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SIMULTANEOUS NUMERICAL BLOW-UP IN A FOUR-COMPONENT SYSTEM OF HEAT EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

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ABSTRACT. This paper investigates the numerical approximation of a system of heat equations with nonlinear boundary conditions. We prove that the solution of a semidiscrete form of above problem blows up in a finite time. We also establish certain conditions under which simultaneous blow-up occurs for the solution of the semidiscrete problem. After showing that the numerical blow-up time converges to the theoretical blow-up time as the mesh size tends to zero, we finally present some numerical results to illustrate key points of our work.

1. INTRODUCTION

In this paper, we focus on analyzing the behavior of a semidiscrete approximation for a system of heat equations with nonlinear boundary conditions:

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$$
(1.1)\quad \begin{cases}
u_t(x,t) = u_{xx}(x,t), \quad v_t(x,t) = v_{xx}(x,t),\\
w_t(x,t) = w_{xx}(x,t), \quad y_t(x,t) = y_{xx}(x,t), \quad (x,t) \in (0,1)\times(0,T),\\
-u_x(0,t) = (u^{p_1}v^{q_2})(0,t),\ -v_x(0,t) = (v^{p_2}w^{q_3})(0,t),\\
-w_x(0,t) = (w^{p_3}y^{q_4})(0,t),\ -y_x(0,t) = (y^{p_4}u^{q_1})(0,t),\ t\in(0,T),\\
u_x(1,t) = (u^{p_1}v^{q_2})(1,t), \quad v_x(1,t) = (v^{p_2}w^{q_3})(1,t),\\
w_x(1,t) = (w^{p_3}y^{q_4})(1,t), \quad y_x(1,t) = (y^{p_4}u^{q_1})(1,t), \quad t\in(0,T),\\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x),\\
w(x,0) = w_0(x), \quad y(x,0) = y_0(x), \quad x\in[0,1],
\end{cases}
$$

where constants $p_j,\ q_j \geq 0\ (j = 1,2,3,4)$ and $u_0(x),\ v_0(x),\ w_0(x),\ y_0(x)$ are positive smooth functions satisfying the compatibility conditions.

Systems of heat equations with nonlinear boundary conditions like (1.1) come from chemical reactions, heat transfer, etc., where $u,\ v,\ w,\ y$ represent concentrations of chemical reactants, temperatures of materials during heat propagation, etc.

The existence and uniqueness of local classical solutions to (1.1) is well known (see, for example, [5]). Here $[0,T)$ is the maximal time interval on which the solution exists. The time T may be finite or not. When T is infinite, we say that the solution (u,v,w,y) exists globally. When T is finite, the solution (u,v,w,y) develops a singularity in finite time, namely

$$\lim_{t\to T}\sup\left\{\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty + \|w(\cdot,t)\|_\infty + \|y(\cdot,t)\|_\infty\right\} = +\infty,$$

where $\|u(\cdot,t)\|_\infty = \max\limits_{0\leq x\leq 1}|u(x,t)|$.

In this case, we say that the solution (u,v,w,y) blows up in finite time, and the time T is called the blow-up time of the solution (u,v,w,y).

Non-simultaneous and simultaneous blow-up for systems of heat equations with nonlinear boundary conditions have attracted significant attention (see, e.g., [1, 2, 6, 8]). Simultaneous blow-up is defined as

$$\lim_{t\to T}\sup\min\left\{\|u(\cdot,t)\|_\infty, \|v(\cdot,t)\|_\infty, \|w(\cdot,t)\|_\infty, \|y(\cdot,t)\|_\infty\right\} = +\infty.$$

Non-simultaneous blow-up means that at least $j \in \{1, 2, 3\}$ components blow up simultaneously while the others remain bounded up to the blow-up time.

