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FIXED POINT THEOREMS ON CONVEX S_B-METRIC SPACE

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ABSTRACT. In this paper, we introduce an extension of S_b-metric space called the convex S_b-metric space, in which we establish and prove some fixed point theorems for a self-map defined on such spaces. we also prove some fixed point results on the topology of the S_b-metric space. Examples are given to establish the validity, applicability, and originality of our work.

1. INTRODUCTION

The Polish Mathematician S. Banach [9] in 1922, proved a theorem that established the existence and uniqueness of a fixed point given necessary conditions. His theorem provides a method for solving many Applied problems in mathematics, Mathematical sciences, Economics, Engineering, and Optimization. Since its inception, many authors and researchers have extended, generalized, unified, modified, and improved on Banach's fixed point theorem in many ways. (see for example [30, 31]). Czerwick [12], extended the Banach contraction principle (BCP) and its generalizations under many contractions [1-25, 27-31]. References [26,32,33], show that several authors have investigated the S-metric and

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S_b-metric spaces by generalizing several results concerning the existence of fixed points. Nizar and Nabil [26], first introduced the S-metric space based on the work of Bakhtin in [10] which is an expansion of b-metric space and later proved some fixed point results under different types of contractions in a complete S_b-metric space. In this work, we introduce the concept of convexity on S_b-metric space through the convex structure. We establish and prove many fixed point theorems.

In[32], Sedghi *et al*. introduced the notion of an S-metric space as follows:

Definition 1.1. *Let X be a non-empty set and $S_\lambda : X^3 \to \mathbb{R}^+$,a function satisfying the following properties:*

(i) $S_\lambda(x,y,z) = 0$*, if, and only if,* $x = y = z$*;*

(ii) $S_\lambda(x,y,z) \leq S_\lambda(x,x,a) + S_\lambda(y,y,a) + S_\lambda(z,z,a)$ *for all* $a,x,y,z \in X$ *(rectangle inequality).*

Then, S_λ is a metric on X and (X, S_λ) is called a S-metric space.

Below is the definition of S_p-metric spaces, a generalization of both S-metric space and S_b-metric spaces.

Definition 1.2. *Let X be a non-empty set and $S_\lambda : X^3 \to \mathbb{R}^+$, with a strictly increasing continuous function,$\beta : [0,\infty) \to [0.\infty)$ such that $\beta(t) \geq t$ for all $t > 0$ and $\beta(0) = 0$, satisfying the following properties:*

(i) $S_\lambda(x,y,z) = 0$*, if, and only if,* $x = y = z$*;*

(ii) $S_\lambda(x,y,z) \leq \beta(S_\lambda(x,x,a) + S_\lambda(y,y,a) + S_\lambda(z,z,a))$ *for all* $a, x.y, z \in X$ *(rectangle inequality).*

Then, S_λ is a metric on X and (X, S_λ) is an S_p metric space.

Remark 1.1.

(i) *If $\beta(z) = z$, S_p-metric space reduces to S-metric space.*

(ii) *If $\beta(z) = bz$, S_p-metric space reduces to S_b-metric space.*

2. Main results

Definition 2.1. *Let (X, S_b) be S_b metric space and $\alpha + \beta + \gamma = 1$. A mapping $\Phi : X^3 \times [0,1] \times [0,1] \times [0,1] \to X$ is said to be a convex structure on X if for each*

$(x, y, z, \alpha, \beta, \gamma) \in X^3 \times [0,1] \times [0,1] \times [0,1]$ *and* $u, v \in X$,

$$S_b(u, v, \Phi(x, y, z, \alpha, \beta, \gamma)) \leq \alpha S_b(u, v, x) + \beta S_b(u, v, y) + \gamma S_b(u, v, z). \tag{2.1}$$

Lemma 2.1. *Let* (X, S_b, Φ) *be a convex* S_b*-metric space, then the following statements hold:*

(i) $S_b(x, y, x) \leq bS_b(y, y, x)$, $S_b(y, x, x) \leq bS_b(y, y, x)$ & $S_b(x, x, y) \leq bS_b(y, y, x)$
(ii) $S_b(x, x, \Phi(x, y, y, \alpha, \beta, \gamma)) + S_b(y, y, \Phi(x, y, y, \alpha, \beta, \gamma)) \leq S_b(x, x, y)$
(iii) $S_b(x, x, \Phi(x, y, y, \alpha, \beta, \gamma)) = (\beta + \gamma)S_b(x, x, y)$
(iv) $S_b(y, y, \Phi(x, y, y, \alpha, \beta, \gamma)) = \alpha S_b(y, y, x)$
(v) $S_b(x, y, \Phi(x, y, y, \alpha, \beta, \gamma)) \leq b(\alpha + 2\beta + 2\gamma)S_b(y, y, x)$

Proof.

(i) $S_b(x, y, x) \leq b[S_b(x, x, x) + S_b(y, y, x) + S_b(x, x, x)] = bS_b(y, y, x)$
$S_b(y, x, x) \leq b[S_b(y, y, x) + S_b(x, x, x) + S_b(x, x, x)] = bS_b(y, y, x)$
$S_b(x, x, y) \leq b[S_b(x, x, x) + S_b(x, x, x) + S_b(y, y, x)] = bS_b(y, y, x)$.

