Advances in Mathematics: Scientific Journal **14**(2) (2025), 141–155
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.14.2.2

ON SOME NEW PROPERTIES OF THE LAMBDA GAMMA FUNCTION

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ABSTRACT. This paper establishes new properties of the lambda gamma function and further introduces a lambda analogue of the Riemann zeta function. We also established a relationship between the lambda analogue of the Riemann zeta function and the lambda analogue of the gamma function and some inequalities involving the relationship.

1. INTRODUCTION

The gamma function is well known for its useful applications in almost all areas of mathematics. In view of this, many mathematicians found the need for further generalization of the function. Among these generalizations are; the multiple gamma function, q-analogue of the gamma function, k-analogue of the gamma function and the recent lambda analogue of the gamma function. Motivated by the gamma function and the recent generalized lambda gamma function, this work is focus on some new properties of the lambda gamma function and its relationship with the Riemann zeta function. The following are some defintions and lemmas used in the work.

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2020 *Mathematics Subject Classification.* 33B15, 26A48, 37C30, 26D07.
Key words and phrases. lambda gamma function; Riemann zeta function; lambda analogue of Riemann zeta function.
Submitted: 31.03.2025; *Accepted:* 15.04.2025; *Published:* 28.04.2025.

Definition 1.1. *For $x > 0$, the gamma function is defined as*

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \tag{1.1}$$

and satisfies the properties

$$\Gamma(x) = \frac{1}{\zeta(x)} \int_0^\infty \frac{u^{x-1}}{e^u - 1} du, \ x > 1, \tag{1.2}$$

and

$$\frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{k=1}^{\infty} \left(1 + \frac{x}{k}\right) e^{\frac{-x}{k}}. \tag{1.3}$$

See [5, p. 64], [5, p. 103], [5, p. 71].

Definition 1.2. *The digamma function, denoted by $\psi(x)$ is defined as*

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x), \tag{1.4}$$

and satisfies the properties

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt \tag{1.5}$$

and

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}} \right) dt, \tag{1.6}$$

where $\gamma = 0.57721\ldots$ is the Euler Mascheroni constant.

The k^{th} derivative of the digamma function known as the polygamma function is defined as

$$\psi^k(x) = \frac{d^k}{dx^k} \psi(x) = \frac{d^{k+1}}{dx^{k+1}} \ln \Gamma(x) = (-1)^{k+1} \int_0^\infty \frac{t^k e^{-xt}}{1 - e^{-t}} dt. \tag{1.7}$$

Definition 1.3. *[9] Let $x > 0$ and $\lambda > 0$. Then the lambda gamma function is defined in different forms as*

$$\Gamma_\lambda(x) = \int_0^\infty t^{x-1} e^{-\lambda t} dt \tag{1.8}$$

$$= \lim_{k \to \infty} \frac{\lambda^{-x} k! k^x}{x(x+1)(x+2)\ldots(x+k)} \tag{1.9}$$

$$= \lambda^{-x} \Gamma(x). \tag{1.10}$$

Lemma 1.1. *[9]. The lambda gamma function satisfies the identity*

$$\Gamma_\lambda(x+1) = \frac{x}{\lambda}\Gamma_\lambda(x). \tag{1.11}$$

Definition 1.4. *[9]. The logarithmic derivative of the lambda gamma function, denoted by $\psi_\lambda(x)$ is defined as*

$$\psi_\lambda(x) = \frac{d}{dx}\ln\Gamma_\lambda(x), \tag{1.12}$$

and satisfies the properties

$$\psi_\lambda(x) = -\ln\lambda + \psi(x) \tag{1.13}$$

$$= -\ln\lambda - \gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1-e^{-t}}dt. \tag{1.14}$$

The existence of the gamma function as an extension of the factorial function for noninteger values has useful applications in diverse areas such as fluid dynamics, statistical mechanics, quantum physics as well as engineering. These importance of the gamma function prompted the need for mathematicians to further generalize the gamma function for the purpose of broadening its domain of applications in sciences. Among these generalizations is the recent lambda gamma function introduced in [9].