In [7], the authors analyzed various blow-up scenarios for system (1.1). Specifically, they examined cases where only one component blows up, where exactly two components blow up, and where blow-up may be either simultaneous or non-simultaneous for any initial data. In particular, they proved that the solution blows up simultaneously in finite time when one of the following conditions is satisfied:

- If $\beta_k = \dfrac{1 - q_{k+1}\beta_{k+1}}{p_k - 1}$ with $\beta_4 = \dfrac{1}{p_4 - 1}$, $p_k \leq 1 < p_4$ for $k = 1, 2, 3$, and if $\beta_1 \geq 0$ and $\beta_k > 0$ for $k = 2, 3$.
- If $p_1, p_2, p_3, p_4 \leq 1$ and $\prod_{n=1}^{4} q_n > \prod_{n=1}^{4}(1 - p_n)$.

The main objective of this paper is to numerically investigate the behavior of the semidiscrete approximation of system (1.1) under these blow-up conditions and to provide an approximation of the blow-up time. Our work is inspired by the studies in [3, 4] on the numerical approximation of heat equations with nonlinear boundary conditions, as well as the references cited therein.

We organise this paper as follows: in the next section, we present a semidiscrete scheme of the problem (1.1). In the third section, we give some properties concerning our semidiscrete scheme. In the fourth section, under some conditions, we prove that the solution of the semidiscrete scheme of (1.1) blows up in finite time. The criteria to identify simultaneous blow-up are proposed in the fifth section. In the sixth section, we show the convergence of the solution of the semidiscrete scheme and the convergence of the blow-up times to the theoretical one when the mesh size goes to zero. Finally, in the last section, we present some numerical experiments.

2. Semidiscrete problem

Let $I \geq 2$ be a positive integer and define the grid $x_i = (i-1)h$, $i = 1, \ldots, I$, where $h = \dfrac{1}{I-1}$ is the mesh parameter. Approximate the solution (u, v, w, y) of the problem (1) by the solution $(U_h(t) = (U_1(t), \ldots, U_I(t))^T, V_h(t) = (V_1(t), \ldots, V_I(t))^T$, $W_h(t) = (W_1(t), \ldots, W_I(t))^T, Y_h(t) = (Y_1(t), \ldots, Y_I(t))^T)$ and approximate the initial data (u_0, v_0, w_0, y_0) of the same problem by $(\varphi_{1,h} = (\varphi_{1,1}, \ldots, \varphi_{1,I})^T$,

$\varphi_{2,h} = (\varphi_{2,1}, \ldots, \varphi_{2,I})^T$, $\varphi_{3,h} = (\varphi_{3,1}, \ldots, \varphi_{3,I})^T$, $\varphi_{4,h} = (\varphi_{4,1}, \ldots, \varphi_{4,I})^T$) of the following system of ODEs which is obtained using the finite difference method

$$U_i'(t) = \delta^2 U_i(t) + \omega_i \left(U_i^{p_1} V_i^{q_2}\right)(t), \quad i = 1, \ldots, I, \ t \in [0, T_h), \tag{2.1}$$

$$V_i'(t) = \delta^2 V_i(t) + \omega_i \left(V_i^{p_2} W_i^{q_3}\right)(t), \quad i = 1, \ldots, I, \ t \in [0, T_h), \tag{2.2}$$

$$W_i'(t) = \delta^2 W_i(t) + \omega_i \left(W_i^{p_3} Y_i^{q_4}\right)(t), \quad i = 1, \ldots, I, \ t \in [0, T_h), \tag{2.3}$$

$$Y_i'(t) = \delta^2 Y_i(t) + \omega_i \left(Y_i^{p_4} U_i^{q_1}\right)(t), \quad i = 1, \ldots, I, \ t \in [0, T_h), \tag{2.4}$$

$$U_i(0) = \varphi_{1,i}, \quad V_i(0) = \varphi_{2,i}, \quad W_i(0) = \varphi_{3,i}, \quad Y_i(0) = \varphi_{4,i}, \quad i = 1, \ldots, I, \tag{2.5}$$

where

$$p_1,\ p_2,\ p_3,\ p_4,\ q_1,\ q_2,\ q_3,\ q_4 \geq 0, \quad \varphi_{1,i},\ \varphi_{2,i},\ \varphi_{3,i},\ \varphi_{4,i} > 0, \quad i = 1, \ldots, I,$$

$$\delta^2 U_i(t) = \frac{U_{i-1}(t) - 2U_i(t) + U_{i+1}(t)}{h^2},\ 2 \leq i \leq I-1,\ t \in [0, T_h),$$
$$\delta^2 U_1(t) = \frac{2U_2(t) - 2U_1(t)}{h^2},\ \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2},\ t \in [0, T_h),$$