(ii) Let $\phi_x = S_b(x, x, \Phi(x, y, y, \alpha, \beta, \gamma))$ and $\phi_y = S_b(y, y, \Phi(x, y, y, \alpha, \beta, \gamma))$, then

$$\begin{aligned}
\phi_x + \phi_y &\leq \alpha S_b(x, x, x) + \beta S_b(x, x, y) + \gamma S_b(x, x, y) \\
&\quad + \alpha S_b(y, y, x) + \beta S_b(y, y, y) + \gamma S_b(y, y, y) \\
&= \beta S_b(x, x, y) + \gamma S_b(x, x, y) + \alpha S_b(y, y, x) \\
&\leq \beta S_b(x, x, y) + \gamma S_b(x, x, y) + \alpha S_b(x, x, y) \\
&= (\alpha + \beta + \gamma)S_b(x, x, y). \\
&= S_b(x, x, y)
\end{aligned}$$

(iii)

$$\begin{aligned}
S_b(x, x, \Phi(x, y, y, \alpha, \beta, \gamma)) &= \alpha S_b(x, x, x) + \beta S_b(x, x, y) + \gamma S_b(x, x, y) \\
&= \beta S_b(x, x, y) + \gamma S_b(x, x, y) \\
&= (\beta + \gamma)S_b(x, x, y).
\end{aligned}$$

(iv)

$$\begin{aligned}
S_b(y, y, \Phi(x, y, y, \alpha, \beta, \gamma)) &= \alpha S_b(y, y, x) + \beta S_b(y, y, y) + \gamma S_b(y, y, y) \\
&= \alpha S_b(y, y, x).
\end{aligned}$$

(v)

$$\begin{aligned} S_b(x,y,\Phi(x,y,y,\alpha,\beta,\gamma)) &\leq \alpha S_b(x,y,x)+\beta S_b(x,y,y)+\gamma S_b(x,y,y) \\ &\leq \alpha S_b(x,x,y)+\beta S_b(x,y,y)+\gamma S_b(x,y,y) \\ &\leq b(\alpha+2\beta+2\gamma)S_b(y,y,x). \end{aligned}$$

□

Definition 2.2. *Let (X, S_b, Φ) be a convex S_b-metric space. For $y \in X$, $r > 0$, the convex S_b-sphere with centre y and radius r is*

$$S_r(y) = \{z \in X : S_b(z,z,\Phi(y,z,z,\alpha,\beta,\gamma)) < r\}.$$

Definition 2.3. *Let (X, S_b, Φ) be a convex S_b-metric space. A sequence $\{x_n\} \subset X$ is convex S_b-convergent to z if the limit of $S_b(x_n, z, \Phi(z,z,z,\alpha,\beta,\gamma))$ tends to zero as n tends to infinity.*

Definition 2.4. *Let (X, S_b, Φ) be a convex S_b-metric space. A sequence $\{x_n\} \subset X$ is said to be a convex S_b-Cauchy sequence if the limit of $S_b(x_n, x_m, \Phi(x_l, x_l, x_l, \alpha, \beta, \gamma))$ tends to zero as n, m, l tends to infinity.*

Example 1. *Let $X = \mathbb{R}$ and S_b be defined by*

$$S_b(x,y,\Phi(z,z,z,\alpha,\beta,\gamma)) = \alpha|2x-(y+z)| + \beta|2y-(z+x)| + \gamma|2z-(x+y)|,$$

then (X, S_b) is a convex S_b-metric space.

Verification:

(i) If $x = y = z$, then $S_b(x,x,\Phi(x,x,x,\alpha,\beta,\gamma)) = \alpha|2x-(x+x)| + \beta|2x-(x+x)| + \gamma|2x-(x+x)| = (\alpha+\beta+\gamma)|2x-(x+x)| = 1 \times |2x-2x| = 0$. Conversely, if $S_b(x,y,\Phi(z,z,z,\alpha,\beta,\gamma)) = 0$, then $\alpha|2x-(y+z)| + \beta|2y-(z+x)| + \gamma|2z-(x+y)| = 0$ which implies $\alpha|2x-(y+z)| = 0$, $\beta|2y-(z+x)| = 0$ and $\gamma|2z-(x+y)| = 0$. Hence, solving simultaneously, $x = y = z$.

