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ON A CHARACTERIZATION OF FREE PRODUCTS OF GROUPS WITH COMMUTING SUBGROUPS BY THE TREES

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ABSTRACT. In this paper, we prove that when a group G acts on a tree Γ such that the quotient graph G/Γ is $path$ 2 and when in G, any element of the stabilizer of one of the segments of this path commutes with those of the stabilizer of the other segment, G may be identified with the free product of groups with commuting subgroups and every free product of groups with commuting subgroups is obtained uniquely in this way. Also, we give here some illustrative trees of this characterization.

1. INTRODUCTION AND RESULTS

Various links between group theory and trees was studied by several authors. R. Möller and J. Vonk in [15] proved that if G is closed subgroup of the automorphism group of a tree Γ and G leaves no non-trivial subtree invariant and fixes no end of Γ, then the subgroup generated by the pointwise stabilizers of $half-trees$ is topologically simple. In their book $Groups\ acting\ on\ graphs$ [2], W. Dicks and M.J. Dunwoody developed powerful techniques for forcing a group to act on a tree. One of their results is a method of building a tree from a graph such that

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the automorphisms group of of this graph acts naturally on the tree. In the same way, R.G. Möller in [14] pointed out how one can, by using this method, reduce questions about ends of graphs to questions about trees. He studied the automorphisms group of regular trees and the action of the automorphisms group of a locally finite graph on the ends of the graph and finally classified the subgroups that have index less than 2^{N_0}.

Now, connections between free constructions of groups and graphs have been a subject of many investigations. J.P. Macmanus in [12] proved that a connected, locally finite, quasi-transitive graph is necessarily accessible. This leads to a complete classification of the finitely generated groups which are quasi-isometric to planar graphs. In particular, such a group is virtually a free product of free surface groups, and then virtually admits a planar Cayley graph. About characterization of free constructions of groups by the trees, Y. Tumartin in [16] proved that only free groups act freely on trees. He established that an action of a group G on a tree Γ defines a decomposition of G called a graph of groups, $(\mathcal{G}, \Gamma)$. Conversely, from a graph of groups $(\mathcal{G}, \Gamma)$, can be defined a group π (called the fundamental group) and a tree Γ' on which π acts. His main result is the statement that these two constructions are inverse to each other, i.e. that a decomposition into a graph of groups uniquely corresponds to an action on a tree. Also in relation to the characterization of free constructions of groups by the trees, let G be a group acting on a tree Γ. According to J.P. Serre in [11], when we know the quotient graph G/Γ as well as the stabilizers G_x $(x \in V(\Gamma))$ and G_y $(y \in E(\Gamma))$ of the vertices and edges, we can distinguish two special cases.

The first case occurs when any G_x and G_y reduces to $\{1\}$ (in which way we said that G acts freely), the group G is then free, see [11, Theorem 4]. This case gives a simple proof of Schreier's theorem, according to which a subgroup of a free group is free. Conversely, every free group acts freely on a tree. See [11, Proposition 15].

The second and last case arises when the quotient graph G/Γ is a segment $T = \overset{P}{\bullet}\overset{y}{\text{———}}\overset{Q}{\bullet}$. For this case G may be identified with the free product with amalgamation $G_P \underset{G_y}{*} G_Q$, see [11, Theorem 6], and every free product of two groups with amalgamation acts on a tree with segment as fundamental domain. See [11, Theorem 7].

In [6], Loginova pointed out a link between free products of groups with amalgamation and free products of groups with commuting subgroups. She established that every free product of groups with commuting subgroups can be written as an amalgamated free product of two amalgamated free products of groups, what we call a double amalgamation of groups.
In this paper we use this link and the characterization of free products of groups with amalgamated subgroups by trees to obtain a characterization of free products of groups with commuting subgroups by trees. That is:

Theorem 1.1. *Let G be a group acting on a graph Γ' and let*

$$T' = \overset{P'}{\bullet}\overset{y_1}{\longrightarrow}\overset{R}{\bullet}\overset{y_2}{\longrightarrow}\overset{Q'}{\bullet}$$

a $path$ 2 be a fundamental domain of Γ' mod G.

Let $G_{P'}$, G_R, $G_{Q'}$, $G_{y_1} = G_{\overline{y_1}}$ and $G_{y_2} = G_{\overline{y_2}}$ be the stabilizers of the vertices and edges of T'. If every element of G_{y_1} commutes with those of G_{y_2}, then the following properties are equivalent:

(1) *Γ' is a tree;*
(2) *there exists a tree Γ (and only one, up to isomorphism) on which G acts with fundamental domain a segment;*
(3) *the homomorphism $G_{P'} \underset{[G_{y_1}, G_{y_2}]}{*} G_{Q'} \longrightarrow G$ induced by the inclusions $G_{P'} \longrightarrow G$ and $G_{Q'} \longrightarrow G$ is an isomorphism.*

Conversely, we prove that every free product of two groups with commuting subgroups acts on a tree with $path$ 2 as fundamental domain. It is:

Theorem 1.2. *Let $G = G_1 \underset{[H,K]}{*} G_2$ be the free product of the groups G_1 and G_2 with commuting subgroups H and K. Then, there exists a tree Γ' (and only one, up to isomorphism) on which G acts with fundamental domain a $path$ 2*

$$T' = \overset{P'}{\bullet}\overset{y_1}{\longrightarrow}\overset{R}{\bullet}\overset{y_2}{\longrightarrow}\overset{Q'}{\bullet},$$

the vertices and edges of this path have $G_{P'} = G_1$; $G_{Q'} = G_2$; $G_R = H \times K$; $G_{y_1} = H$ and $G_{y_2} = K$ as their respective stabilizers.

The illustrative trees given in this paper can help to derive the following.

Corollary 1.1. *Let $G = G_1 \underset{[H,K]}{*} G_2$ be the free product of the groups G_1 and G_2 with commuting subgroups H and K. Let Γ' be a tree on which acts G with T', a $path\ 2$ as a fundamental domain. Then we have:*

(1) *every segment in Γ' is a result of the action of an element of G on a segment of T'. More precisely, for a given segment in Γ', there exists another segment in Γ' connected to the first one such that the resultant $path\ 2$ is obtained from the action of an element of G on T' and,*

(2) *every $path\ 2$ of Γ' is not necessarily a result of the action of an element of G on T'.*

2. Preliminaries notions and results

In this section, we collect some imformations and properties on some free constructions and groups acting on graphs which will be usefull in this paper. See [4, 7–9, 13] for free constructions and [1, 3, 10, 11, 16] for groups acting on trees, for more details.