$\omega_1 = \omega_I = \dfrac{2}{h}$ and $\omega_i = 0,\ i = 2, \ldots, I-1$. Here $[0, T_h)$ is the maximal time interval on which $\max\{\|U_h(t)\|_\infty, \|V_h(t)\|_\infty,\ \|W_h(t)\|_\infty, \|Y_h(t)\|_\infty\}$ is finite, where $\|U_h(t)\|_\infty = \max\limits_{1 \leq i \leq I} |U_i(t)|$. When the time T_h is finite, we say that the solution (U_h, V_h, W_h, Y_h) blows up in a finite time and the time T_h is called the blow-up time of the solution (U_h, V_h, W_h, Y_h).

3. Properties of the semidiscrete scheme

In this section, we present auxiliary results for the problem (2.1)–(2.5).

Definition 3.1. *We define* $(\underline{U_h}, \underline{V_h}, \underline{W_h}, \underline{Y_h}) \in \left(C^1([0, T_h), \mathbb{R}^I)\right)^4$ *as a lower solution to* (2.1)–(2.5) *if*

$$\begin{aligned}
\underline{U_i}'(t) &\leq \delta^2 \underline{U_i}(t) + \omega_i \underline{U_i}^{p_1}(t) \underline{V_i}^{q_2}(t), \quad i = 1, \ldots, I,\ t \in (0, T_h), \\
\underline{V_i}'(t) &\leq \delta^2 \underline{V_i}(t) + \omega_i \underline{V_i}^{p_2}(t) \underline{W_i}^{q_3}(t), \quad i = 1, \ldots, I,\ t \in (0, T_h), \\
\underline{W_i}'(t) &\leq \delta^2 \underline{W_i}(t) + \omega_i \underline{W_i}^{p_3}(t) \underline{Y_i}^{q_4}(t), \quad i = 1, \ldots, I,\ t \in (0, T_h),
\end{aligned}$$

$$\begin{aligned}
\underline{Y_i}'(t) &\leq \delta^2\underline{Y_i}(t) + \omega_i\underline{Y_i}^{p_4}(t)\underline{U_i}^{q_1}(t), \quad i=1,\dots,I,\ t\in(0,T_h),\\
0<\underline{U_i}(0) &\leq \varphi_{1,i}, \quad 0<\underline{V_i}(0)\leq\varphi_{2,i},\\
0 &< \underline{W_i}(0)\leq\varphi_{3,i},\ 0<\underline{Y_i}(0)\leq\varphi_{4,i}, \quad i=1,\dots,I,
\end{aligned}$$

where (U_h, V_h, W_h, Y_h) *is the solution of* (2.1)–(2.5). *Similarly, we define* $(\overline{U_h}, \overline{V_h}, \overline{W_h}, \overline{Y_h}) \in \left(C^1([0,T_h),\mathbb{R}^I)\right)^4$ *as an upper solution to* (2.1)–(2.5) *if these inequalities are reversed.*

The following lemma provides a semidiscrete version of the maximum principle.