(ii) We are required to show that:

$$\begin{aligned} S_b(x,y,\Phi(z,z,z,\alpha,\beta,\gamma)) &\leq b[S_b(x,x,\Phi(a,a,a,\alpha,\beta,\gamma)) \\ &+ S_b(y,y,\Phi(a,a,a,\alpha,\beta,\gamma)) \\ &+ S_b(z,z,\Phi(a,a,a,\alpha,\beta,\gamma))] \end{aligned}$$

(2.2) $S_b(x, y, \Phi(z, z, z, \alpha, \beta, \gamma)) = \alpha|2x - (y + z)| + \beta|2y - (z + x)| + \gamma|2z - (x + y)|$

(2.3) $S_b(x, x, \Phi(a, a, a, \alpha, \beta, \gamma)) = \alpha|2x - (x + a)| + \beta|2x - (a + x)| + \gamma|2a - (x + x)|$

(2.4) $S_b(y, y, \Phi(a, a, a, \alpha, \beta, \gamma)) = \alpha|2y - (y + a)| + \beta|2y - (a + y)| + \gamma|2a - (y + y)|$

(2.5) $S_b(z, z, \Phi(a, a, a, \alpha, \beta, \gamma)) = \alpha|2z - (z + a)| + \beta|2z - (a + z)| + \gamma|2a - (z + z)|$

Expressions (2.7), (2.8), (2.9) and (2.10) imply (2.6) for all $x, y, z, a \in X$ and $b > 1$.

Theorem 2.1. *Let X be a complete convex S_b-metric space and $T : X \to X$ a map for which there exist the real number, a satisfying $0 \leq k < 1$ such that for all $x, y \in X$,*

(2.6) $$S_b(Tx, Ty, \Phi(Tz, Tz, Tz, \alpha, \beta, \gamma)) \leq kS_b(x, y, \Phi(z, z, z, \alpha, \beta, \gamma)).$$

Then T has a unique fixed point.

Proof. Suppose T satisfies condition (2.6) and $x_0 \in X$ be an arbitrary point and define a sequence x_n by $x_n = T^n x_0$, then

$$\begin{aligned}
& S_b(x_n, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) \\
= \; & S_b(Tx_{n-1}, Tx_n, \Phi(Tx_n, Tx_n, Tx_n, \alpha, \beta, \gamma)) \\
\leq \; & kS_b(x_{n-1}, x_n, \Phi(x_n, x_n, x_n, \alpha, \beta, \gamma)) \\
= \; & k\alpha S_b(x_{n-1}, x_n, x_n) + k\beta S_b(x_{n-1}, x_n, x_n) + k\gamma S_b(x_{n-1}, x_n, x_n) \\
= \; & k(\alpha + \beta + \gamma) S_b(x_{n-1}, x_n, x_n) \\
= \; & kS_b(x_{n-1}, x_n, x_n).
\end{aligned}$$

Also,

$$\begin{aligned}
& S_b(x_n, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) \\
= \; & \alpha S_b(x_n, x_{n+1}, x_{n+1}) + \beta S_b(x_n, x_{n+1}, x_{n+1}) \gamma S_b(x_n, x_{n+1}, x_{n+1}) \\
= \; & (\alpha + \beta + \gamma) S_b(x_n, x_{n+1}, x_{n+1}) \\
= \; & S_b(x_n, x_{n+1}, x_{n+1}).
\end{aligned}$$

Then, (2.7) and (2.8) implies

$$\begin{aligned} S_b(x_n, x_{n+1}, x_{n+1}) &\leq kS_b(x_{n-1}, x_n, x_n) \leq k^2 S_b(x_{n-2}, x_{n-1}, x_{n-1}) \\ &\leq k^3 S_b(x_{n-3}, x_{n-2}, x_{n-2}) \\ &\vdots \\ &\leq k^n S_b(x_0, x_1, x_1). \end{aligned}$$

Taking the limit of $S_b(x_n, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma))$ as $n \to \infty$, we have

$$\lim_{n\to\infty} S_b(x_n, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) = \lim_{n\to\infty} k^n S_b(x_0, x_1, x_1) = 0. \tag{2.7}$$

Using (ii) of Definition 2.1 repeatedly with $n < m < l$, we obtain:

$$\lim_{n,m,l\to\infty} S_b(x_n, x_m, \Phi(x_l, x_l, x_l, \alpha, \beta, \gamma)) = 0. \tag{2.8}$$

So, $\{x_n\}$ is a convex S_b-Cauchy Sequence.

By completeness of (X, S_b), there exist $x_o \in X$ such that x_n is convex S_b-convergent to x_o. Suppose $Tx_o \neq x_o$

$$S_b(x_n, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma)) \leq kS_b(x_{n-1}, x_o, \Phi(x_o, x_o, x_o, \alpha, \beta, \gamma)). \tag{2.9}$$

Taking the limit as $n \to \infty$ and using the fact that function is convex S_b-continuous in its variables, we get

$$S_b(x_o, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma)) \leq kS_b(x_o, x_o, \Phi(x_o, x_o, x_o, \alpha, \beta, \gamma)). \tag{2.10}$$

Hence,

$$S_b(x_o, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma)) \leq 0. \tag{2.11}$$

This is a contradiction. So, $Tx_o = x_o$.