2.1. **Some Free Constructions of Groups.**

Definition 2.1 (Free product of groups with amalgamation). *Let G_1, G_2 and H be abstract groups and let $f_i : H \longrightarrow G_i$ $(i = 1, 2)$ be monomorphisms of groups. A free product of G_1 and G_2 with amalgamated subgroup H is defined to be a pushout*

$$\begin{tikzcd} H \arrow[r, "f_2"] \arrow[d, "f_1"'] & G_2 \arrow[d, "\varphi_2"] \\ G_1 \arrow[r, "\varphi_1"] & G \end{tikzcd}$$

in the category of groups, i.e. a group G together with homomorphisms $\varphi_i : G_i \longrightarrow G$ $(i = 1, 2)$, satisfying the following universal property: for any pair of homomorphisms $\psi_1 : G_1 \longrightarrow G'$ and $\psi_2 : G_2 \longrightarrow G'$ into a group G' with $\psi_1 f_i = \psi_2 f_2$, there exists a unique homomorphism $\psi : G \longrightarrow G'$ such that $\psi\varphi_1 = \psi_1$ and $\psi\varphi_2 = \psi_2$ i.e. the *following diagram is commutative:*

$$\begin{tikzcd}
& H \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\
G_1 \arrow[r, "\varphi_1"] \arrow[dr, "\psi_1"'] & G \arrow[d, dotted, "\exists!\psi"] & G_2 \arrow[l, "\varphi_2"'] \arrow[dl, "\psi_2"] \\
& G' &
\end{tikzcd}$$

The free product of groups G_1 and G_2 with amalgamated subgroup H is unique (up to isomorphism) and we denote this group by $G_1 \underset{H}{\star} G_2$.

Definition 2.2 (Free product of groups with commuting subgroups)**.** *Let H be a subgroup of a group G_1 and let K be a subgroup of a group G_2. The group $G = (G_1 * G_2,\ [H, K] = 1)$ generated by all the generators of groups G_1 and G_2 and defined by all the relators of groups G_1 and G_2 together with all the relations of the form $[h, k] = 1$, for all $h \in H$ and $k \in K$, is called the free product of groups G_1 and G_2 with commuting subgroups H and K. In other words, G is the free product of groups G_1 and G_2 modulo the normal closure of $[H, K]$, the commutator of the subgroups H and K, in the free product $G_1 * G_2$ i.e. $G = (G_1 * G_2)/([H, K])^{(G_1 * G_2)}$. The free product of groups G_1 and G_2 with commuting subgoups H and K is unique (up to isomorphism) and we denote this group by $G = G_1 \underset{[H,K]}{*} G_2$.*

Remark 2.1. *Loginova in [6] studied the residual finiteness of free products of groups with commuting subgroups. She established that the free product $G = G_1 \underset{[H,K]}{*} G_2$ of groups G_1 and G_2 with commuting subgroups H and K can be written as a double amalgamation: $(G_1 \underset{H}{*} (H \times K)) \underset{H \times K}{*} ((H \times K) \underset{K}{*} G_2)$.*

2.2. **Group acting on graph.**

Definition 2.3. *A (oriented) graph Γ consists of a set $X = V(\Gamma)$, a set $Y = E(\Gamma)$ and two maps*

$$\begin{array}{ccc} Y & \longrightarrow & X \times X \\ y & \longmapsto & (o(y), t(y)) \end{array}$$

and

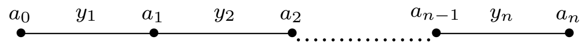
$$\begin{array}{ccc} Y & \longrightarrow & Y \\ y & \longmapsto & \overline{y} \end{array}$$

which satisfy the following conditions: for each $y \in Y$, we have $\overline{\overline{y}} = y$, $\overline{y} \neq y$ and $o(y) = t(\overline{y})$.

An element $P \in X$ is called a $vertex$ of Γ, an element $y \in Y$ is called an (oriented) $edge$, and $\overline{y}$ is called the $inverse\ edge$. For any $y \in Y$, the vertex $o(y)$ is called the $origin$ of y and the vertex $t(y)$ its $terminus$. These two vertices are called the $extremities$ of y.

Definition 2.4. *A $path$ in a graph is a finite or infinite sequence of edges which join a sequence of vertices which are all distinct. In other words, a path is a sequence of non-repeated vertices connected through edges present in a graph.*

Remark 2.2. *When a path contains n edges ($n \in \mathbb{N}$), it is of $length\ n$ and is denoted $path\ n$. In this paper, a $path\ n$*



is often designated by $a_0 - a_1 - a_2 - ... - a_n$. Note that a $segment$ is a $path$ 1 and a $circuit$ is a path such that the origin of the first edge is the terminus of the last edge.

Definition 2.5. *A graph is said to be connected if any two vertices are the extremities of a $path$.*

Definition 2.6. *A tree is a connected non-empty graph without circuits (in other words, a tree is a graph in which any two vertices are connected by exactly one path).*

Definition 2.7. *An action of a group G on a graph Γ is an application*
$\cdot : \underset{(g,x)}{G \times \Gamma} \underset{\longmapsto}{\longrightarrow} \underset{g\cdot x}{\Gamma}$ *such that for any $g, h \in G$, $P \in V(\Gamma)$ and $y \in E(\Gamma)$, we have:*

$i)$ $1 \cdot P = P$ and $1 \cdot y = y$ (1 is the identity of G);

$ii)$ $g \cdot (h \cdot P) = (gh) \cdot P$ and $g \cdot (h \cdot y) = (gh) \cdot y$.

If $i : E(\Gamma) \longrightarrow E(\Gamma)$ is the map $y \longmapsto i(y) = \overline{y}$, then $g.i(y) = i(g.y)$.

Definition 2.8. *Let Γ be a graph on which acts a group G. Let $x \in \Gamma$ i.e., $x \in V(\Gamma)$ or $x \in E(\Gamma)$. The $stabilizer$ of x is the set*

$$G_x = \{g \in G, g.x = x\}.$$

G_x is a subgroup of G.

Remark 2.3. *If $y \in E(\Gamma)$, then $G_y = G_{\overline{y}}$. Indeed, let $y \in E(\Gamma)$. For any $g \in G_y$, $g.\overline{y} = \overline{g.y} = \overline{y}$. So, $G_y \subseteq G_{\overline{y}}$. Conversely, for any $g \in G_{\overline{y}}$, $g.y = g.\overline{\overline{y}} = \overline{g.\overline{y}} = \overline{\overline{y}} = y$. Thus, $G_{\overline{y}} \subseteq G_y$. Finally, $G_y = G_{\overline{y}}$.*

Definition 2.9. *Let Γ be a graph on which acts a group G. An $inversion$ is a pair (g, y) where $g \in G$ and $y \in E(\Gamma)$ such that $g.y = \overline{y}$. If there is no such pair we say that G acts $without\ inversion$.*

Let G be a group that acts on a graph Γ. We define on Γ the relation $\mathcal{R}$ as follows: $x\mathcal{R}y \Leftrightarrow \exists g \in G$ such that $y = g.x$. The relation $\mathcal{R}$ as defined is an equivalence relation.