The newly established lambda gamma function can be used to obtain solutions for integrals of the form $\int_0^\infty f(t)\exp(-\lambda t)dt,$ which elementary solution does not exist. More importantly, in a similar way with the classical gamma function, the integrand in (1.8) also describes exponentially decay proccesses in time or space which reveals the usefulness of the lambda gamma function in analysis [4, p. v].

Recent results presented an extended concept of special matrix functions as a modified degenerate gamma matrix functions and some properties of the new function. The results also presents limit representation as well as asymptotic equality and an infinite product representation. The established function of matrix arguement is an analogue of the gamma function which charaterizes the lambda analogue of the gamma function established in [9]. The function reduces to the gamma matrix functions as $\lambda \to 0^+$. See [1], [2], [8].

In this paper, the goal is to establish some new properties of the lambda gamma function and further introduces a lambda analogue of the Riemann zeta function. We also establish a relationship betweeen the lambda gamma function and the lambda analogue of the Riemann zeta function.The results in this paper serves as an extension of the results obtained in [9] which provides further properties that can support findings in the lambda gamma function related problems. For example, if $f(t)$ is a power function and $g_\lambda(t) = -\lambda t$, then by change of variables one will obtain

$$\int_0^\infty t^x e^{-\lambda t} dt = \frac{\Gamma(x+1)}{\lambda^{x+1}},$$

in which case the ordinary gamma function is then considered for the final solution with the approperiate choice of λ.

2. Preliminaries

Lemma 2.1. *[5, p. 94] For $t > 0$ and $u > 0$, the Frullani's integral representation for $\ln t$ is given as*

$$\ln t = \int_0^\infty \left(\frac{e^{-u}}{u} - \frac{e^{-ut}}{u} \right) du. \tag{2.1}$$

Lemma 2.2. *[5, p. 103] For $x > 1$, the Riemann zeta function is defined as*

$$\zeta(x) = \sum_{m=1}^\infty \frac{1}{m^x}. \tag{2.2}$$

Lemma 2.3. *[5, p. 103] For $x > 1$, the Riemann zeta function, $\zeta(x)$ satisfies the integral representation,*

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{u^{x-1}}{e^u - 1} du. \tag{2.3}$$

Lemma 2.4. *(Cauchy product of series) [5, p. 14] Let $\sum_{i=0}^\infty a_i$ and $\sum_{i=0}^\infty b_j$ be two infinite series. Then the Cauchy product of the two series is defined as*

$$\left(\sum_{i=0}^\infty a_i \right) \left(\sum_{j=0}^\infty b_j \right) = \sum_{k=0}^\infty \sum_{p=0}^m a_p b_{k-p}. \tag{2.4}$$

Lemma 2.5. *[3] Let T be a function. If T' is completely monotonic on $(0, \infty)$, then the function e^{-T} is also compeletely monotonic on $(0, \infty)$.*

Lemma 2.6. *[3] Let α_i and β_i be real numbers where $i = 1, 2, 3 \ldots, n$ such that $0 < \alpha_1 \le \alpha_2 \le \alpha_3 \le \cdots \le \alpha_n$, $0 < \beta_1 \le \beta_2 \le \beta_3 \cdots \le \beta_n$ and $\sum_{i=1}^m \alpha_i \le \sum_{i=1}^m \beta_i$. If f is decreasing and convex on $\mathbb{R}$, then $\sum_{i=1}^\infty f(\beta_i) \le \sum_{i=1}^\infty f(\alpha_i)$.*

Lemma 2.7. *[6] Let f, g be nonneggative functions of a real variable and m and n are real numbers. If f, g are integrable functions, then the inequality*

$$\left(\int_a^b g(t)(f(t))^m dt\right)\left(\int_a^b g(t)(f(t))^n dt\right) \geq \left(\int_a^b g(t)(f(t))^{\frac{m+n}{2}} dt\right)^2, \tag{2.5}$$

holds.