Lemma 3.1. *Let* $e_h, c_h, \alpha_h, \beta_h, \lambda_h, \gamma_h, \mu_h, \eta_h \in C^0([0,T_h),\mathbb{R}^I)$ *and* $U_h, V_h, W_h, Y_h \in C^1([0,T_h),\mathbb{R}^I)$ *such that*

$$\begin{aligned}
U_i'(t) - \delta^2U_i(t) - e_i(t)U_i(t) - c_i(t)V_i(t) \geq 0, \quad i=1,\dots,I,\ t\in(0,T_h),\\
V_i'(t) - \delta^2V_i(t) - \alpha_i(t)V_i(t) - \beta_i(t)W_i(t) \geq 0, \quad i=1,\dots,I,\ t\in(0,T_h),\\
W_i'(t) - \delta^2W_i(t) - \lambda_i(t)W_i(t) - \gamma_i(t)Y_i(t) \geq 0, \quad i=1,\dots,I,\ t\in(0,T_h),\\
Y_i'(t) - \delta^2Y_i(t) - \mu_i(t)Y_i(t) - \eta_i(t)U_i(t) \geq 0, \quad i=1,\dots,I,\ t\in(0,T_h),\\
U_i(0)\geq 0,\ V_i(0)\geq 0,\ W_i(0)\geq 0,\ Y_i(0)\geq 0, \qquad i=1,\dots,I.
\end{aligned}$$

Then, it follows that

$$U_i(t)\geq 0,\ V_i(t)\geq 0,\ W_i(t)\geq 0,\ Y_i(t)\geq 0, \quad i=1,\dots,I,\ t\in[0,T_h).$$

Proof. Let $T_0 < T_h$ and let $(N_h(t), M_h(t), K_h(t), L_h(t)) = (\mathrm{e}^{\nu t}U_h(t), \mathrm{e}^{\nu t}V_h(t), \mathrm{e}^{\nu t}W_h(t), \mathrm{e}^{\nu t}Y_h(t))$ where ν is a real. We find that $(N_h(t), M_h(t), K_h(t), L_h(t))$ satisfies the following inequalities : for $i=1,\dots,I,\ t\in(0,T_h)$

$$N_i'(t) - \delta^2N_i(t) - (e_i(t)+\nu)N_i(t) - c_i(t)M_i(t) \geq 0, \tag{3.1}$$

$$M_i'(t) - \delta^2M_i(t) - (\alpha_i(t)+\nu)M_i(t) - \beta_i(t)K_i(t) \geq 0, \tag{3.2}$$

$$K_i'(t) - \delta^2K_i(t) - (\lambda_i(t)+\nu)K_i(t) - \gamma_i(t)L_i(t) \geq 0, \tag{3.3}$$

$$L_i'(t) - \delta^2L_i(t) - (\mu_i(t)+\nu)L_i(t) - \eta_i(t)N_i(t) \geq 0, \tag{3.4}$$

$$N_i(0)\geq 0,\ M_i(0)\geq 0,\ K_i(0)\geq 0,\ L_i(0)\geq 0. \tag{3.5}$$

Let

$$m = \min\Big\{ \min_{1\le i\le I,\ t\in[0,T_0]} N_i(t),\ \min_{1\le i\le I,\ t\in[0,T_0]} M_i(t), \\ \min_{1\le i\le I,\ t\in[0,T_0]} K_i(t),\ \min_{1\le i\le I,\ t\in[0,T_0]} L_i(t) \Big\}$$

Since for $i \in \{1,\dots,I\}$, N_i, M_i, K_i and L_i arecontinuous functions on a compact, we can assume that $m = N_{i_0}(t_{i_0})$, for a certain $i_0 \in \{1,\dots,I\}$. Assume $m < 0$ and $\nu < 0$ such that:

$$(e_{i_0}(t_{i_0}) + \nu) < 0, \quad (\alpha_{i_0}(t_{i_0}) + \nu) < 0, \quad (\lambda_{i_0}(t_{i_0}) + \nu) < 0 \quad \text{and} \quad (\mu_{i_0}(t_{i_0}) + \nu) < 0.$$

If $t_{i_0} = 0$, then $N_{i_0}(0) < 0$, which contradicts (3.5), hence $t_{i_0} \neq 0$; if $1 \le i_0 \le I$, we have

$$N'_{i_0}(t_{i_0}) = \lim_{k\to 0} \frac{N_{i_0}(t_{i_0}) - N_{i_0}(t_{i_0}-k)}{k} \le 0 \quad \text{and} \quad \delta^2 N_{i_0}(t_{i_0}) \ge 0.$$

