounded, we easily deduce that there exists a constant b^i independent of $\widetilde{M}$ such that

$$\left\|d\left(\mathcal{T}_i\left(\widetilde{M}\right)\right)(\eta_1,\eta_2,\eta_3)\right\|_{\mathscr{L}(\mathbb{R}^3,\mathbb{R})}\leq b^i. \tag{4.40}$$

The domain $\mathscr{P}'$ is convex, being a parallelepiped. For $(\eta_1,\eta_2,\eta_3),(\eta_1',\eta_2',\eta_3')\in\mathscr{P}'$ if for all $i=1,2,3,4$ $\frac{\partial \mathcal{T}_i(\widetilde{M})}{\partial \eta_j}$ are defined on the segment $\left[(\eta_1,\eta_2,\eta_3),(\eta_1',\eta_2',\eta_3')\right]\equiv\left\{(\eta_1,\eta_2,\eta_3)+\alpha\left(\eta_1'-\eta_1,\eta_2'-\eta_2,\eta_3'-\eta_3\right)\mid(0\leq\alpha\leq1)\right\}$, then by the mean value inequality for all $i=1,2,3,4$,

$$\begin{aligned}&\left|\mathcal{T}_i\left(\widetilde{M}\right)(\eta_1,\eta_2,\eta_3)-\mathcal{T}_i\left(\widetilde{M}\right)(\eta_1',\eta_2',\eta_3')\right|\\&\leq b^i\left\|(\eta_1,\eta_2,\eta_3)-(\eta_1',\eta_2',\eta_3')\right\|_{\mathbb{R}^3}.\end{aligned} \tag{4.41}$$

Then for all $\varepsilon>0$, with

$$\eta_\varepsilon=\min_{1\leq i\leq4}\frac{\varepsilon}{b^i}, \tag{4.42}$$

we have

$$\begin{aligned}&\left\|(\eta_1,\eta_2,\eta_3)-(\eta_1',\eta_2',\eta_3')\right\|_{\mathbb{R}^3}<\eta_\varepsilon\\&\Longrightarrow\left\|\mathcal{T}\left(\widetilde{M}\right)(\eta_1,\eta_2,\eta_3)-\mathcal{T}\left(\widetilde{M}\right)(\eta_1',\eta_2',\eta_3')\right\|_{\mathbb{R}^4}\leq\varepsilon\end{aligned} \tag{4.43}$$

The $\frac{\partial \mathcal{T}_i(\widetilde{M})}{\partial \eta_j}$ may not be defined only at points of a plane, hence the later points are adherent to the domain of $\left(\frac{\partial \mathcal{T}_i(\widetilde{M})}{\partial \eta_j}\right)_{i,j}$. Therefore by continuity of $\mathcal{T}\left(\widetilde{M}\right)$ on $\mathscr{P}'$, (4.43) can be extended to $\mathscr{P}'$:

$$\begin{aligned}
&\forall \varepsilon > 0, \exists \eta_\varepsilon > 0, \forall \widetilde{M} \in \mathscr{M}_R, \forall \left(\eta_1, \eta_2, \eta_3\right), \left(\eta_1', \eta_2', \eta_3'\right) \in \mathscr{P}' : \\
&\|(\eta_1, \eta_2, \eta_3) - (\eta_1', \eta_2', \eta_3')\| < \eta_\varepsilon \\
\Longrightarrow &\left\| \mathcal{T}\left(\widetilde{M}\right)(\eta_1, \eta_2, \eta_3) - \mathcal{T}\left(\widetilde{M}\right)(\eta_1', \eta_2', \eta_3') \right\|_{\mathbb{R}^4} \leq \varepsilon.
\end{aligned}$$

Therefore $\mathcal{T}\left(\mathscr{M}_R\right)$ is equicontinuous in $C\left(\mathscr{P}'; \mathbb{R}^4\right)$ and by Arzelà-Ascoli theorem, $\mathcal{T}\left(\mathscr{M}_R\right)$ is relatively compact in $C\left(\mathscr{P}'; \mathbb{R}^4\right)$ i.e. $\mathcal{T}$ is compact on $\mathscr{M}_R$. □