To show the uniqueness, suppose $x_1 \neq x_2$ is such that $Tx_1 = x_1$ and $Tx_2 = x_2$ then

$$S_b(Tx_1, Tx_2, \Phi(Tx_2, Tx_2, Tx_2, \alpha, \beta, \gamma)) \leq kS_b(x_1, x_2, \Phi(x_2, x_2, x_2, \alpha, \beta, \gamma)). \tag{2.12}$$

Since $Tx_1 = x_1$ and $Tx_2 = x_2$, we have

$$S_b(x_1, x_2, \Phi(x_2, x_2, x_2, \alpha, \beta, \gamma)) \leq 0, \tag{2.13}$$

which implies that $x_1 = x_2$. □

Remark 2.1. *Convex S_b metric space is an extension of S_b metric space. This will probably extend application in real life.*

Corollary 2.1. *Let (X, S_b) be a convex S_b metric space, for all $x, y, z, a \in X$. Then,*

(*i*) $S_b(x, y, \Phi(z, z, z, \alpha, \beta, \gamma)) = 0 \iff x = y = z$;
(*ii*) $S_b(x, y, \Phi(z, z, z, \alpha, \beta, \gamma)) \leq S_b(x, x, \Phi(a, a, a, \alpha, \beta, \gamma))+$
$S_b(y, y, \Phi(a, a, a, \alpha, \beta, \gamma)) + S_b(z, z, \Phi(a, a, a, \alpha, \beta, \gamma))$.

Proof.

(*i*) If $S_b(x, y, \Phi(z, z, z, \alpha, \beta, \gamma)) = 0$, then

$$\alpha S_b(x, y, z) + \beta S_b(x, y, z) + \gamma S_b(x, y, z) = 0,$$

which implies $(\alpha + \beta + \gamma)S_b(x, y, z) = 0$, and further $S_b(x, y, z) = 0$.

Conversely, if $x = y = z$, then, $S_b(x, y, z) = 0$ (Since (X, S_b) is S_b metric space) and $S_b(x, y, z) = 0$. This implies $(\alpha + \beta + \gamma)S_b(x, y, z) = 0$, and further $\alpha S_b(x, y, z) + \beta S_b(x, y, z) + \gamma S_b(x, y, z) = 0$. Hence,

$$S_b(x, y, \Phi(z, z, z, \alpha, \beta, \gamma)) = 0.$$

(*ii*)

$$\begin{aligned} S_b(x, y, \Phi(z, z, z, \alpha, \beta, \gamma)) &= \alpha S_b(x, y, z) + \beta S_b(x, y, z) + \gamma S_b(x, y, z) \\ &= (\alpha + \beta + \gamma)S_b(x, y, z) \\ &= S_b(x, y, z). \end{aligned}$$

$$\begin{aligned} S_b(x, x, \Phi(a, a, a, \alpha, \beta, \gamma)) &= \alpha S_b(x, x, a) + \beta S_b(x, x, a) + \gamma S_b(x, x, a) \\ &= (\alpha + \beta + \gamma)S_b(x, x, a) \\ &= S_b(x, x, a). \end{aligned}$$

$$\begin{aligned} S_b(y, y, \Phi(a, a, a, \alpha, \beta, \gamma)) &= \alpha S_b(y, y, a) + \beta S_b(x, x, a) + \gamma S_b(y, y, a) \\ &= (\alpha + \beta + \gamma)S_b(x, x, a) \\ &= S_b(y, y, a). \end{aligned}$$

$$\begin{aligned} S_b(z, z, \Phi(a, a, a, \alpha, \beta, \gamma)) &= \alpha S_b(z, z, a) + \beta S_b(z, z, a) + \gamma S_b(z, z, a) \\ &= (\alpha + \beta + \gamma) S_b(z, z, a) \\ &= S_b(z, z, a). \end{aligned}$$

Expressions (2.17), (2.18), (2.19) and (2.20) imply

$$\begin{aligned} S_b(x, y, \Phi(z, z, z, \alpha, \beta, \gamma)) \leq\ & S_b(x, x, \Phi(a, a, a, \alpha, \beta, \gamma)) \\ & + S_b(y, y, \Phi(a, a, a, \alpha, \beta, \gamma)) \\ & + S_b(z, z, \Phi(a, a, a, \alpha, \beta, \gamma)). \end{aligned}$$

□

Theorem 2.2. *Let X be a complete convex S_b-metric space and $T : X \to X$ a map for which there exist the real number, k satisfying $0 \leq k < \frac{1}{2}$ such that for all $x, y \in X$,*

$$\begin{aligned} S_b(Tx, Ty, \Phi(Tz, Tz, Tz, \alpha, \beta, \gamma)) &\leq k[S_b(x, Tx, \Phi(Tx, Tx, Tx, \alpha, \beta, \gamma)) \\ &+ S_b(y, Ty, \Phi(Ty, Ty, Ty, \alpha, \beta, \gamma))]. \end{aligned}$$

Then T has a unique fixed point.