Lemma 2.8. *[6]. Let $p > 1$ and $q > 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for integrable functions, $f, g : [a, b] \to \mathbb{R}$. the inequality*

$$\left|\int_a^b f(x)g(x)dx\right| \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}}, \tag{2.6}$$

holds.

3. Results and Discussions

Proposition 3.1. *The lambda gamma function has an Infinite product representation of the form*

$$\frac{1}{\Gamma_\lambda(x)} = \lambda^x x e^{x\gamma} \prod_{k=1}^{\infty}\left(1 + \frac{x}{k}\right) e^{\frac{-x}{k}}, \tag{3.1}$$

for $x > 0$ and $\lambda > 0$.

Proof. Using the identity (1.10), we have

$$\frac{1}{\Gamma_\lambda(x)} = \lambda^x \frac{1}{\Gamma(x)}. \tag{3.2}$$

Now, substituting (1.3) into (3.2) gives (3.1), which completes the proof. □

Proposition 3.2. *The $\lambda-$diagamma function has the following integral representations*

$$\psi_\lambda(x) = \int_0^\infty \left(\frac{e^{-\lambda t}}{t} - \frac{e^{-xt}}{1 - e^{-t}}\right) dt \tag{3.3}$$

$$= \int_0^\infty \left(e^{-\lambda t} - (1 + t)^{-x}\right) \frac{dt}{t} \tag{3.4}$$

$$= \int_0^1 \frac{e^{\frac{-\lambda(1-t)}{t}} - t^x}{t(1 - t)} dt. \tag{3.5}$$

Proof. By using equations (1.14), (2.1) and (1.13), we obtain

$$\begin{aligned}\psi_\lambda(x) &= -\ln\lambda + \psi(x)\\ &= -\int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-\lambda t}}{t}\right) + \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right) dt\\ &= \int_0^\infty \left(\frac{e^{-\lambda t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right) dt,\end{aligned}$$

which proves (3.3).

Next, let $s+1 = e^t$. Then $\frac{1}{s+1} = e^{-t}$ and $\ln(s+1) = t$. Then substituting these into (3.3) gives

$$\begin{aligned}\psi_\lambda(x) &= \int_0^\infty \left[\frac{\left(\frac{1}{s+1}\right)^\lambda}{\ln(s+1)} - \frac{\left(\frac{1}{s+1}\right)^x}{1-\frac{1}{s+1}}\right]\frac{ds}{s+1}\\ &= \int_0^\infty \left[\frac{\left(\frac{1}{s+1}\right)^\lambda}{(s+1)\ln(s+1)} - \frac{\left(\frac{1}{s+1}\right)^x}{s}\right] ds\\ &= \int_0^\infty \left[\frac{s\left(\frac{1}{s+1}\right)^\lambda}{(s+1)\ln(s+1)} - \left(\frac{1}{s+1}\right)^x\right]\frac{ds}{s}.\end{aligned}$$

Now, spliting the integrals gives

$$\psi_\lambda(x) = \int_0^\infty \frac{\left(\frac{1}{s+1}\right)^\lambda}{(s+1)\ln(s+1)} ds - \int_0^\infty \left(\frac{1}{s+1}\right)^x \frac{ds}{s}. \tag{3.6}$$

By substituting $s+1 = e^t$ into the first integral gives

$$\int_0^\infty \frac{\left(\frac{1}{s+1}\right)^\lambda}{(s+1)\ln(s+1)} ds = \int_0^\infty \frac{e^{-\lambda t}}{t} dt. \tag{3.7}$$

Substituting (3.7) into (3.6) gives

$$\psi_\lambda(x) = \int_0^\infty (e^{-\lambda t} - (t+1)^{-x})\frac{dt}{t},$$

which proves (3.4).