4.4. **Stable convex set under the operator.**

Proposition 4.5. *For all $\widetilde{M} \in \mathscr{M}_R$, we have $\mathcal{T}\left(\widetilde{M}\right) \in E^4$ and*

$$\mathscr{N}\left(\mathcal{T}\left(\widetilde{M}\right)\right) \leq 4cS\left(1 + 2 \cdot \max\left\{4T; \frac{2}{c}\left(b_1 - a_1\right); \frac{1}{c}\left(b_2 - a_2\right)\right\}\right) R^2 + \max\left\{\left\|\overline{N_1^0}\right\|_1; \left\|\overline{N_1^-}\right\|_1; \left\|\overline{N_2^0}\right\|_1; \left\|\overline{N_2^-}\right\|_1; \left\|\overline{N_3^0}\right\|_1; \left\|\overline{N_3^+}\right\|_1; \left\|\overline{N_4^0}\right\|_1; \left\|\overline{N_4^+}\right\|_1\right\}. \tag{4.44}$$

Proof. We have $\widetilde{M} \in \mathscr{M}_R \subset E^4 \implies \mathcal{T}\left(\widetilde{M}\right) \in E^4$. (proposition (4.2)). Using (4.30), (4.33), (4.35) and (4.37) we obtain

$$\left\|\frac{\partial \mathcal{T}\left(\widetilde{M}\right)}{\partial \eta_1}\right\| \leq 4cS\left(1 + 2 \cdot \max\left\{4T; \frac{2}{c}\left(b_1 - a_1\right); \frac{1}{c}\left(b_2 - a_2\right)\right\}\right) \left(\mathscr{N}\left(\widetilde{M}\right)\right)^2 + \max\left\{\left\|\overline{N_2^0}\right\|_1; \left\|\overline{N_2^-}\right\|_1; \left\|\overline{N_3^0}\right\|_1; \left\|\overline{N_3^+}\right\|_1; \left\|\overline{N_4^0}\right\|_1; \left\|\overline{N_4^+}\right\|_1\right\}. \tag{4.45}$$

Similarly we have

$$\left\|\frac{\partial \mathcal{T}\left(\widetilde{M}\right)}{\partial \eta_2}\right\| \leq 4cS\left(1 + 2 \cdot \max\left\{2T; \frac{1}{c}\left(b_1 - a_1\right); \frac{1}{c}\left(b_2 - a_2\right)\right\}\right) \left(\mathscr{N}\left(\widetilde{M}\right)\right)^2 + \max\left\{\left\|\overline{N_1^0}\right\|_1; \left\|\overline{N_1^-}\right\|_1; \left\|\overline{N_3^0}\right\|_1; \left\|\overline{N_3^+}\right\|_1; \left\|\overline{N_4^0}\right\|_1; \left\|\overline{N_4^+}\right\|_1\right\}. \tag{4.46}$$

$$\left\| \frac{\partial \mathcal{T}\left(\widetilde{M}\right)}{\partial \eta_3} \right\| \leq 4cS \left(1 + 2 \cdot \max\left\{2T; \frac{1}{c}\left(b_1 - a_1\right); \frac{1}{c}\left(b_2 - a_2\right)\right\}\right) \left(\mathscr{N}\left(\widetilde{M}\right)\right)^2$$

$$+ \max\left\{\left\|\overline{N_1^0}\right\|_1; \left\|\overline{N_1^-}\right\|_1; \left\|\overline{N_2^0}\right\|_1; \left\|\overline{N_2^-}\right\|_1; \left\|\overline{N_4^0}\right\|_1; \left\|\overline{N_4^+}\right\|_1\right\}. \tag{4.47}$$

By taking $\widetilde{N} = \mathcal{T}\left(\widetilde{M}\right)$ and using (4.27), (4.45), (4.46) and (4.47), we obtain for $\widetilde{M} \in \mathscr{M}_R$ (4.44). □