For any $x \in \Gamma$, the class $[x]$ of x modulo $\mathcal{R}$ is defined to be:

$$[x] = \{g.x;\ g \in G\}.$$

Definition 2.10. *Every equivalence class of $\mathcal{R}$ as defined above is called a $G-orbite$. For any x belonging to a graph Γ, we put $\mathcal{O}(x)$ the $orbite$ of x.*

$$[x] = \mathcal{O}(x) = \{y \in \Gamma;\ y\mathcal{R}x\} = \{g.x;\ g \in G\}.$$

If G acts on Γ without inversion, the $quotient\ graph$ noted by G/Γ is the set of orbites of Γ. The set of vertices (resp. of edges) of G/Γ is the quotient of $V(\Gamma)$ (resp. de $E(\Gamma)$) under the action of G.

Throughout this paper, every action of group on a graph is without inversion.

In the following, we present the notion of graph morphism. It is in fact an application between two graphs which preserves the structure of these graphs (that is to say which preserves the adjacency relations present in the initial graph). Note that in [5, Theorem 2.7], K. K. Williams proved that in the category of graphs, a graph morphism is an isomorphism if and only if it is bijective as a function. We can therefore limit ourselves to the following definitions as described in [10] p. 19.

Definition 2.11. *The $morphism$ from a graph Γ to a graph Γ' is a map $\alpha : \Gamma \longrightarrow \Gamma'$ which takes edges to edges and the origin and terminus of $\alpha(e)$ are, respectively, the images of the origin and terminus of edge e.*

A morphism $\alpha : \Gamma \longrightarrow \Gamma'$ is an $isomorphism$ if there is a morphism $\beta : \Gamma' \longrightarrow \Gamma$ such that $\alpha \circ \beta$ and $\beta \circ \alpha$ are identity maps.

Let G be a group acting on a graph Γ and let T be a segment or a path 2 in Γ. In this paper, $[T]$ denotes the set $\{g.T; g \in G\}$.

Proposition 2.1 ([11]). *Let Γ be a connected graph on which acts without inversion a group G. Every subtree T' of G/Γ lifts to a subtree of Γ.*

Definition 2.12. *Let G be a group acting on a graph Γ. The $fundamental\ domain$ of Γ modulo G, is every subgraph T of Γ such that $T \longrightarrow G/\Gamma$ is an isomorphism.*

Remark 2.4. *If a group G acts on a graph Γ with a subgraph T as a fundamental domain, then $G.T = \Gamma$. In fact $G.T = G.(G/\Gamma) = \Gamma$.*

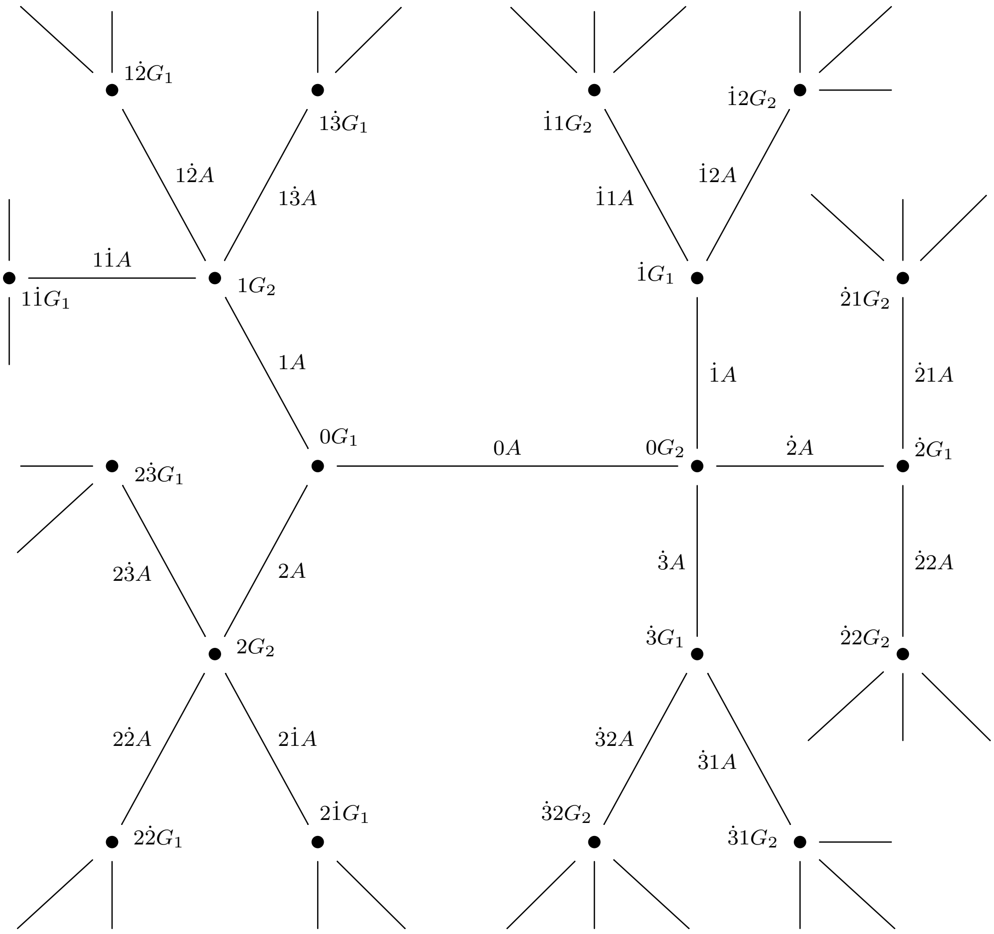
If a group G acts on a graph with a segment T as a fundamental domain, any segment of this graph is a result of the action of an element of G on T. Indeed, if T' is any segment in this graph, it has the same orbite with T, since the last one is a fundamental domain of this graph modulo G. So, $[T'] = [T]$ and then $T' = g.T$ with $g \in G$.

J.P. Serre in [11] (Theorem 6 and Theorem 7) established a convenient equivalence between amalgamated free products of groups and groups acting on trees with a segment as fundamental domain. In the following, we give more details about the tree of free product of groups with amalgamated subgroup $G = \mathbb{Z}/3\mathbb{Z} \underset{A}{*} \mathbb{Z}/4\mathbb{Z}$ with $A = \{0\}$. The corresponding tree is briefly presented in [11], example 4.2 p.35.

Put $G_1 = \mathbb{Z}/3\mathbb{Z} = \{0; 1; 2\}$, $G_2 = \mathbb{Z}/4\mathbb{Z} = \{0; \dot{1}; \dot{2}; \dot{3}\}$ and $A = \{0\}$. The tree on which acts G is given on the next page.

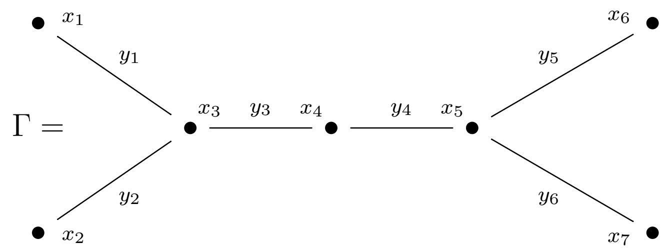
Note that in this tree, any vertex is indistinguishable to some other vertices; for example, $2G_2,\ 2\dot{1}G_2,\ 2\dot{2}G_2$ and $2\dot{3}G_2$ are geometrically at the identical position.