Moreover through direct computation, we obtain the following results

$$N'_{i_0}(t_{i_0}) - \delta^2 N_{i_0}(t_{i_0}) - (e_{i_0}(t_{i_0}) + \nu) N_{i_0}(t_{i_0}) - c_{i_0}(t_{i_0}) M_{i_0}(t_{i_0}) < 0,$$

but this inequality contradicts (3.1) which completes proof. □

Lemma 3.2. *Let* $(\underline{U_h},\underline{V_h},\underline{W_h},\underline{Y_h}), (\overline{U_h},\overline{V_h},\overline{W_h},\overline{Y_h}) \in \left(C^1([0,T_h),\mathbb{R}^I)\right)^4$ *be lower and upper solutions of* (2.1)–(2.5) *respectively such that,* $(\underline{U_h}(0),\underline{V_h}(0),\underline{W_h}(0),\underline{Y_h}(0)) \le (\overline{U_h}(0),\overline{V_h}(0),\overline{W_h}(0),\overline{Y_h}(0))$, *then*

$$(\underline{U_h}(t),\underline{V_h}(t),\underline{W_h}(t),\underline{Y_h}(t)) \le (\overline{U_h}(t),\overline{V_h}(t),\overline{W_h}(t),\overline{Y_h}(t)).$$

Proof. Let us define

$$(F_h(t),G_h(t),L_h(t),H_h(t)) = (\overline{U_h}(t),\overline{V_h}(t),\overline{W_h}(t),\overline{Y_h}(t)) - (\underline{U_h}(t),\underline{V_h}(t),\underline{W_h}(t),\underline{Y_h}(t)).$$

By straightforward computations and applying the Mean Value Theorem, we obtain for $i = 1,\dots,I$:

$$F'_i(t) - \delta^2 F_i(t) - p_1\omega_i \overline{V}_i^{q_2}(t)(\xi_i(t))^{p_1-1} F_i(t) \\ - q_2\omega_i \underline{U}_i^{p_1}(t)(\rho_i(t))^{q_2-1} G_i(t) \ge 0, \tag{3.6}$$

$$G_i'(t) - \delta^2 G_i(t) - p_2\omega_i \overline{W}_i^{q_3}(t)(\rho_i(t))^{p_2-1}G_i(t) \tag{3.7}$$
$$-q_3\omega_i \underline{V}_i^{p_2}(t)(\varrho_i(t))^{q_3-1}L_i(t) \geq 0,$$
$$L_i'(t) - \delta^2 L_i(t) - p_3\omega_i \overline{Y}_i^{q_4}(t)(\varrho_i(t))^{p_3-1}L_i(t) \tag{3.8}$$
$$-q_4\omega_i \underline{W}_i^{p_3}(t)(\lambda_i(t))^{q_4-1}H_i(t) \geq 0,$$
$$H_i'(t) - \delta^2 H_i(t) - p_4\omega_i \overline{U}_i^{q_1}(t)(\lambda_i(t))^{p_4-1}H_i(t) \tag{3.9}$$
$$-q_1\omega_i \underline{Y}_i^{p_4}(t)(\xi_i(t))^{q_1-1}F_i(t) \geq 0,$$
$$F_i(0) \geq 0,\ G_i(0) \geq 0,\ L_i(0) \geq 0,\ H_i(t) \geq 0, \tag{3.10}$$

where $\xi_i(t)$, $\rho_i(t)$, $\varrho_i(t)$ and $\lambda_i(t)$ lie, respectively, between $\overline{U_i}(t)$ and $\underline{U_i}(t)$, between $\overline{V_i}(t)$ and $\underline{V_i}(t)$, between $\overline{W_i}(t)$ and $\underline{W_i}(t)$ and between $\overline{Y_i}(t)$ and $\underline{Y_i}(t)$, for $i \in \{1, \dots, I\}$.