Next, using (3.4) with the substitution $s = \frac{1}{1+t}$, we have

$$\begin{aligned}\psi_\lambda(x) &= -\int_1^0 \left(e^{\frac{-\lambda(1-s)}{s}} - s^x\right) \frac{s\left(1+\frac{1-s}{s}\right)^2}{1-s} ds \\ &= -\int_1^0 \left(e^{\frac{-\lambda(1-s)}{s}} - s^x\right) \frac{1}{s(1-s)} ds \\ &= \int_0^1 \left(e^{\frac{-\lambda(1-s)}{s}} - s^x\right) \frac{1}{s(1-s)} ds \\ &= \int_0^1 \frac{e^{\frac{-\lambda(1-s)}{s}} - s^x}{s(1-s)} ds.\end{aligned}$$

which gives (3.5). □

Remark 3.1. *As a result of Proposition 3.2, the following are obtained.*

$$\int_0^\infty \left(e^{-\lambda t} - \frac{1}{1+t}\right) \frac{dt}{t} = \int_0^\infty \left(\frac{e^{-\lambda t}}{t} - \frac{e^{-t}}{1-e^{-t}}\right) \frac{dt}{t} \tag{3.8}$$

$$= \int_0^1 \frac{e^{\frac{-\lambda(1-t)}{t}} - t}{t(1-t)} dt \tag{3.9}$$

$$= -(\ln \lambda + \gamma). \tag{3.10}$$

Corollary 3.1. *The $\lambda-$digamma function satisfies the identity*

$$\psi_\lambda(x+1) = \int_0^\infty \left(\frac{e^{-\lambda t}}{t} - \frac{e^{-xt}}{e^t - 1}\right) dt. \tag{3.11}$$

Proof. By Proposition 3.2,

$$\begin{aligned}\psi_\lambda(x) &= \int_0^\infty \left(\frac{e^{-\lambda t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right) dt \\ &= \int_0^\infty \left(\frac{e^{-\lambda t}}{t} - \frac{e^{-(x-1)t}}{e^t - 1}\right) dt.\end{aligned}$$

Then by replacing x with $x+1$, we obtain (3.11). □

Theorem 3.1. *Let $x > 1$, $\lambda > 0$. Then the lambda gamma function satisfies the relationship,*

$$\Gamma_\lambda(x) = \frac{1}{\zeta(x)} \int_0^\infty \frac{u^{x-1}}{e^{\lambda u} - 1} du. \tag{3.12}$$

Proof. By using (1.8) with $t = mu$ where $m \in \mathbb{N}$, we have

$$\Gamma_\lambda(x) = m^x \int_0^\infty u^{x-1} e^{-\lambda m u} du.$$

Rearranging the equation gives

$$\frac{1}{m^x} = \frac{1}{\Gamma_\lambda(x)} \int_0^\infty u^{x-1} e^{-\lambda m u} du.$$

Now, introducing summation with respect to m gives

$$\sum_{m=1}^{\infty} \frac{1}{m^x} = \frac{1}{\Gamma_\lambda(x)} \int_0^\infty u^{x-1} \sum_{m=1}^{\infty} e^{-\lambda m u} du. \tag{3.13}$$

Since $\sum_{m=1}^{\infty} e^{-\lambda m u}$ is a geometric series, we have

$$\sum_{m=1}^{\infty} e^{-\lambda m u} = \frac{1}{e^{\lambda u} - 1}. \tag{3.14}$$

By substituting (3.14) into (3.13) and rearranging (3.13), we obtain the relation

$$\Gamma_\lambda(x) = \frac{1}{\zeta(x)} \int_0^\infty \frac{u^{x-1}}{e^{\lambda u} - 1} du. \tag{3.15}$$

□

Definition 3.1. *For $x > 1$ and $\lambda > 0$, the lambda analogue of the Riemann zeta function is defined as*

$$\zeta_\lambda(x) = \sum_{m=1}^{\infty} \frac{\lambda^x}{m^x}. \tag{3.16}$$

Corollary 3.2. *The lambda analogue of the Riemann zeta function satisfies the identity*

$$\zeta_\lambda(x) = \lambda^x \zeta(x). \tag{3.17}$$

Proof. The identity (3.17) follows from (3.16). □

Proposition 3.3. *For $x > 1$, the lambda analogue of the Riemann zeta function satisfies the integral representation*

$$\zeta_\lambda(x) = \frac{1}{\Gamma_\lambda(x)} \int_0^\infty \frac{u^{x-1}}{e^u - 1} du. \tag{3.18}$$