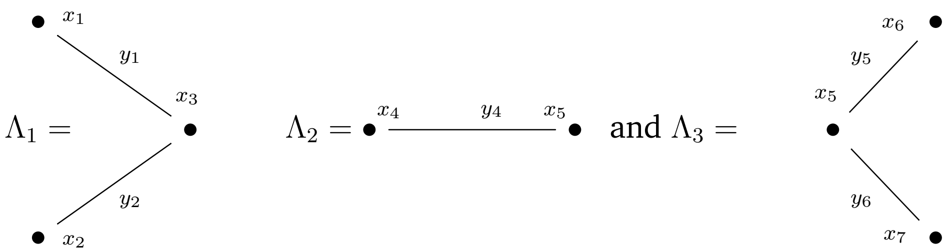
In the following section, we extend the notion of contraction of disjoint subtrees in a graph presented in [10] p. 26 to the contraction of subtrees which are not necessarily disjoint. Let Γ be any connected graph and let Λ be a subgraph which is a union of a family Λ_i, $i \in I$ of subtrees where any two subtrees Λ_i and Λ_j of this family are disjoint or have at most one common vertex. We shall define a new graph called a $contraction$ of Λ in Γ or a $subtrees\ contraction$ of Λ in Γ denoted by Γ/Λ as follow. Each Λ_i gives one vertex a_i of Γ/Λ and each vertex of Γ outside Λ also gives one vertex of Γ/Λ. If P is a common vertex of Λ_i and Λ_j, it becomes an edge $P = e_{ij}$ between a_i and a_j in Γ/Λ. The edge set of Γ/Λ is defined as the union of the set of e_{ij} and the set of edges of Γ which are not in Λ. The origin and terminus of any edge of Γ/Λ is defined from the corresponding map on Γ by passing to quotients (see [10] p. 26) and those of e_{ij} are a_i and a_j respectively. The map $e \longmapsto \overline{e}$ in Γ and $e_{ij} \longmapsto \overline{e_{ij}} = e_{ji}$ defines the inverse of any edge of Γ/Λ.

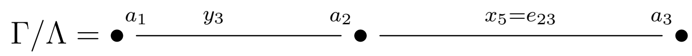


In the example given on the next page, $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$.

Using [10, Corollary 13 p. 27] or [11, Corollary 2 p. 23] we get the following proposition.







Proposition 2.2. *Let Γ be a graph and Λ a subgraph which is a union of family $\Lambda_i,\ i \in I$ of subtrees where every two subtrees Λ_i and Λ_j of this family are disjoint or have at most one common vertex. Γ is a tree if and only if Γ/Λ is one.*

Proof. Let us distinguish two cases:

i. If the family $\Lambda_i,\ i \in I$ is such that every two subtrees Λ_i and Λ_j are disjoint, then by [10, Corollary 13 p. 27], Γ is a tree if and only if Γ/Λ is one.

ii. Assume that in Γ, there exists two subtrees Λ_i and Λ_j of Λ having exactly one common vertex P_{ij}. Then, the vertex P_{ij} stretches into an edge e_{ij} between these subtrees to make them disjoint. Note that the obtained graph Γ_1 after this process is a tree if and only if Γ is one. Indeed:
   - If Γ is connected then Γ_1 is too since there is no cancelled edge by adding the edge e_{ij}. Conversely, if Γ is not connected then Γ_1 is not as well, since the added edge is not between two different vertices of Γ.
   - If Γ does not have a circuit then Γ_1 does not have one, since the added edge is not between two different vertices of Γ. Conversely, if Γ have a circuit then Γ_1 has one, since there is no cancelled edge by adding the edge e_{ij}.

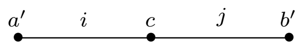
By doing the same thing iteratively for all two subtrees Λ_i and Λ_j of Λ having exactly one common vertex, we obtain a graph Γ' containing a disjoint union of family Λ_i, $i \in I$. By the previous justification, Γ is a tree if and only if Γ' is one. Using again [10, Corollary 13 p. 27], Γ' is a tree if and only if Γ'/Λ is one. Observe that $\Gamma'/\Lambda = \Gamma/\Lambda$ to finish the proof.

$\square$

3. Proof of Theorem 1.1 and Theorem 1.2

3.1. Proof of Theorem 1.1. We first prove the following lemmas.

Lemma 3.1. *Let G be a group acting on a graph Γ' and let*

$T' =$ 

a $path$ 2 be a fundamental domain of Γ' mod G.

Let G_c, $G_i = G_{\bar{i}}$ and $G_j = G_{\bar{j}}$ be the stabilizers of the vertex c and the edges i and j of T'. If any element of G_i commutes with those of G_j, then $G_c = G_i \times G_j$.

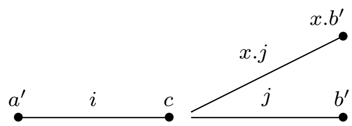
Proof. Assume that any element of G_i commutes with those of G_j. Let prove that $G_c = G_i \times G_j$.

(1) $G_i \times G_j \subset G_c$. Indeed, let $x \in G_i \times G_j = \{x_1x_2 / x_1 \in G_i,\ x_2 \in G_j$ and $x_1x_2 = x_2x_1\}$. There exist $x_1 \in G_i$ and $x_2 \in G_j$ such that $x = x_1x_2$ and $x_1x_2 = x_2x_1$. Then, $x_1 \in G_i \Rightarrow x_1 \in G_c$ since stabilizing an edge means stabilizing its vertices. $G_i = G_{a'} \cap G_c$. Also, $x_2 \in G_j \Rightarrow x_2 \in G_c$ since $G_j = G_{b'} \cap G_c$. Thus, $x = x_1x_2 \in G_c$.

(2) Let now prove that $G_c \subset G_i \times G_j$. Let $x \in G_c$. Put $T'_1 = x.T'$. We can distinguish the following cases:

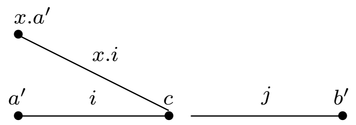
   (a) $T'_1 = T'$ in this case, $x \in G_i$; $x \in G_j$ and so $x \in G_i \times G_j$.

   (b) $T'_1 = a' - c - x.b'$ in



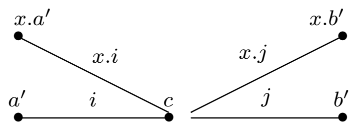
In this case, $x \in G_i$ and $x \notin G_j$. And, $x \in G_i \Rightarrow x \in G_i \times G_j$.

(c) $T_1' = x.a' - c - b'$ in



In this case, $x \in G_j$ and $x \notin G_i$. And, $x \in G_j \Rightarrow x \in G_i \times G_j$.