We can rewrite (3.6)–(3.10) as

$$F_i'(t) - \delta^2 F_i(t) - e_i(t)F_i(t) - c_i(t)G_i(t) \geq 0, \quad i = 1, \dots, I,\ t \in (0, T_h),$$
$$G_i'(t) - \delta^2 G_i(t) - \theta_i(t)G_i(t) - \beta_i(t)L_i(t) \geq 0, \quad i = 1, \dots, I,\ t \in (0, T_h),$$
$$L_i'(t) - \delta^2 L_i(t) - \vartheta_i(t)L_i(t) - \gamma_i(t)H_i(t) \geq 0, \quad i = 1, \dots, I,\ t \in (0, T_h),$$
$$H_i'(t) - \delta^2 H_i(t) - \mu_i(t)H_i(t) - \eta_i(t)F_i(t) \geq 0, \quad i = 1, \dots, I,\ t \in (0, T_h),$$

where

$$e_i(t) = p_1\omega_i \overline{V}_i^{q_2}(t)(\xi_i(t))^{p_1-1}, c_i(t) = q_2\omega_i \underline{U}_i^{p_1}(t)(\rho_i(t))^{q_2-1},$$
$$\theta_i(t) = p_2\omega_i \overline{W}_i^{q_3}(t)(\rho_i(t))^{p_2-1},$$
$$\beta_i(t) = q_3\omega_i \underline{V}_i^{p_2}(t)(\varrho_i(t))^{q_3-1}, \vartheta_i(t) = p_3\omega_i \overline{Y}_i^{q_4}(t)(\varrho_i(t))^{p_3-1},$$
$$\gamma_i(t) = q_4\omega_i \underline{W}_i^{p_3}(t)(\lambda_i(t))^{q_4-1}, \mu_i(t) = p_4\omega_i \overline{U}_i^{q_1}(t)(\lambda_i(t))^{p_4-1},$$
$$\eta_i(t) = q_1\omega_i \underline{Y}_i^{p_4}(t)(\xi_i(t))^{q_1-1}, for\ i = 1, \dots, I, \forall t \in (0, T_h).$$

According to Lemma 3.1, $F_i(t) \geq 0$, $G_i(t) \geq 0$, $L_i(t) \geq 0$, $H_i(t) \geq 0$, for $i = 1, \dots, I$, $\forall t \in (0, T_h)$. This concludes the proof. The lemma below reveals the positivity of the solution of the semidiscrete problem. □

Lemma 3.3. *Let* $(U_h, V_h, W_h, Y_h) \in \left(C^1([0, T_h), \mathbb{R}^I)\right)^4$ *be the solution of* (2.1)–(2.5) *with an initial data* $(\varphi_{1,h}, \varphi_{2,h}, \varphi_{3,h}, \varphi_{4,h})$ *lower solution such that* $0 < \varphi_{1,i} < \varphi_{1,i+1}$, $0 < \varphi_{2,i} < \varphi_{2,i+1}$, $0 < \varphi_{3,i} < \varphi_{3,i+1}$ *and* $0 < \varphi_{4,i} < \varphi_{4,i+1}$, *for* $i = 1, \dots, I - 1$.

Then, it follows that

$$(U_i(t),V_i(t),W_i(t),Y_i(t)) \geq (\varphi_{1,i},\varphi_{2,i},\varphi_{3,i},\varphi_{4,i}), \quad i=1,\ldots,I, \quad t\in(0,T_h).$$

Proof. As $(\varphi_{1,h},\varphi_{2,h},\varphi_{3,h},\varphi_{4,h})$ is a lower solution of (2.1)–(2.5), it follows from Lemma 3.2 that $(U_i(t),V_i(t),W_i(t),Y_i(t)) \geq (\varphi_{1,i},\varphi_{2,i},\varphi_{3,i},\varphi_{4,i}) > 0$, $i=1,\ldots,I$, $t\in(0,T_h)$. □

The lemma below shows that the solution of the semidiscrete problem is increasing in space.