Let us set

$$p \equiv 4cS \left(1 + 2 \cdot \max\left\{4T; \frac{2}{c}\left(b_1 - a_1\right); \frac{1}{c}\left(b_2 - a_2\right)\right\}\right) \tag{4.48}$$

and

$$q'' \equiv \max\left\{\left\|\overline{N_1^0}\right\|_1; \left\|\overline{N_1^-}\right\|_1; \left\|\overline{N_2^0}\right\|_1; \left\|\overline{N_2^-}\right\|_1;\right.$$

$$\left.\left\|\overline{N_3^0}\right\|_1; \left\|\overline{N_3^+}\right\|_1; \left\|\overline{N_4^0}\right\|_1; \left\|\overline{N_4^+}\right\|_1\right\}. \tag{4.49}$$

Equations (3.25) yield

$$\begin{aligned} \frac{\partial \overline{N_1^0}}{\partial \eta_2}\left(\eta_2, \eta_3\right) &= -c\frac{\partial}{\partial x} N_1^0\left(x, y\right) + c\frac{\partial}{\partial y} N_1^0\left(x, y\right), \\ \left(x, y\right) &= \left(-c\eta_2 - c\eta_3, c\eta_2 - c\eta_3\right) \\ \frac{\partial \overline{N_1^0}}{\partial \eta_3}\left(\eta_2, \eta_3\right) &= -c\frac{\partial}{\partial x} N_1^0\left(x, y\right) - c\frac{\partial}{\partial y} N_1^0\left(x, y\right), \\ \left(x, y\right) &= \left(-c\eta_2 - c\eta_3, c\eta_2 - c\eta_3\right) \end{aligned} \tag{4.50}$$

$$\begin{aligned} \frac{\partial \overline{N_1^-}}{\partial \eta_2}\left(\eta_2, \eta_3\right) &= \frac{\partial}{\partial t} N_1^-\left(t, y\right) + c\frac{\partial}{\partial y} N_1^-\left(t, y\right), \\ \left(t, y\right) &= \left(\frac{1}{c}a_1 + \eta_2 + \eta_3, c\eta_2 - c\eta_3\right) \end{aligned} \tag{4.51}$$

(4.52)

$$\frac{\partial \overline{N_1^-}}{\partial \eta_3}(\eta_2, \eta_3) = \frac{\partial}{\partial t} N_1^-(t,y) - c\frac{\partial}{\partial y} N_1^-(t,y),$$
$$(t,y) = \left(\frac{1}{c}a_1 + \eta_2 + \eta_3, c\eta_2 - c\eta_3\right)$$

$$\left\|\overline{N_1^0}\right\|_\infty \leq \left\|N_1^0\right\|_1 ; \left\|\frac{\partial \overline{N_1^0}}{\partial \eta_2}\right\|_\infty \leq 2c\left\|N_1^0\right\|_1 ; \left\|\frac{\partial \overline{N_1^0}}{\partial \eta_3}\right\|_\infty \leq 2c\left\|N_1^0\right\|_1$$
$$\left\|\overline{N_1^-}\right\|_\infty \leq \left\|N_1^-\right\|_1 ; \left\|\frac{\partial \overline{N_1^-}}{\partial \eta_2}\right\|_\infty \leq (1+c)\left\|N_1^-\right\|_1 ; \left\|\frac{\partial \overline{N_1^-}}{\partial \eta_3}\right\|_\infty \leq (1+c)\left\|N_1^-\right\|_1$$

(4.53) $$\left\|\overline{N_1^0}\right\|_1 \leq \max\{1; 2c\}\left\|N_1^0\right\|_1 ; \left\|\overline{N_1^-}\right\|_1 \leq (1+c)\left\|N_1^-\right\|_1$$