Proof. Suppose T satisfies condition (2.21) and $x_0 \in X$ be an arbitrary point and define a sequence x_n by $x_n = T^n x_0$, then

$$\begin{aligned} & S_b(x_n, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) \\ = \ & S_b(Tx_{n-1}, Tx_n, \Phi(Tx_n, Tx_n, Tx_n, \alpha, \beta, \gamma)) \\ \leq \ & k[S_b(x_n, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) \\ + \ & S_b(x_{n-1}, x_n, \Phi(x_n, x_n, x_n, \alpha, \beta, \gamma))] \\ = \ & k[S_b(x_{n-1}, x_n, x_n) + S_b(x_n, x_{n+1}, x_{n+1})] \\ = \ & \frac{k}{1-k} S_b(x_{n-1}, x_n, x_n). \end{aligned}$$

Setting $a = \frac{k}{1-k}$, we have

$$S_b(x_n, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) = aS_b(x_{n-1}, x_n, x_n) \tag{2.14}$$

$$\begin{aligned}
& S_b(x_n, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) \\
= \ & \alpha S_b(x_n, x_{n+1}, x_{n+1}) + \beta S_b(x_n, x_{n+1}, x_{n+1})\gamma S_b(x_n, x_{n+1}, x_{n+1}) \\
= \ & (\alpha + \beta + \gamma) S_b(x_n, x_{n+1}, x_{n+1}) \\
= \ & S_b(x_n, x_{n+1}, x_{n+1}).
\end{aligned}$$

Expressions (2.23) and (2.24) imply

$$\begin{aligned}
S_b(x_n, x_{n+1}, x_{n+1}) & \leq aS_b(x_{n-1}, x_n, x_n) \leq a^2 S_b(x_{n-2}, x_{n-1}, x_{n-1}) \\
& \leq a^3 S_b(x_{n-3}, x_{n-2}, x_{n-2}) \\
& \vdots \\
& \leq a^n S_b(x_0, x_1, x_1).
\end{aligned}$$

Taking the limit of $S_b(x_n, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma))$ as $n \to \infty$, we have

$$\lim_{n\to\infty} S_b(x_n, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) = \lim_{n\to\infty} a^n S_b(x_0, x_1, x_1) = 0. \tag{2.15}$$

Using (ii) of Definition 2.1 repeatedly with $n < m < l$, we obtain:

$$\lim_{n,m,l\to\infty} S_b(x_n, x_m, \Phi(x_l, x_l, x_l, \alpha, \beta, \gamma)) = 0. \tag{2.16}$$

So, $\{x_n\}$ is a convex S_b-Cauchy Sequence.

By completeness of), there exist $x_o \in X$ such that x_n is convex S_b-convergent to x_o. Suppose $Tx_o \neq x_o$

$$\begin{aligned}
S_b(x_n, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma)) & \leq k[S_b(x_{n-1}, x_n, \Phi(x_n, x_n, x_n, \alpha, \beta, \gamma)) \\
& + S_b(x_o, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma))].
\end{aligned}$$

Taking the limit as $n \to \infty$ and using the fact that function is convex S_b-continuous in its variables, we get

$$S_b(x_o, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma)) \leq 2kS_b(x_o, x_o, \Phi(x_o, x_o, x_o, \alpha, \beta, \gamma)). \tag{2.17}$$

Hence,

$$S_b(x_o, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma)) \leq 0. \tag{2.18}$$

This is a contradiction. So, $Tx_o = x_o$.

To show the uniqueness, suppose $x_1 \neq x_2$ is such that $Tx_1 = x_1$ and $Tx_2 = x_2$ then

$$\begin{aligned} S_b(Tx_1, Tx_2, \Phi(Tx_2, Tx_2, Tx_2, \alpha, \beta, \gamma)) &\leq k[S_b(x_1, Tx_1, \Phi(Tx_1, Tx_1, Tx_1, \alpha, \beta, \gamma)) \\ &+ S_b(x_2, Tx_2, \Phi(Tx_2, Tx_2, Tx_2, \alpha, \beta, \gamma))]. \end{aligned}$$

Since $Tx_1 = x_1$ and $Tx_2 = x_2$, we have

$$S_b(x_1, x_2, \Phi(x_2, x_2, x_2, \alpha, \beta, \gamma)). \leq 0. \tag{2.19}$$

which implies that $x_1 = x_2$. □

Theorem 2.3. *Let X be a complete convex S_b-metric space and $T : X \to X$ a map for which there exist the real number, k satisfying $0 \leq k < \frac{1}{3}$ such that for all $x, y \in X$,*

$$\begin{aligned} S_b(Tx, Ty, \Phi(Tz, Tz, Tz, \alpha, \beta, \gamma)) &\leq k[S_b(x, Tx, \Phi(Tx, Tx, Tx, \alpha, \beta, \gamma)) \\ &+ S_b(y, Ty, \Phi(Ty, Ty, Ty, \alpha, \beta, \gamma)) \\ &+ S_b(z, Tz, \Phi(Tz, Tz, Tz, \alpha, \beta, \gamma))]. \end{aligned}$$

Then T has a unique fixed point.