(d) $T_1' = x.a' - c - x.b'$ in



In this case, $x \in G_c$, $x \notin G_i$ and $x \notin G_j$.
Consider the path $T_2' = a' - c - x.b'$. There exists $x_1 \in G$ such that $x_1.T' = T_2'$. So $x_1^{-1}.T_2' = T'$. See that $x_1^{-1} \in G_i$ and $x_1^{-1}x \in G_j$. Considering $x = x_1 x_1^{-1} x$ we can take $x_2 = x_1^{-1}x \in G_j$, so that $x = x_1 x_2$. And since $x_1 x_2 = x_2 x_1$ by the hypothesis, clearly $x \in G_i \times G_j$. Finally the equality $G_c = G_i \times G_j$ holds.

□

Lemma 3.2. *Let G be a group acting on a tree Γ' with a $path$ 2 as a fundamental domain. Then there exists a tree Γ (and only one, up to isomorphism) on which G acts with a segment as fundamental domain.*

Proof. Let G be a group acting on a tree Γ' with

$$T' = \overset{a'}{\bullet}\overset{i}{\text{———————}}\overset{c}{\bullet}\overset{j}{\text{———————}}\overset{b'}{\bullet}$$

a $path$ 2 as a fundamental domain. We designate this action by $(.)$. Let determine a tree Γ on which G acts with the segment

$$T = \overset{a}{\bullet}\overset{c}{\text{———————————}}\overset{b}{\bullet}$$

as a fundamental domain. This action will be designated by $(\circ)$.

Put

$$T_1 = \overset{a'}{\bullet}\overset{i}{\text{———————————}}\overset{c}{\bullet}\ ;\ T_2 = \overset{c}{\bullet}\overset{j}{\text{———————————}}\overset{b'}{\bullet}\ .$$

According to [11, Theorem 7], there exists a tree Γ_1 (subtree of Γ') on which the subset $G_{a'} \underset{G_i}{*} G_c$ of G acts with a segment as fundamental domain. Since two segments in a graph are isomorphic, without lose to the generality, consider that this fundamental domain is T_1. By contraction, we put tree Γ_1 to a vertex a. Argue similarly, the subset $G_c \underset{G_j}{*} G_{b'}$ of G acts on a tree Γ_2 (subtree of Γ') with segment T_2 as fundamental domain. Again by contraction, we put Γ_2 to a vertex b. Now, c is a common part of Γ_1 and of Γ_2, it stretches into an edge between a and b.

Note that Γ' is made up of "multiples" of Γ_1 and Γ_2, that is to say $\Gamma' = \{g.\Gamma_1 \cup g'.\Gamma_2,\ g, g' \in G\}$. Indeed, when G acts on Γ', the action of an element $g \in G$ on T', gives a $path$ 2 in Γ' such that any segment in $g.T'$ will be a result of the action of g on either T_1 or T_2.

Now, $g.T_1$ and $g.T_2$ are segments in $g.\Gamma_1$ and in $g.\Gamma_2$ respectively. So

$$\Gamma' = G.T' = G.T_1 \cup G.T_2 = G.\Gamma_1 \cup G.\Gamma_2.$$

For any $g \in G$, contract in Γ' subtree $g.\Gamma_1$ to a vertex $g \circ a$; a subtree $g.\Gamma_2$ to a vertex $g \circ b$ and the edge between $g \circ a$ and $g \circ b$ is $g \circ c$.

Put $\Gamma = \Gamma'/\Lambda$ where Λ is the union of subtrees $g.\Gamma_1$ and of $g.\Gamma_2$, $g \in G$. See that the graph Γ is defined by:

$$V(\Gamma) = \{g \circ a,\ g \circ b;\ g \in G\};\quad E(\Gamma) = \{g \circ c;\ g \in G\}.$$

Note that Γ is a tree as a subtrees contraction in the tree Γ'. See Proposition 2.2.

($\circ$) defines an action of G on Γ as we illustrate in the following

$$G \times \Gamma \longrightarrow \Gamma\ \ ; (g,\ \overset{g'\circ a}{\bullet}\!\!\xrightarrow[]{g'\circ c}\!\!\overset{g'\circ b}{\bullet}\ ) \longmapsto \overset{gg'\circ a}{\bullet}\!\!\xrightarrow[]{gg'\circ c}\!\!\overset{gg'\circ b}{\bullet} \quad \text{with } g' \in G.$$

See that $G/\Gamma = \overset{a}{\bullet}\!\!\overset{c}{\text{———}}\!\!\overset{b}{\bullet}$ and then $T = \overset{a}{\bullet}\!\!\overset{c}{\text{———}}\!\!\overset{b}{\bullet}$ is the fundamental domain of Γ mod G. $\square$

We are now ready to prove the Theorem 1.1.

Proof. Loginova in [6] proved that:

$$G_{P'} \underset{[G_{y_1}, G_{y_2}]}{*} G_{Q'} = (G_{P'} \underset{G_{y_1}}{*} (G_{y_1} \times G_{y_2})) \underset{G_{y_1} \times G_{y_2}}{*} (G_{Q'} \underset{G_{y_2}}{*} (G_{y_1} \times G_{y_2})).$$

1 $\Leftrightarrow$ **2**) The implication 1$\Rightarrow$2 results from the Lemma 3.2.

The implication 2$\Rightarrow$1 follows from the fact that, a given graph is a tree if and only if one contraction of subtrees in this graph is a tree (see Proposition 2.2).

Note that, Γ is a tree which is a subtrees contraction of the graph Γ' as described in the proof of Lemma 3.2.

2 $\Leftrightarrow$ **3**) Let G be a group acting on a tree Γ with the segment

$$T = \overset{P}{\bullet}\!\overset{y}{\text{——————}}\!\overset{Q}{\bullet}$$

as a fundamental domain.

In accordance with the proof of Lemma 3.2, P is obtained by contraction of one subtree Γ_1 of Γ' on which acts the subgroup $G_{P'} \underset{G_{y_1}}{*} G_R$ of G with $T_1 = \overset{P'}{\bullet}\!\overset{y_1}{\text{——————}}\!\overset{R}{\bullet}$ as a fundamental domain; Q is obtained by contraction of one subtree Γ_2 of Γ' on which acts the subgroup $G_R \underset{G_{y_2}}{*} G_{Q'}$ of G with $T_2 = \overset{R}{\bullet}\!\overset{y_2}{\text{——————}}\!\overset{Q'}{\bullet}$ as a fundamental domain; $y = R$.

Through the Theorem 6 in [11], Γ is a tree if and only if

$$L = G_P \underset{G_y}{*} G_Q \longrightarrow G$$

is an isomorphism.

Let determine $G_P,\ G_y$ and G_Q.

Firstly, we determine $G_y = G_R$:

$G_y = G_{y_1} \times G_{y_2}$ by Lemma 3.1.

Secondly, we determine G_P and G_Q:

See that if $g \in G_{P'} \underset{G_{y_1}}{*} G_R$, we have $g.\Gamma_1 = \Gamma_1$ then $g \circ P = P$ and if $g \notin G_{P'} \underset{G_{y_1}}{*} G_R$, we have $g.\Gamma_1 \neq \Gamma_1$, and then $g \circ P \neq P$. Recall that (.) and ($\circ$) denote the action of G on Γ' and on Γ respectively, (as describe in the proof of Lemma 3.2). So, when G acts on Γ, $G_P = G_{P'} \underset{G_{y_1}}{*} G_R$.