Similarly, (3.26), (3.27), (3.28) yield inequalities which with (4.53) yield

(4.54) $$q'' \leq q,$$

where

(4.55) $$q \equiv \max\Big\{\max\{1;2c\}\left\|N_1^0\right\|_1 ; (1+c)\left\|N_1^-\right\|_1 ; \max\{1;2c\}\left\|N_2^0\right\|_1 ; \max\{2;(1+c)\}\left\|N_2^-\right\|_1 ; \max\{1;2c\}\left\|N_3^0\right\|_1 ; \max\{2;(1+c)\}\left\|N_3^+\right\|_1 ; \max\{1;2c\}\left\|N_4^0\right\|_1 ; (2+c)\left\|N_4^+\right\|_1\Big\};$$

Now, (4.44) yields

(4.56) $$\mathscr{N}\left(\mathcal{T}\left(\widetilde{M}\right)\right) \leq pR^2 + q.$$

Proposition 4.6. *Suppose*

(4.57) $$pq \leq \frac{1}{4}$$

and

(4.58) $$\frac{1-\sqrt{1-4pq}}{2p} \leq R \leq \frac{1+\sqrt{1-4pq}}{2p}.$$

Then $\mathcal{T}(\mathscr{M}_R) \subset \mathscr{M}_{\mathcal{R}}$.

Proof. For $\widetilde{M} \in \mathscr{M}_R$ we have (4.56). Thus to have $\mathcal{T}\left(\widetilde{M}\right) \in \mathscr{M}_{\mathcal{R}}, \forall \widetilde{M} \in \mathscr{M}_R$,it is enough that $R > 0$ satisfies the inequality $pR^2 + q \leq R$ i.e. $pR^2 - R + q \leq 0$.

Now $R > 0$ satisfies the preceding inequality if we have:

$$\begin{cases} 1 - 4pq \geq 0 \\ \frac{1-\sqrt{1-4pq}}{2p} \leq R \leq \frac{1+\sqrt{1-4pq}}{2p}. \end{cases} \tag{4.59}$$

□

4.5. Existence theorem.

Proposition 4.7. *Suppose $pq \leq \frac{1}{4}$. Then the system Σ^1 (3.1-3.12) has a non-negative solution $\widetilde{N} = \left(\widetilde{N_1}, \widetilde{N_2}, \widetilde{N_3}, \widetilde{N_4}\right) \in C\left(\ \mathscr{P}'\ ; \mathbb{R}^4\right)$ such that the partial derivatives $\frac{\partial \widetilde{N_i}}{\partial \eta_1}, \frac{\partial \widetilde{N_i}}{\partial \eta_2}, \frac{\partial \widetilde{N_i}}{\partial \eta_3}$ are defined in $\mathring{\mathscr{P}}'$, except possibly on a finite number of planes, are continuous and bounded for $i = 1, 2, 3, 4$, and such that*

$$\max_{1\leq i\leq 4}\left\{\left\|\widetilde{N_i}\right\|_\infty, \left\|\frac{\partial \widetilde{N_i}}{\partial \eta_1}\right\|_\infty, \left\|\frac{\partial \widetilde{N_i}}{\partial \eta_2}\right\|_\infty, \left\|\frac{\partial \widetilde{N_i}}{\partial \eta_3}\right\|_\infty\right\} \leq \frac{1+\sqrt{1-4pq}}{2p}. \tag{4.60}$$

Proof. For any $\mathscr{M}_R = \left\{\widetilde{M} = \left(\widetilde{M_i}\right)_{i=1}^4 \in E^4 : \mathscr{N}\left(\widetilde{M}\right) \leq R\right\}, (R > 0)$, such that $\left(1 - \sqrt{1-4pq}\right)/2p \leq R \leq \left(1 + \sqrt{1-4pq}\right)/2p$, $\mathscr{M}_{\mathcal{R}}$ is a non-empty convex subset of $C\left(\mathscr{P}'; \mathbb{R}^4\right)$. From propositions 4.1, 4.4 and 4.6, $\mathcal{T}$ is continuous and compact on $\mathscr{M}_{\mathcal{R}}$, and $\mathcal{T}\left(\mathscr{M}_{\mathcal{R}}\right) \subset \mathscr{M}_{\mathcal{R}}$. Thus according to Schauder's theorem 4.1, $\mathcal{T}$ has a fixed point $\widetilde{N} \in \mathscr{M}_{\mathcal{R}}$.