Proof. Suppose T satisfies condition (2.31) and $x_0 \in X$ be an arbitrary point and define a sequence x_n by $x_n = T^n x_0$, then

$$\begin{aligned} & S_b(x_n, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) \\ = \;& S_b(Tx_{n-1}, Tx_n, \Phi(Tx_n, Tx_n, Tx_n, \alpha, \beta, \gamma)) \\ \leq \;& 2k[S_b(x_n, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) \\ + \;& S_b(x_{n-1}, x_n, \Phi(x_n, x_n, x_n, \alpha, \beta, \gamma))] \\ = \;& k[2S_b(x_{n-1}, x_n, x_n) + S_b(x_n, x_{n+1}, x_{n+1})] \\ = \;& \frac{k}{1-2k} S_b(x_{n-1}, x_n, x_n). \end{aligned}$$

Setting $a = \frac{k}{1-2k}$, we have

$$S_b(x_n, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) = aS_b(x_{n-1}, x_n, x_n), \tag{2.20}$$

$$\begin{aligned}
& S_b(x_n, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) \\
= \;& \alpha S_b(x_n, x_{n+1}, x_{n+1}) + \beta S_b(x_n, x_{n+1}, x_{n+1}) \gamma S_b(x_n, x_{n+1}, x_{n+1}) \\
= \;& (\alpha + \beta + \gamma) S_b(x_n, x_{n+1}, x_{n+1}) \\
= \;& S_b(x_n, x_{n+1}, x_{n+1}).
\end{aligned}$$

Expressions (2.33) and (2.34) imply

$$\begin{aligned}
S_b(x_n, x_{n+1}, x_{n+1}) &\leq aS_b(x_{n-1}, x_n, x_n) \leq a^2 S_b(x_{n-2}, x_{n-1}, x_{n-1}) \\
&\leq a^3 S_b(x_{n-3}, x_{n-2}, x_{n-2}) \\
&\vdots \\
&\leq a^n S_b(x_0, x_1, x_1).
\end{aligned}$$

Taking the limit of $S_b(x_n, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma))$ as $n \to \infty$, we have

$$\lim_{n\to\infty} S_b(x_n, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) = \lim_{n\to\infty} a^n S_b(x_0, x_1, x_1) = 0. \tag{2.21}$$

Using (ii) of Definition 2.1 repeatedly with $n < m < l$, we obtain:

$$\lim_{n,m,l\to\infty} S_b(x_n, x_m, \Phi(x_l, x_l, x_l, \alpha, \beta, \gamma)) = 0. \tag{2.22}$$

So, $\{x_n\}$ is a convex S_b-Cauchy Sequence. By completeness of (X, S), there exist $x_o \in X$ such that x_n is convex S_b-convergent to x_o. Suppose $Tx_o \neq x_o$,

$$\begin{aligned}
S_b(x_n, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma)) \;\leq\;& k[S_b(x_{n-1}, x_n, \Phi(x_n, x_n, x_n, \alpha, \beta, \gamma)) \\
+\;& 2S_b(x_o, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma))].
\end{aligned}$$

Taking the limit as $n \to \infty$ and using the fact that function is convex S_b-continuous in its variables, we get

$$S_b(x_o, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma)) \leq 3kS_b(x_o, x_o, \Phi(x_o, x_o, x_o, \alpha, \beta, \gamma)). \tag{2.23}$$

Hence,

$$S_b(x_o, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma)) \leq 0. \tag{2.24}$$

This is a contradiction. So, $Tx_o = x_o$.

To show the uniqueness, suppose $x_1 \neq x_2$ is such that $Tx_1 = x_1$ and $Tx_2 = x_2$ then

$$\begin{aligned} S_b(Tx_1, Tx_2, \Phi(Tx_2, Tx_2, Tx_2, \alpha, \beta, \gamma)) &\leq k[S_b(x_1, Tx_1, \Phi(Tx_1, Tx_1, Tx_1, \alpha, \beta, \gamma)) \\ &+ 2S_b(x_2, Tx_2, \Phi(Tx_2, Tx_2, Tx_2, \alpha, \beta, \gamma))]. \end{aligned}$$

Since $Tx_1 = x_1$ and $Tx_2 = x_2$, we have

$$S_b(x_1, x_2, \Phi(x_2, x_2, x_2, \alpha, \beta, \gamma)). \leq 0, \tag{2.25}$$

which implies that $x_1 = x_2$. □

Theorem 2.4. *Let X be a complete convex S_b-metric space and $T : X \to X$ a map for which there exist the real number, k satisfying $0 \leq k < 1$ such that for all $x, y \in X$,*

$$\begin{aligned} S_b(Tx, Ty, \Phi(Tz, Tz, Tz, \alpha, \beta, \gamma)) &\leq k[S_b(x, Ty, \Phi(Ty, Ty, Ty, \alpha, \beta, \gamma)) \\ &+ S_b(y, Tx, \Phi(Tx, Tx, Tx, \alpha, \beta, \gamma))]. \end{aligned}$$

Then T has a unique fixed point.