Similarly, we prove that $G_Q = G_{Q'} \underset{G_{y_2}}{*} G_R$.

According to the determination of $G_P,\ G_y$ and G_Q above, we get that Γ is a tree if and only if

$$(G_{P'} \underset{G_{y_1}}{*} (G_{y_1} \times G_{y_2})) \underset{G_{y_1} \times G_{y_2}}{*} (G_{Q'} \underset{G_{y_2}}{*} (G_{y_1} \times G_{y_2})) \longrightarrow G$$

is an isomorphism.

Applying Proposition 1 in [6],it follows that Γ is a tree if and only if

$$G_{P'} \underset{[G_{y_1}, G_{y_2}]}{*} G_{Q'} \longrightarrow G$$

is an isomorphism. Finally, the theorem is proven. $\square$

3.2. **Proof of Theorem 1.2 and Corollary 1.1.** ***Proof of Theorem 1.2***

Proof. Let $G = G_1 \underset{[H,K]}{*} G_2$. We define the graph Γ' on which acts G as follows:

$$V(\Gamma') = (G/G_1) \cup (G/H \times K) \cup (G/G_2)$$

and

$$E(\Gamma') = (G/H) \cup (\overline{G/H}) \cup (G/K) \cup (\overline{G/K}).$$

G acts obviously on Γ' as follows:

$$\begin{array}{ccc} G \times \Gamma' & \longrightarrow & \Gamma' \\ (g_1; g_2.x) & \longmapsto & (g_1 g_2).x \end{array}$$

for all $x \in G_1 \cup G_2 \cup (H \times K)$ and $g_1, g_2 \in G$.

With this action of G on Γ' (denoted by $(.)$), we see that the stabilizers of the vertices $1.G_1$, $1.H \times K$ and $1.G_2$ are respectively the groups G_1, $H \times K$ and G_2 and those of the edges $1.H$ and $1.K$ are respectively H and K.

Now, put $P' = 1.G_1$, $R = 1.H \times K$, $Q' = 1.G_2$, $y_1 = 1.H$ and $y_2 = 1.K$.

The $path\ 2$

$$T' = \overset{P'}{\bullet}\!\overset{y_1}{\longrightarrow}\!\overset{R}{\bullet}\!\overset{y_2}{\longrightarrow}\!\overset{Q'}{\bullet}$$

is a fundamental domain of Γ' modulo G. Applying $3) \Longrightarrow 1)$ of Theorem 1.1, we get that Γ' is a tree.

Let us prove the uniqueness. Assume that G acts on a tree Γ'' with fundamental domain a $path\ 2$

$$T'' = \overset{P''}{\bullet}\!\overset{y_1'}{\longrightarrow}\!\overset{R'}{\bullet}\!\overset{y_2'}{\longrightarrow}\!\overset{Q''}{\bullet}\ .$$

We denote this action by $(\bullet)$. Consider the graph isomorphism

$$\alpha_1 : T' \longrightarrow T'', \;\; P' \mapsto P'', \; Q' \mapsto Q'', \; R \mapsto R', \; y_1 \mapsto y_1', \; y_2 \mapsto y_2'.$$

Since T' and T'' are fundamental domain of Γ' and Γ'' for the actions $(.)$ and $(\bullet)$ respectively, α_1 can extend to a morphism $\alpha : \Gamma' \longrightarrow \Gamma''$ as follows:

$$\alpha(g.x) = g \bullet \alpha_1(x) \text{ for all } x \in T' \text{ for all } g \in G.$$

Let $\beta_1 : T'' \longrightarrow T'$ the reciprocal isomorphism of α_1. As above, β_1 can also extend to a morphism $\beta : \Gamma'' \longrightarrow \Gamma'$. Clearly, that $\beta \circ \alpha = Id_{\Gamma'}$ and $\alpha \circ \beta = Id_{\Gamma''}$ and then Γ' and Γ'' are isomorphic. $\square$

Proof of Corollary 1.1

Proof. Let T_1 be a segment in a tree Γ' on which acts a group G with T' a path 2 as fundamental domain. Assume that T' is the union of the segments T'_1 and T'_2.

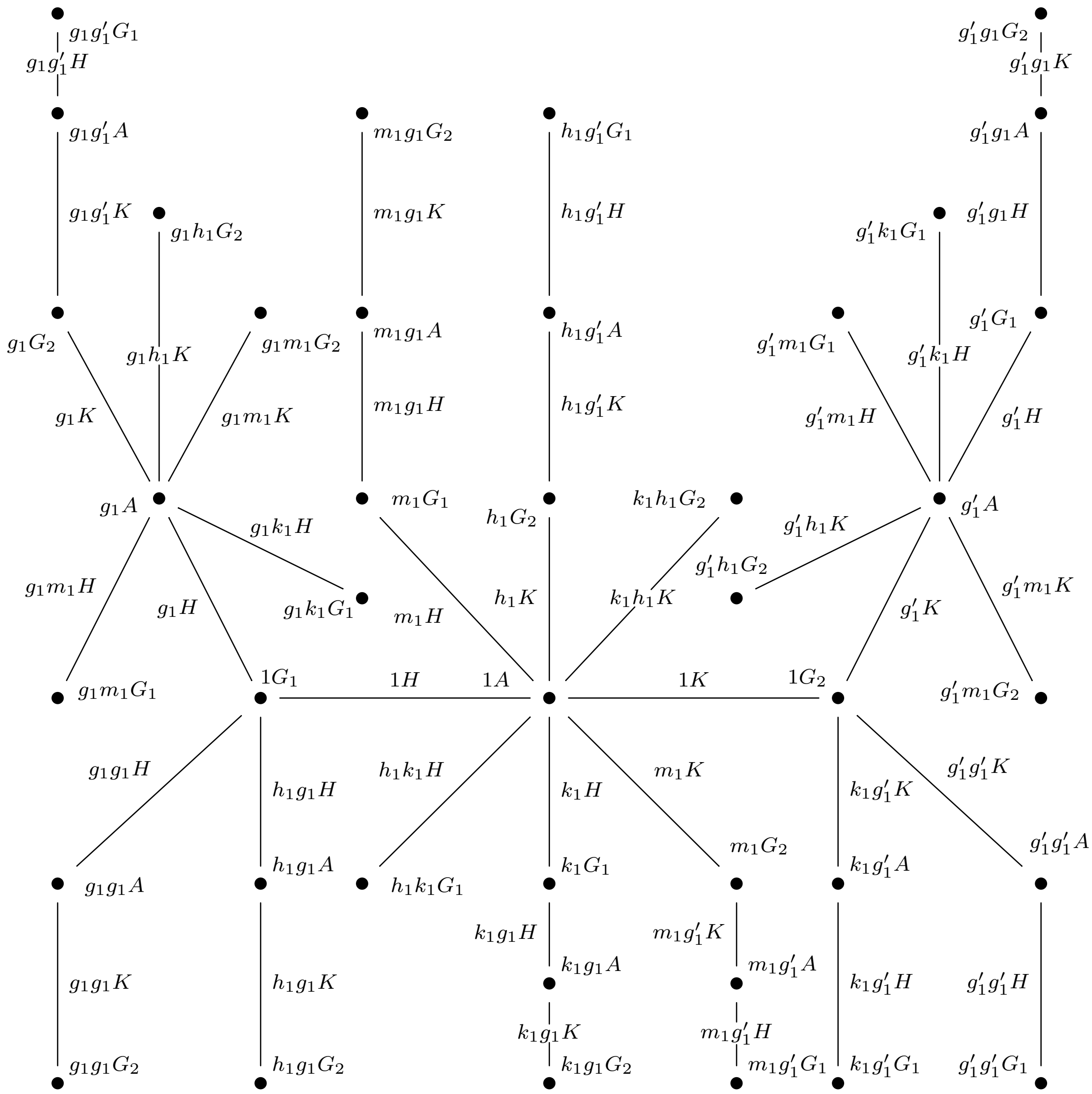
(1) Since $G/\Gamma' \cong T'$ then $[T_1] = [T'_1]$ or $[T_1] = [T'_2]$. Without lose to the generality, assume that $[T_1] = [T'_1]$. Therefore, there exists $g \in G$ such that $T_1 = g.T'_1$. Further, the segment $g.T'_2$ is connected to T_1 and the resultant path 2 is $g.T'$.
(2) $T_1 = g.T'_1$ with $g \in G$. Let g' be a nonidentity element belonging to the stabilizer of common vertex between T'_1 and T'_2. Then $gg'.T'_1 = T_2$ is connected to $g.T'_1 = T_1$. Finally, the path 2 which is the union of T_1 and T_2 is the result of action of elements of G on T'_1.