We thus have $\mathscr{N}\left(\widetilde{N}\right) \leq R \leq \left(1 + \sqrt{1-4pq}\right)/2p$. From subsection 4.1, $\widetilde{N} = \left(\widetilde{N_1}, \widetilde{N_2}, \widetilde{N_3}, \widetilde{N_4}\right)$ is a solution of problem Σ^1. $\widetilde{N} = \left(\widetilde{N_1}, \widetilde{N_2}, \widetilde{N_3}, \widetilde{N_4}\right)$ is non-negative from theorem 3.1. On the other hand, $\widetilde{N} \in \mathscr{M}_R \implies \widetilde{N} \in E^4$, hence $\widetilde{N} \in C\left(\ \mathscr{P}'\ ; \mathbb{R}^4\right)$ and $\frac{\partial \widetilde{N_i}}{\partial \eta_1}, \frac{\partial \widetilde{N_i}}{\partial \eta_2}, \frac{\partial \widetilde{N_i}}{\partial \eta_3}$ are defined in $\mathring{\mathscr{P}}'$, except possibly on a finite number of planes, are continuous and bounded $i = 1, 2, 3, 4$. □

For $u \in C\left([0; T] \times [a_1, b_1] \times [a_2, b_2]; \mathbb{R}\right)$ let us put

$$\|u\|_\infty = \sup_{(t,x,y)\in[0;T]\times[a_1,b_1]\times[a_2,b_2]} |u(t, x, y)|, \tag{4.61}$$

and for $N = (N_i)_{i=1}^4 \in C\left([0; T] \times [a_1, b_1] \times [a_2, b_2]; \mathbb{R}^4\right)$

$$\|N\| = \max_{1\leq i\leq 4} \|N_i\|_\infty. \tag{4.62}$$

For $u : [0;T] \times [a_1, b_1] \times [a_2, b_2] \longrightarrow \mathbb{R}$ with domain $D_u \subset [0;T] \times [a_1, b_1] \times [a_2, b_2]$ such that u is bounded on D_u, let us put

$$\|u\|_\infty = \sup_{(t,x,y)\in D_u} |u(t,x,y)|, \tag{4.63}$$

and for $N = (N_i)_{i=1}^4 \in C(D_N; \mathbb{R}^4)$,

$$\|N\| = \max_{1\le i\le 4} \|N_i\|_\infty . \tag{4.64}$$

4.6. **Proof of the main theorem 2.1.**

Proof. From proposition 4.7, and the change of variables $\mathscr{F}$, we deduce that the problem Σ^0 has a solution $N = (N_i)_{i=1}^4 \equiv N(t,x,y)$ such that $N(t,x,y) = \widetilde{N}(\eta_1, \eta_2, \eta_3)$ where $\widetilde{N}$ is a solution of Σ^1.

We have

$$\left\|\widetilde{N_i}\right\|_\infty = \sup_{(\eta_1,\eta_2,\eta_3)\in\mathscr{P}'} \left|\widetilde{N_i}(\eta_1,\eta_2,\eta_3)\right| = \sup_{(t,x,y)\in\mathscr{P}} |N_i(t,x,y)| = \|N_i\|_\infty$$

hence (4.60) yields

$$\|N\| = \max_{1\le i\le 4} \{\|N_i\|_\infty\} \le \frac{1+\sqrt{1-4pq}}{2p}. \tag{4.65}$$