Proof. Suppose T satisfies condition (2.31) and $x_0 \in X$ be an arbitrary point and define a sequence x_n by $x_n = T^n x_0$, then

$$\begin{aligned} & S_b(x_n, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) \\ = \; & S_b(Tx_{n-1}, Tx_n, \Phi(Tx_n, Tx_n, Tx_n, \alpha, \beta, \gamma)) \\ \leq \; & k[S_b(x_{n-1}, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) \\ + \; & S_b(x_n, x_n, \Phi(x_n, x_n, x_n, \alpha, \beta, \gamma))] \\ \leq \; & k[S_b(x_{n-1}, x_{n-1}, x_n) + 2S_b(x_n, x_n, x_{n+1})]. \end{aligned}$$

Hence,

$$S_b(x_n, x_n, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) = \frac{k}{1-2k} S_b(x_{n-1}, x_{n-1}, x_n).$$

Setting $a = \frac{k}{1-2k}$, we have

$$S_b(x_n, x_n, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) = aS_b(x_{n-1}, x_{n-1}, x_n), \tag{2.26}$$

$$
\begin{aligned}
& S_b(x_n, x_n, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) \\
= \ & \alpha S_b(x_n, x_n, x_{n+1}) + \beta S_b(x_n, x_n, x_{n+1}) \gamma S_b(x_n, x_n, x_{n+1}) \\
= \ & (\alpha + \beta + \gamma) S_b(x_n, x_n, x_{n+1}) \\
= \ & S_b(x_n, x_n, x_{n+1}).
\end{aligned}
$$

Expressions (2.33) and (2.34) imply

$$
\begin{aligned}
S_b(x_n, x_n, x_{n+1}) &\leq a S_b(x_{n-1}, x_{n-1}, x_n) \leq a^2 S_b(x_{n-2}, x_{n-2}, x_{n-1}) \\
&\leq a^3 S_b(x_{n-3}, x_{n-3}, x_{n-2}) \\
&\vdots \\
&\leq a^n S_b(x_0, x_0, x_1).
\end{aligned}
$$

Taking the limit of $S_b(x_n, x_n, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma))$ as $n \to \infty$, we have

(2.27) $$\lim_{n \to \infty} S_b(x_n, x_n, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) = \lim_{n \to \infty} a^n S_b(x_0, x_0, x_1) = 0.$$

Using (ii) of Definition 2.1 repeatedly with $n < m < l$, we obtain:

(2.28) $$\lim_{n,m,l \to \infty} S_b(x_n, x_m, \Phi(x_l, x_l, x_l, \alpha, \beta, \gamma)) = 0.$$

So, $\{x_n\}$ is a convex S_b-Cauchy Sequence. By completeness of (X, S), there exist $x_o \in X$ such that x_n is convex S_b-convergent to x_o. Suppose $Tx_o \neq x_o$,

$$
\begin{aligned}
S_b(x_n, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma)) \leq \ & k[S_b(x_{n-1}, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma)) \\
+ \ & S_b(x_o, x_n, \Phi(x_n, x_n, x_n, \alpha, \beta, \gamma))].
\end{aligned}
$$

Taking the limit as $n \to \infty$ and using the fact that function is convex S_b-continuous in its variables, we get

(2.29) $$
\begin{aligned}
& S_b(x_o, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma)) \\
\leq & S_b(x_o, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma)).
\end{aligned}
$$

Hence,

(2.30) $$S_b(x_o, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma)) \leq 0.$$

This is a contradiction. So, $Tx_o = x_o$.

To show the uniqueness, suppose $x_1 \neq x_2$ is such that $Tx_1 = x_1$ and $Tx_2 = x_2$ then

$$\begin{aligned} S_b(Tx_1, Tx_2, \Phi(Tx_2, Tx_2, Tx_2, \alpha, \beta, \gamma)) &\leq k[S_b(x_1, Tx_2, \Phi(Tx_2, Tx_2, Tx_2, \alpha, \beta, \gamma)) \\ &+ S_b(x_2, Tx_1, \Phi(Tx_1, Tx_1, Tx_1, \alpha, \beta, \gamma))]. \end{aligned}$$

Since $Tx_1 = x_1$ and $Tx_2 = x_2$, we have

$$S_b(x_1, x_2, \Phi(x_2, x_2, x_2, \alpha, \beta, \gamma)). \leq 0, \tag{2.31}$$

which implies that $x_1 = x_2$. □

Theorem 2.5. *Let X be a complete convex S_b-metric space and $T : X \to X$ a map for which there exist the real number, k satisfying $0 \leq k < \frac{1}{4}$ such that for all $x, y \in X$,*

$$\begin{aligned} S_b(Tx, Ty, \Phi(Tz, Tz, Tz, \alpha, \beta, \gamma)) &\leq k[S_b(x, Ty, \Phi(Ty, Ty, Ty, \alpha, \beta, \gamma)) \\ &+ S_b(y, Tz, \Phi(Tz, Tz, Tz, \alpha, \beta, \gamma)) \\ &+ S_b(z, Tx, \Phi(Tx, Tx, Tx, \alpha, \beta, \gamma))]. \end{aligned}$$

Then T has a unique fixed point.