Proof. We begin by substituting $\lambda t = mu, m \in \mathbb{N}$ into (1.8) which gives

$$\Gamma_\lambda(x) = \int_0^\infty \left(\frac{mu}{\lambda}\right)^{x-1} e^{-mu}\frac{m}{\lambda}du = \frac{m^x}{\lambda^x}\int_0^\infty u^{x-1}e^{-mu}du.$$

Rearranging gives

$$\frac{\lambda^x}{m^x} = \frac{1}{\Gamma_\lambda(x)}\int_0^\infty u^{x-1}e^{-mu}du. \tag{3.19}$$

Summing both sides of (3.19) from $m = 1$ to ∞ gives

$$\sum_{m=1}^{\infty}\frac{\lambda^x}{m^x} = \frac{1}{\Gamma_\lambda(x)}\int_0^\infty u^{x-1}\sum_{m=1}^{\infty}e^{-mu}du. \tag{3.20}$$

Now, substituting $\sum_{m=1}^{\infty}e^{-mu} = \frac{1}{e^u-1}$ into (3.20) yields

$$\zeta_\lambda(x) = \frac{1}{\Gamma_\lambda(x)}\int_0^\infty \frac{u^{x-1}}{e^u-1}du. \tag{3.21}$$

This completes the proof. □

Proposition 3.4. *Let $x > 1$. Then the integral identity*

$$\int_0^\infty \frac{t}{e^{\lambda t}-1}dt = \frac{\pi^2}{6\lambda^2} \tag{3.22}$$

is valid.

Proof. Using (3.21), we proceed as

$$\int_0^\infty \frac{u^{x-1}}{e^u-1}du = \zeta_\lambda(x)\Gamma_\lambda(x) = \lambda^x\zeta(x)\lambda^{-x}\Gamma(x) = \zeta(x)\Gamma(x).$$

Now, substituting $x = 2$ and using identity (3.21) at $\lambda = 1$ gives

$$\int_0^\infty \frac{u}{e^u-1}du = \zeta(2)\Gamma(2) = \frac{\pi^2}{6}.$$

Next, letting $u = \lambda t$, we have

$$\int_0^\infty \frac{u}{e^u-1}du = \lambda^2\int_0^\infty \frac{t}{e^{\lambda t}-1}dt. \tag{3.23}$$

Rearranging (3.23), we obtain

$$\int_0^\infty \frac{t}{e^{\lambda t}-1}dt = \frac{1}{\lambda^2}\int_0^\infty \frac{u}{e^u-1}du = \frac{\pi^2}{6\lambda^2}.$$

This completes the proof. □

Proposition 3.5. *Let $x > 1$. Then the product $\zeta_\lambda(x)\zeta(x)$ has a double sum representation of the form:*

$$\frac{\zeta_\lambda(x)\zeta(x)}{\lambda^x} = \sum_{m=1}^{\infty}\sum_{i=1}^{m}\frac{(m-i)^{-x}}{i^x}, \tag{3.24}$$

for $n \in \mathbb{N}$, $m \in \mathbb{N}$.

Proof. Using (2.2) and (2.4), we obtain

$$\begin{aligned}\zeta_\lambda(x)\zeta(x) &= \left(\sum_{m=1}^{\infty}\frac{\lambda^x}{m^x}\right)\left(\sum_{m=1}^{\infty}\frac{1}{m^x}\right) = \lambda^x\sum_{m=1}^{\infty}\left[\sum_{i=1}^{m}\frac{1}{i^x(m-i)^x}\right]\\ &= \lambda^x\sum_{m=1}^{\infty}\sum_{i=1}^{m}\frac{(m-i)^{-x}}{i^x},\end{aligned}$$

which completes the proof. □

Proposition 3.6. *For $x > 0$, the lambda analogue of the Riemann zeta function satisfies the identity*

$$\frac{\zeta_\lambda(x)}{\lambda^x} = \sqrt{\sum_{m=1}^{\infty}\sum_{i=1}^{m}\frac{(m-i)^{-x}}{i^x}}. \tag{3.25}$$