One has from the inverse of the change of variables,

$$N(t,x,y) = \widetilde{N}\left(\frac{1}{c}x; \frac{1}{2}t - \frac{1}{2c}x + \frac{1}{2c}y; \frac{1}{2}t - \frac{1}{2c}x - \frac{1}{2c}y\right),$$

hence

$$\begin{cases} \frac{\partial N_i}{\partial t} = \frac{1}{2}\frac{\partial \widetilde{N_i}}{\partial \eta_2} + \frac{1}{2}\frac{\partial \widetilde{N_i}}{\partial \eta_3}; \\ \frac{\partial N_i}{\partial x} = \frac{1}{c}\frac{\partial \widetilde{N_i}}{\partial \eta_1} - \frac{1}{2c}\frac{\partial \widetilde{N_i}}{\partial \eta_2} - \frac{1}{2c}\frac{\partial \widetilde{N_i}}{\partial \eta_3}; \\ \frac{\partial N_i}{\partial y} = \frac{1}{2c}\frac{\partial \widetilde{N_i}}{\partial \eta_2} - \frac{1}{2c}\frac{\partial \widetilde{N_i}}{\partial \eta_3}. \end{cases} \tag{4.66}$$

We deduce that

$$\begin{aligned} \left\|\frac{\partial N_i}{\partial t}\right\|_\infty &\le \frac{1}{2}\left\|\frac{\partial \widetilde{N_i}}{\partial \eta_2}\right\|_\infty + \frac{1}{2}\left\|\frac{\partial \widetilde{N_i}}{\partial \eta_3}\right\|_\infty \\ \left\|\frac{\partial N_i}{\partial t}\right\|_\infty &\le \frac{1+\sqrt{1-4pq}}{2p} \end{aligned} \tag{4.67}$$

$$\left\|\frac{\partial N_i}{\partial x}\right\|_\infty \leq \frac{1}{c}\left\|\frac{\partial \widetilde{N_i}}{\partial \eta_1}\right\|_\infty + \frac{1}{2c}\left\|\frac{\partial \widetilde{N_i}}{\partial \eta_2}\right\|_\infty + \frac{1}{2c}\left\|\frac{\partial \widetilde{N_i}}{\partial \eta_3}\right\|_\infty$$

$$\left\|\frac{\partial N_i}{\partial x}\right\|_\infty \leq \frac{2}{c}\frac{1+\sqrt{1-4pq}}{2p} \tag{4.68}$$

$$\left\|\frac{\partial N_i}{\partial y}\right\|_\infty \leq \frac{1}{2c}\left\|\frac{\partial \widetilde{N_i}}{\partial \eta_2}\right\|_\infty + \frac{1}{2c}\left\|\frac{\partial \widetilde{N_i}}{\partial \eta_3}\right\|_\infty$$

$$\left\|\frac{\partial N_i}{\partial y}\right\|_\infty \leq \frac{1}{c}\frac{1+\sqrt{1-4pq}}{2p}. \tag{4.69}$$

Thus we can deduce (2.20).

Also, $\frac{\partial \widetilde{N_i}}{\partial \eta_1}, \frac{\partial \widetilde{N_i}}{\partial \eta_2}, \frac{\partial \widetilde{N_i}}{\partial \eta_3}$ are defined except possibly on a finite number of planes including the four planes with respective equations

$$-c\eta_2 - c\eta_3 = a_1, -c\eta_1 - 2c\eta_3 = a_2, c\eta_1 + 2c\eta_2 = b_2, 2c\eta_1 + c\eta_2 + c\eta_3 = b_1. \tag{4.70}$$

We deduce that $\frac{\partial N_i}{\partial t}, \frac{\partial N_i}{\partial x}, \frac{\partial N_i}{\partial y}$ are defined except possibly on a finite number of planes including the transformed of the planes with equations (4.70) by the inverse $\mathscr{F}^{-1}$; direct calculations using $\mathscr{F}^{-1}$ give the equations (2.19).

Suppose that the problem Σ^0 have two solutions M and N satisfying

$$\begin{cases} \|M\| = \max\limits_{1\leq i\leq 4} \|M_i\|_\infty \leq \frac{1+\sqrt{1-4pq}}{2p} \\ \|N\| = \max\limits_{1\leq i\leq 4} \|N_i\|_\infty \leq \frac{1+\sqrt{1-4pq}}{2p} \end{cases}. \tag{4.71}$$

$\widetilde{M}, \widetilde{N}$ defined on $\mathscr{P}'$ by $\widetilde{M}(\eta_1,\eta_2,\eta_3) = M(t,x,y)$ and $\widetilde{N}(\eta_1,\eta_2,\eta_3) = N(t,x,y)$ are solutions of problem Σ^1.