$\square$

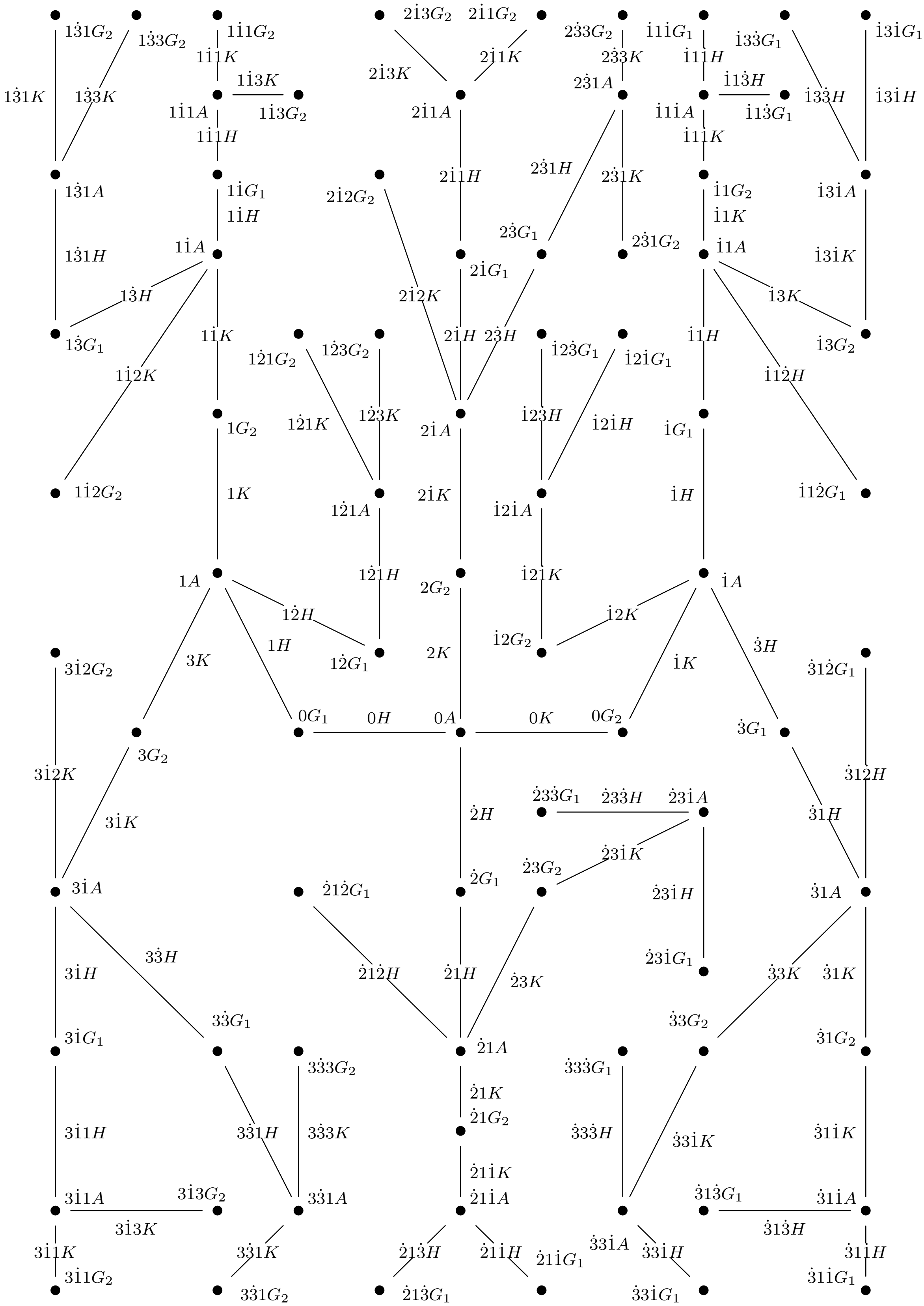
4. Illustrative Trees

4.1. **Tree of free product of groups with commuting subgroups.** Consider the following free product of groups with commuting subgroups: $G = G_1 \underset{[H,K]}{*} G_2 = \{1; g_1; g'_1; h_1; k_1; m_1; ...\}$ with $h_1 \in H$, $k_1 \in K$, $m_1 \in (H \times K) - (H \cup K)$, $g_1 \in G_1 - H$ and $g'_1 \in G_2 - K$. Put $A = H \times K = \{1; h_1; k_1; m_1; \ldots\}$. The corresponding tree on the following page.