Proof. We start with the product,

$$\zeta_\lambda(x)\zeta_\lambda(x) = \lambda^x\zeta(x)\zeta_\lambda(x) = \lambda^{2x}\sum_{m=1}^{\infty}\sum_{i=1}^{m}\frac{(m-i)^{-x}}{i^x}.$$

Now, we have

$$\zeta_\lambda^2(x) = \lambda^{2x}\sum_{m=1}^{\infty}\sum_{i=1}^{m}\frac{(m-i)^{-x}}{i^x}. \tag{3.26}$$

By taking the square root of both sides and rearranging gives (3.25), which ends the proof. □

Theorem 3.2. *Let $x > 0$ and $0 < u < 1$. Then the function*

$$\phi_{\lambda,u}(x) = \frac{\Gamma_\lambda(ux+1)}{[\Gamma_\lambda(x+1)]^u} \tag{3.27}$$

is decreasing.

Proof. By taking logarithm of $\phi_{\lambda,u}(x)$, we obtain

$$\ln \phi_{\lambda,u}(x) = \ln \Gamma_\lambda(ux+1) - \ln \Gamma_\lambda(x+1).$$

Differentiating $\ln \phi_{\lambda,u}(x)$ with respect to x yields

$$\begin{aligned}\frac{\phi'_{\lambda,u}(x)}{\phi_{\lambda,u}(x)} &= \frac{u\Gamma'_\lambda(ux+1)}{\Gamma_\lambda(ux+1)} - \frac{u\Gamma_\lambda(x+1)}{\Gamma_\lambda(x+1)} = u\psi_\lambda(ux+1) - u\psi_\lambda(x+1)\\ &= u[\psi_\lambda(ux+1) - \psi_\lambda(x+1)] \leq 0.\end{aligned}$$

This completes the proof. □

Corollary 3.3. *Let $x > 0$ and $0 < u < 1$. Then the function $\phi_{\lambda,u}(x)$ satisfies the inequality*

$$\frac{1}{\Gamma_\lambda(x+1)} \leq \frac{\Gamma_\lambda(ux+1)}{[\Gamma_\lambda(x+1)]^{u-1}} \leq \frac{1}{\lambda}. \tag{3.28}$$

Proof. By Theorem 3.2, we obtain

$$\lim_{x\to 0} \phi_{\lambda,u}(x) \geq \phi_{\lambda,u}(x) \geq \lim_{x\to 1} \phi_{\lambda,u}(x),$$

which yields (3.28). □

Remark 3.2. *Theorem 3.2 and Corollary 3.3 are similar to results obtained for the $p,k-$ analogue of the gamma function in Lemma 2.3 and Theorem 2.4 of [10].*

Theorem 3.3. *Let α_i and β_i be real numbers where $i = 1,2,3\ldots,n$ such that $0 < \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \cdots \leq \alpha_n$, $0 < \beta_1 \leq \beta_2 \leq \beta_3 \cdots \leq \beta_n$ and $\sum_{i=1}^m \alpha_i \leq \sum_{i=1}^m \beta_i$ for $m = 1,2,3,\ldots n$. Then the function*

$$f_\lambda(x) = \prod_{i=1}^m \frac{\Gamma_\lambda(x+\beta_i)}{\Gamma_\lambda(x+\alpha_i)} \tag{3.29}$$

is logarithmically completely monotonic on $(0,\infty)$.