We have $\left\|\widetilde{N_i}\right\|_\infty = \|N_i\|_\infty$ and $\left\|\widetilde{M_i}\right\|_\infty = \|M_i\|_\infty$, hence $\left\|\widetilde{N}\right\| = \max_{1\leq i\leq 4}\left\|\widetilde{N_i}\right\|_\infty = \max_{1\leq i\leq 4}\|N_i\|_\infty = \|N\|$ and $\left\|\widetilde{M}\right\| = \|M\|$.

Equations (4.71) yields $\left\|\widetilde{M}\right\| \leq \left(1+\sqrt{1-4pq}\right)/2p, \left\|\widetilde{N}\right\| \leq \left(1+\sqrt{1-4pq}\right)/2p$. Hence (4.19) yields

$$\left\|\mathcal{T}\left(\widetilde{M}\right) - \mathcal{T}\left(\widetilde{N}\right)\right\| \leq p' \cdot 2 \cdot \frac{1+\sqrt{1-4pq}}{2p}\left\|\widetilde{M}-\widetilde{N}\right\|$$

$$\leq \frac{p'}{p}\cdot\left(1+\sqrt{1-4pq}\right)\left\|\widetilde{M}-\widetilde{N}\right\|. \tag{4.72}$$

But $\widetilde{M}$ and $\widetilde{N}$ are fixed points of $\mathcal{T}$, hence (4.72) is written

$$\left\|\widetilde{M}-\widetilde{N}\right\| \leq \frac{p'}{p} \cdot \left(1+\sqrt{1-4pq}\right) \left\|\widetilde{M}-\widetilde{N}\right\|, \tag{4.73}$$

i.e.,

$$\left(1-\frac{p'}{p} \cdot \left(1+\sqrt{1-4pq}\right)\right) \left\|\widetilde{M}-\widetilde{N}\right\| \leq 0. \tag{4.74}$$

From (4.19) and (4.48) we have

$$\frac{p'}{p} = \frac{\max\left\{T, \frac{1}{c}\left(b_1-a_1\right), \frac{1}{c}\left(b_2-a_2\right)\right\}}{1+2\cdot\max\left\{4T; \frac{2}{c}\left(b_1-a_1\right); \frac{1}{c}\left(b_2-a_2\right)\right\}} \equiv \frac{m_1}{1+2m_2}, \tag{4.75}$$

$m_1 \leq m_2$; $\frac{p'}{p} = \frac{m_1}{1+2m_2} = \frac{1}{\frac{1}{m_2}+2} \cdot \frac{m_1}{m_2}$; and successively

$$\begin{cases} \frac{1}{\frac{1}{m_2}+2} < \frac{1}{2} \\ \frac{m_1}{m_2} \leq 1 \end{cases};$$

$$\frac{p'}{p} = \frac{1}{\frac{1}{m_2}+2} \cdot \frac{m_1}{m_2} < \frac{1}{2}.$$

On the other hand $1+\sqrt{1-4pq} < 2$, hence $\frac{p'}{p} \cdot \left(1+\sqrt{1-4pq}\right) < 1$, i.e.,

$$1-\frac{p'}{p} \cdot \left(1+\sqrt{1-4pq}\right) > 0. \tag{4.76}$$

From (4.74) and (4.76) we deduce that

$$\left\|\widetilde{M}-\widetilde{N}\right\|. \leq 0 \tag{4.77}$$

Therefore $\widetilde{M} = \widetilde{N}$ and $M = N$. Hence the uniqueness. □

5. Conclusion

We show that under some condition on the data, the initial-boundary value problem in a rectangle for the two dimension Broadwell's 4-velocity model has a continuous unique non-negative solution bounded with its first partial derivatives.

We provide a bound for the solution and the derivatives. Our perspectives are now to study the case where the data are general.

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