Lemma 3.4. *Let* $(U_h,V_h,W_h,Y_h) \in \left(C^1([0,T_h),\mathbb{R}^I)\right)^4$ *be the solution of* (2.1)–(2.5). *Then we have*

$$(U_{i+1}(t),V_{i+1}(t),W_{i+1}(t),Y_{i+1}(t)) > (U_i(t),V_i(t),W_i(t),Y_i(t)),$$

$i=1,\ldots,I-1$, $t\in(0,T_h)$.

Proof. We argue by contradiction. Assume that t_0 is the first $t>0$, such that $(R_i,F_i,S_i,K_i)(t) = (U_{i+1}-U_i, V_{i+1}-V_i, W_{i+1}-W_i, Y_{i+1}-Y_i)(t) > 0$, for $1\leq i\leq I-1$, but $R_{i_0}(t_0)=U_{i_0+1}(t_0)-U_{i_0}(t_0)=0$ or $F_{i_0}(t_0)=V_{i_0+1}(t_0)-V_{i_0}(t_0)=0$ or $S_{i_0}(t_0)=W_{i_0+1}(t_0)-W_{i_0}(t_0)=0$ or $K_{i_0}(t_0)=Y_{i_0+1}(t_0)-Y_{i_0}(t_0)=0$, for a certain $i_0\in\{1,\ldots,I-1\}$. Assume that $R_{i_0}(t_0)=U_{i_0+1}(t_0)-U_{i_0}(t_0)=0$. Without lost of generality, we can suppose that i_0 is the smallest integer which satisfies the above equality. We get

$$\text{(3.11)} \quad \begin{cases} R_1'(t) - \dfrac{R_2(t)-3R_1(t)}{h^2} - \dfrac{2}{h}U_1^{p_1}(t)V_1^{q_2}(t) = 0, \\ R_i'(t) - \dfrac{R_{i+1}(t)-2R_i(t)-R_{i-1}(t)}{h^2} = 0, \ 2\leq i\leq I-2, \\ R_{I-1}'(t) - \dfrac{R_{I-2}(t)-3R_{I-1}(t)}{h^2} - \dfrac{2}{h}U_I^{p_1}(t)V_I^{q_2}(t) = 0. \end{cases}$$

According to the hypotheses on t_0, we have the following inequalities:

$$R_{i_0}'(t_0) = \lim_{\epsilon\to 0}\frac{R_{i_0}(t_0)-R_{i_0}(t_0-\epsilon)}{\epsilon} \leq 0,$$

$$\delta^2 R_{i_0}(t_0) = \frac{R_{i_0+1}(t_0)-2R_{i_0}(t_0)+R_{i_0-1}(t_0)}{h^2} > 0, \text{ if } \quad 2\leq i_0\leq I-2,$$

$$\delta^2 R_{i_0}(t_0) = \frac{R_{i_0+1}(t_0)-3R_{i_0}(t_0)}{h^2} > 0, \text{ if } i_0=1,$$

$$\delta^2 R_{i_0}(t_0) = \frac{-3R_{i_0}(t_0) + R_{i_0-1}(t_0)}{h^2} > 0, \text{ if } i_0 = I-1,$$

which implies

$$R'_{i_0}(t_0) - \delta^2 R_{i_0}(t_0) < 0, \text{ if } \quad 2 \le i_0 \le I-2,$$

$$R'_{i_0}(t_0) - \delta^2 R_{i_0}(t_0) - \frac{2}{h}\left(U_{i_0}^{p_1}(t) V_{i_0}^{q_2}(t)\right) < 0, \quad \text{if} \quad i_0 = 1,$$

$$R'_{i_0}(t_0) - \delta^2 R_{i_0}(t_0) - \frac{2}{h}\left(U_{i_0}^{p_1}(t) V_{i_0}^{q_2}(t)\right) < 0, \quad \text{if} \quad i_0 = I-1.$$

Thus, we have a contradiction with (3.11), which leads to the desired result. □

The lemma below shows that the solution of the semidiscrete problem is increasing in time.