Proof. Let

$$T_\lambda(x) = \ln f_\lambda(x) = \sum_{i=1}^m \left[\ln \Gamma_\lambda(x+\beta_i) - \ln \Gamma_\lambda(x+\alpha_i)\right].$$

Differentiating $T_\lambda(x)$, we obtain

$$T'_\lambda(x) = \sum_{i=1}^{m}\left[\frac{\Gamma'_\lambda(x+\beta_i)}{\Gamma_\lambda(x+\beta_i)} - \frac{\Gamma'_\lambda(x+\alpha_i)}{\Gamma_\lambda(x+\alpha_i)}\right]$$
$$= \sum_{i=1}^{m}[\psi_\lambda(x+\beta_i) - \psi_\lambda(x+\alpha_i)].$$

Diffentiating $T_\lambda(x)$ continuously for k times yields

$$T_\lambda^{(k)}(x) = \sum_{i=1}^{m}[\psi_\lambda^{(k)}(x+\beta_i) - \psi_\lambda^{(k)}(x+\alpha_i)]. \tag{3.30}$$

Since $\psi'_\lambda(x) = \psi'(x)$, we have $\psi_\lambda^{(k)}(x) = \psi^{(k)}(x)$. By substituting (1.7) into (3.30) , we obtain

$$T_\lambda^{(k)}(x) = \sum_{i=1}^{m}\left[(-1)^{k+1}\int_0^\infty \frac{t^k e^{-(x+\beta_i)t}}{1-e^{-t}}dt - (-1)^{k+1}\int_0^\infty \frac{t^k e^{-(x+\alpha_i)t}}{1-e^{-t}}dt\right]$$
$$= (-1)^{k+1}\sum_{i=1}^{m}\left[\int_0^\infty \frac{t^k e^{-(x+\beta_i)t}}{1-e^{-t}}dt - \int_0^\infty \frac{t^k e^{-(x+\alpha_i)t}}{1-e^{-t}}dt\right]$$
$$= (-1)^{k+1}\sum_{i=1}^{m}\int_0^\infty \frac{t^k}{1-e^{-t}}\left[e^{-(x+\beta_i)t} - e^{-(x+\alpha_i)t}\right]dt$$
$$= (-1)^{k+1}\int_0^\infty \frac{t^k}{1-e^{-t}}\sum_{i=1}^{m}\left[e^{-(x+\beta_i)t} - e^{-(x+\alpha_i)t}\right]dt$$
$$= (-1)^{k+1}\int_0^\infty \frac{t^k e^{-xt}}{1-e^{-t}}\sum_{i=1}^{m}\left[e^{-\beta_i t} - e^{-\alpha_i t}\right]dt.$$

By Lemma 2.6, we have

$$\sum_{i=1}^{m}\left[e^{-\beta_i t} - e^{-\alpha_i t}\right] \leq 0,$$

which gives

$$(-1)^k T_\lambda^{(k)}(x) = (-1)^{2k+1}\int_0^\infty \frac{t^k e^{-xt}}{1-e^{-t}}\sum_{i=1}^{m}\left[e^{-\beta_i t} - e^{-\alpha_i t}\right]dt \geq 0.$$

This completes the proof. □

Remark 3.3. *Theorem 3.3 generalizes the results of Theorem* 10 *in [3].*

Remark 3.4. *By Lemma 2.5, we conclude that* $e^{-T_\lambda(x)}$ *is also completely monotonic.*

Theorem 3.4. *The inequality*

$$(x+1)\frac{\zeta_\lambda(x+2)}{\zeta_\lambda(x+1)} \geq x\frac{\zeta_\lambda(x+1)}{\zeta_\lambda(x)} \tag{3.31}$$

holds.

Proof. Using Lemma 2.7 with $m = x-1$, $n = x+1$, $f(u) = u$ and $g(u) = \frac{1}{e^u-1}$, we have

$$\left(\int_0^\infty \frac{u^{x-1}}{e^u-1}du\right)\left(\int_0^\infty \frac{u^{x+1}}{e^u-1}du\right) \geq \left(\int_0^\infty \frac{u^{x}}{e^u-1}du\right)^2. \tag{3.32}$$

By identity (3.18), we have

$$\Gamma_\lambda(x)\zeta_\lambda(x)\Gamma_\lambda(x+2)\zeta_\lambda(x+2) \geq \Gamma_\lambda^2(x+1)\zeta_\lambda^2(x+1). \tag{3.33}$$