4.2. **Illustrative example of tree of free product of groups with commuting subgroups.** Let present a concrete example of a tree on which acts a free product of groups with commuting subgroups. Consider the following free product of groups with commuting subgroups:



$G = \mathbb{Z}/4\mathbb{Z} \underset{[2\mathbb{Z}/4\mathbb{Z},2\mathbb{Z}/4\mathbb{Z}]}{*} \mathbb{Z}/4\mathbb{Z}$. Put $G_1 = \mathbb{Z}/4\mathbb{Z} = \{0;1;2;3\}$, $G_2 = \mathbb{Z}/4\mathbb{Z} = \{\dot{0};\dot{1};\dot{2};\dot{3}\}$, $H = 2\mathbb{Z}/4\mathbb{Z} = \{0;2\} \subset G_1$ and $K = 2\mathbb{Z}/4\mathbb{Z} = \{\dot{0};\dot{2}\} \subset G_2$. Recall that in $G = G_1 \underset{[H,K]}{*} G_2$, $0 = \dot{0}$. The corresponding tree is the following:


$1\dot{3}1G_2$
$1\dot{3}3G_2$
$1\dot{1}1G_2$
$2\dot{1}3G_2$
$2\dot{1}1G_2$
$2\dot{3}3G_2$
$\dot{1}1\dot{1}G_1$
$\dot{1}3\dot{3}G_1$
$\dot{1}3\dot{1}G_1$
$1\dot{1}1K$
$2\dot{1}1K$
$2\dot{3}3K$
$\dot{1}1\dot{1}H$
$1\dot{3}1K$
$1\dot{3}3K$
$1\dot{1}3K$
$2\dot{1}3K$
$2\dot{3}1A$
$\dot{1}1\dot{3}H$
$\dot{1}3\dot{3}H$
$\dot{1}3\dot{1}H$
$1\dot{1}1A$
$1\dot{1}3G_2$
$2\dot{1}1A$
$\dot{1}1\dot{1}A$
$\dot{1}1\dot{3}G_1$
$1\dot{1}1H$
$\dot{1}1\dot{1}K$
$2\dot{3}1H$
$2\dot{1}1H$
$2\dot{3}1K$
$1\dot{3}1A$
$1\dot{1}G_1$
$2\dot{1}2G_2$
$\dot{1}1G_2$
$\dot{1}3\dot{1}A$
$1\dot{1}H$
$\dot{1}1K$
$1\dot{3}1H$
$1\dot{1}A$
$2\dot{3}G_1$
$2\dot{3}1G_2$
$\dot{1}1A$
$\dot{1}3\dot{1}K$
$2\dot{1}G_1$
$1\dot{3}H$
$2\dot{1}2K$
$\dot{1}3K$
$1\dot{3}G_1$
$1\dot{1}K$
$1\dot{2}1G_2$
$1\dot{2}3G_2$
$2\dot{1}H$
$2\dot{3}H$
$\dot{1}2\dot{3}G_1$
$\dot{1}2\dot{1}G_1$
$\dot{1}1H$
$\dot{1}3G_2$
$1\dot{1}2K$
$\dot{1}1\dot{2}H$
$1G_2$
$1\dot{2}1K$
$1\dot{2}3K$
$2\dot{1}A$
$\dot{1}2\dot{3}H$
$\dot{1}2\dot{1}H$
$\dot{1}G_1$
$1\dot{1}2G_2$
$1K$
$1\dot{2}1A$
$2\dot{1}K$
$\dot{1}2\dot{1}A$
$\dot{1}H$
$\dot{1}1\dot{2}G_1$
$1A$
$1\dot{2}1H$
$2G_2$
$\dot{1}2\dot{1}K$
$\dot{1}A$
$1\dot{2}H$
$\dot{1}2K$
$1H$
$\dot{3}H$
$3\dot{1}2G_2$
$3K$
$1\dot{2}G_1$
$2K$
$\dot{1}2G_2$
$\dot{1}K$
$\dot{3}1\dot{2}G_1$
$0G_1$
$0H$
$0A$
$0K$
$0G_2$
$\dot{3}G_1$
$3G_2$
$3\dot{1}2K$
$\dot{3}1\dot{2}H$
$\dot{2}3\dot{3}G_1$
$\dot{2}3\dot{3}H$
$\dot{2}3\dot{1}A$
$\dot{2}H$
$3\dot{1}K$
$\dot{3}1H$
$\dot{2}3\dot{1}K$
$\dot{2}3G_2$
$\dot{2}G_1$
$3\dot{1}A$
$\dot{2}1\dot{2}G_1$
$\dot{2}3\dot{1}H$
$\dot{3}1A$
$3\dot{3}H$
$\dot{2}3\dot{1}G_1$
$3\dot{1}H$
$\dot{2}1\dot{2}H$
$\dot{2}1H$
$\dot{2}3K$
$\dot{3}3K$
$\dot{3}1K$
$3\dot{3}G_1$
$\dot{3}3G_2$
$3\dot{1}G_1$
$\dot{2}1A$
$3\dot{3}3G_2$
$\dot{3}3\dot{3}G_1$
$\dot{3}1G_2$
$\dot{2}1K$
$\dot{2}1G_2$
$3\dot{1}1H$
$3\dot{3}1H$
$3\dot{3}3K$
$\dot{3}3\dot{3}H$
$\dot{3}3\dot{1}K$
$\dot{3}1\dot{1}K$
$\dot{2}1\dot{1}K$
$3\dot{1}1A$
$3\dot{1}3G_2$
$3\dot{3}1A$
$\dot{2}1\dot{1}A$
$\dot{3}1\dot{3}G_1$
$\dot{3}1\dot{1}A$
$3\dot{1}3K$
$\dot{3}1\dot{3}H$
$3\dot{1}1K$
$3\dot{3}1K$
$\dot{2}1\dot{3}H$
$\dot{2}1\dot{1}H$
$\dot{2}1\dot{1}G_1$
$\dot{3}3\dot{1}A$
$\dot{3}3\dot{1}H$
$\dot{3}1\dot{1}H$
$3\dot{1}1G_2$
$3\dot{3}1G_2$
$\dot{2}1\dot{3}G_1$
$\dot{3}3\dot{1}G_1$
$\dot{3}1\dot{1}G_1$

- Note that the relation $2\dot{2} = \dot{2}2$ in $G_1 \underset{[H,K]}{*} G_2$ is obviously visible on the above tree. For example, $3\dot{2}G_1 = 12\dot{2}G_1 = 1\dot{2}2G_1 = 1\dot{2}G_1$. Likewise, $\dot{3}2G_2 = \dot{1}2G_2$, $2\dot{2}G_1 = \dot{2}G_1$, $\dot{2}2G_2 = 2G_2$, $\dot{3}G_1 = \dot{1}\dot{2}G_1 = \dot{3}2G_1 = \dot{1}\dot{2}2G_1 = 12\dot{2}G_1$ and $3G_2 = 12G_2 = 3\dot{2}G_2 = 12\dot{2}G_2 = 1\dot{2}2G_2$.
- Note also that, $3A = 12A = 1A$ and likewise $\dot{3}A = \dot{1}A$, $3H = 1H$ and $\dot{3}K = \dot{1}K$.
- Any vertex is then indistinguishable to some other vertices; for example $1\dot{1}A$, $1\dot{3}A$ and $1\dot{1}2A$ are at the identical position.
- The path 2 $\overset{1\dot{1}G_1}{\bullet}\overset{1\dot{1}H}{\text{———}}\overset{1\dot{1}A}{\bullet}\overset{1\dot{3}H}{\text{———}}\overset{1\dot{3}G_1}{\bullet}$ is not a result of the action of an element of G on the fundamental domain. This illustrates the second point of the Corollary 1.1.

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