Proof. Suppose T satisfies condition (2.31) and $x_0 \in X$ be an arbitrary point and define a sequence x_n by $x_n = T^n x_0$, then

$$\begin{aligned} & S_b(x_n, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) \\ = \; & S_b(Tx_{n-1}, Tx_n, \Phi(Tx_n, Tx_n, Tx_n, \alpha, \beta, \gamma)) \\ \leq \; & k[S_b(x_{n-1}, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) \\ + \; & S_b(x_n, x_{n+1}, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma))] \\ = \; & k[S_b(x_{n-1}, x_{n-1}, x_n) + 3S_b(x_n, x_n, x_{n+1})] \\ = \; & \frac{k}{1-3k} S_b(x_{n-1}, x_{n-1}, x_n). \end{aligned}$$

Setting $a = \frac{k}{1-3k}$, we have

$$S_b(x_n, x_n, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) = aS_b(x_{n-1}, x_{n-1}, x_n), \tag{2.32}$$

$$
\begin{aligned}
& S_b(x_n, x_n, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) \\
= \;& \alpha S_b(x_n, x_n, x_{n+1}) + \beta S_b(x_n, x_n, x_{n+1}) \gamma S_b(x_n, x_n, x_{n+1}) \\
= \;& (\alpha + \beta + \gamma) S_b(x_n, x_n, x_{n+1}) \\
= \;& S_b(x_n, x_n, x_{n+1}).
\end{aligned}
$$

Expressions (2.33) and (2.34) imply

$$
\begin{aligned}
S_b(x_n, x_n, x_{n+1}) &\leq aS_b(x_{n-1}, x_{n-1}, x_n) \leq a^2 S_b(x_{n-2}, x_{n-2}, x_{n-1}) \\
&\leq a^3 S_b(x_{n-3}, x_{n-3}, x_{n-2}) \\
&\vdots \\
&\leq a^n S_b(x_0, x_0, x_1).
\end{aligned}
$$

Taking the limit of $S_b(x_n, x_n, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma))$ as $n \to \infty$, we have

(2.33) $$\lim_{n \to \infty} S_b(x_n, x_n, \Phi(x_{n+1}, x_{n+1}, x_{n+1}, \alpha, \beta, \gamma)) = \lim_{n \to \infty} a^n S_b(x_0, x_0, x_1) = 0.$$

Using (ii) of Definition 2.1 repeatedly with $n < m < l$, we obtain:

(2.34) $$\lim_{n,m,l \to \infty} S_b(x_n, x_m, \Phi(x_l, x_l, x_l, \alpha, \beta, \gamma)) = 0.$$

So, $\{x_n\}$ is a convex S_b-Cauchy Sequence. By completeness of (X, S), there exist $x_o \in X$ such that x_n is convex S_b-convergent to x_o. Suppose $Tx_o \neq x_o$

$$
\begin{aligned}
S_b(x_n, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma)) &\leq k[S_b(x_{n-1}, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma)) \\
&+ S_b(x_o, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma)) \\
&+ S_b(x_o, x_n, \Phi(x_n, x_n, x_n, \alpha, \beta, \gamma))].
\end{aligned}
$$

Taking the limit as $n \to \infty$ and using the fact that function is convex S_b-continuous in its variables, we get

(2.35) $$
\begin{aligned}
&(1 - 2k) S_b(x_o, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma)) \\
&\leq kS_b(x_o, x_o, \Phi(x_o, x_o, x_o, \alpha, \beta, \gamma)).
\end{aligned}
$$

Hence,

(2.36) $$S_b(x_o, Tx_o, \Phi(Tx_o, Tx_o, Tx_o, \alpha, \beta, \gamma)) \leq 0.$$

This is a contradiction. So, $Tx_o = x_o$.

To show the uniqueness, suppose $x_1 \neq x_2$ is such that $Tx_1 = x_1$ and $Tx_2 = x_2$ then

$$\begin{aligned} S_b(Tx_1, Tx_2, \Phi(Tx_2, Tx_2, Tx_2, \alpha, \beta, \gamma)) &\leq k[S_b(x_1, Tx_2, \Phi(Tx_2, Tx_2, Tx_2, \alpha, \beta, \gamma)) \\ &+ S_b(x_2, Tx_2, \Phi(Tx_2, Tx_2, Tx_2, \alpha, \beta, \gamma)) \\ &+ S_b(x_2, Tx_1, \Phi(Tx_1, Tx_1, Tx_1, \alpha, \beta, \gamma))]. \end{aligned}$$

Since $Tx_1 = x_1$ and $Tx_2 = x_2$, we have

$$S_b(x_1, x_2, \Phi(x_2, x_2, x_2, \alpha, \beta, \gamma)). \leq 0, \tag{2.37}$$

which implies that $x_1 = x_2$. □

Conclusion

In conclusion, a new abstract space is introduced in this research work and some contractive mappings are established and used to prove some fixed point results on the newly introduced space. Examples are given to validate the originality and applicability of our results.

Conflict of interest

There is no conflict of interest of any kind, financial or non-financial type.

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