Rearranging (3.33) gives

$$\frac{\zeta_\lambda(x)\zeta_\lambda(x+2)\Gamma_\lambda(x+2)}{\Gamma_\lambda(x+1)} \geq \frac{\zeta_\lambda^2(x+1)\Gamma_\lambda(x+1)}{\Gamma_\lambda(x)}. \tag{3.34}$$

Using (1.11) with (3.34), we obtain

$$\frac{x+1}{\lambda}\zeta_\lambda(x)\zeta_\lambda(x+2) \geq \frac{x}{\lambda}\zeta_\lambda^2(x+1). \tag{3.35}$$

Rearranging (3.35) proves (3.31). □

Remark 3.5. *When $\lambda = 1$, we obtain the results of Theorem 2.3 in [7].*

Remark 3.6. *Inequality* (3.31) *implies that the function $x\frac{\zeta_\lambda(x+1)}{\zeta_\lambda(x)}$ is increasing.*

Theorem 3.5. *The inequality*

$$\frac{\Gamma_\lambda\left(\frac{w}{p}+\frac{r}{q}\right)}{\Gamma_\lambda(w)\Gamma_\lambda(r)} \leq \frac{\zeta_\lambda^{\frac{1}{p}}(w)\zeta_\lambda^{\frac{1}{q}}(r)}{\zeta_\lambda\left(\frac{w}{p}+\frac{r}{q}\right)}, \tag{3.36}$$

holds.

Proof. Using (3.18) and (2.6) with $f(u) = \frac{u^{\frac{w-1}{p}}}{(e^u-1)^{\frac{1}{p}}}$ and $g(u) = \frac{u^{\frac{r-1}{q}}}{(e^u-1)^{\frac{1}{q}}}$.

We proceed as

$$\left(\int_0^\infty \frac{u^{\frac{w-1}{p}}}{(e^u-1)^{\frac{1}{p}}}du\right)\left(\int_0^\infty \frac{u^{\frac{r-1}{q}}}{(e^u-1)^{\frac{1}{q}}}du\right)$$
$$\leq \left[\int_0^\infty \left(\frac{u^{\frac{w-1}{p}}}{(e^u-1)^{\frac{1}{p}}}\right)^p du\right]^{\frac{1}{p}} \left[\int_0^\infty \left(\frac{u^{\frac{r-1}{q}}}{(e^u-1)^{\frac{1}{q}}}\right)^q du\right]^{\frac{1}{q}}.$$

Simplifying the inequality results

$$\int_0^\infty \frac{u^{\frac{w}{p}+\frac{r}{q}-1}}{e^u-1}du \leq \left[\int_0^\infty \frac{u^{w-1}}{e^u-1}du\right]^{\frac{1}{p}} \left[\int_0^\infty \frac{u^{r-1}}{e^u-1}du\right]^{\frac{1}{q}},$$

which implies that

$$\Gamma_\lambda\left(\frac{w}{p}+\frac{r}{q}\right)\zeta_\lambda\left(\frac{w}{p}+\frac{r}{q}\right) \leq \Gamma_\lambda^{\frac{1}{p}}(w)\zeta_\lambda^{\frac{1}{p}}(w)\Gamma_\lambda^{\frac{1}{q}}(r)\zeta_\lambda^{\frac{1}{q}}(r).$$

Rearranging the inequality gives

$$\frac{\Gamma_\lambda\left(\frac{w}{p}+\frac{r}{q}\right)}{\Gamma_\lambda^{\frac{1}{p}}(w)\Gamma_\lambda^{\frac{1}{q}}(r)} \leq \frac{\zeta_\lambda^{\frac{1}{p}}(w)\zeta_\lambda^{\frac{1}{q}}(r)}{\zeta_\lambda\left(\frac{w}{p}+\frac{r}{q}\right)},$$

which completes the proof. □

Remark 3.7. *Theorem 3.5 is similar to results of Theorem 3.3 obtained for the k-analogue of the Riemann zeta function in [6].*

Conflict of interest

The authors have no conflict of interest regarding the work.

Acknowledgment

The research received no specific grant from any funding agency in the public, commecial or profit sectors.

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