Lemma 3.5. *Let* $(U_h, V_h, W_h, Y_h) \in \left(C^1([0,T_h), \mathbb{R}^I)\right)^4$ *be the solution of* (2.1)–(2.5). *Then we have* $(U'_i(t), V'_i(t), W'_i(t), Y'_i(t)) > 0$*,* $i = 1, \dots, I$*,* $t \in (0, T_h)$*.*

Proof. Let us define

$$\begin{aligned}(F_h(t), G_h(t), L_h(t), H_h(t)) &= (U_h(t), V_h(t), W_h(t), Y_h(t)) \\ &- (U_h(t+\varepsilon), V_h(t+\varepsilon), W_h(t+\varepsilon), Y_h(t+\varepsilon)).\end{aligned}$$

Using straightforward computations and the Mean Value Theorem, we obtain the following for $i = 1, \dots, I$

$$F'_i(t) - \delta^2 F_i(t) - p_1 \omega_i V_i^{q_2}(t) (\xi_i(t))^{p_1 - 1} F_i(t) - q_2 \omega_i U_i^{p_1}(t+\varepsilon) (\rho_i(t))^{q_2 - 1} G_i(t) \ge 0,$$

$$G'_i(t) - \delta^2 G_i(t) - p_2 \omega_i W_i^{q_3}(t) (\rho_i(t))^{p_2 - 1} G_i(t) - q_3 \omega_i V_i^{p_2}(t+\varepsilon) (\varrho_i(t))^{q_3 - 1} L_i(t) \ge 0,$$

$$L'_i(t) - \delta^2 L_i(t) - p_3 \omega_i Y_i^{q_4}(t) (\varrho_i(t))^{p_3 - 1} L_i(t) - q_4 \omega_i W_i^{p_3}(t+\varepsilon) (\lambda_i(t))^{q_4 - 1} H_i(t) \ge 0,$$

$$H'_i(t) - \delta^2 H_i(t) - p_4 \omega_i U_i^{q_1}(t) (\lambda_i(t))^{p_4 - 1} H_i(t) - q_1 \omega_i Y_i^{p_4}(t+\varepsilon) (\xi_i(t))^{q_1 - 1} F_i(t) \ge 0,$$

$$F_i(0) \ge 0, \ G_i(0) \ge 0, \ L_i(0) \ge 0, \ H_i(t) \ge 0,$$

where $\xi_i(t)$, $\rho_i(t)$, $\varrho_i(t)$ and $\lambda_i(t)$ lie, respectively, between $U_i(t)$ and $U_i(t+\varepsilon)$, between $V_i(t)$ and $V_i(t+\varepsilon)$, between $W_i(t)$ and $W_i(t+\varepsilon)$ and between $Y_i(t)$ and $Y_i(t+\varepsilon)$, for $i \in \{1, \dots, I\}$.

From Lemma 3.1 we get,

$$F_i(t) \geq 0, \quad G_i(t) \geq 0, \quad L_i(t) \geq 0, \quad H_i(t) \geq 0, \quad i = 1, \dots, I, \quad \forall t \in (0, T_h).$$

This concludes the proof. □

4. Semidiscrete blow-up solution

In this section, under certain assumptions, we provide the conditions for the global existence of the solution of the semidiscrete problem, and we also show that this solution (U_h, V_h, W_h, Y_h) of (2.1)–(2.5) blows up in finite time. We characterize the blow-up or global existence of the solution (U_h, V_h, W_h, Y_h) of (2.1)–(2.5) in terms of the matrix B where

$$B = \begin{pmatrix} p_1 & q_2 & 0 & 0 \\ 0 & p_2 & q_3 & 0 \\ 0 & 0 & p_3 & q_4 \\ q_1 & 0 & 0 & p_4 \end{pmatrix}$$

For convenience, we define $p_{4+a} = p_a$ and $q_{4+a} = q_a$, for all integers a. Let $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be the solution of

$$\begin{pmatrix} p_1 - 1 & q_2 & 0 & 0 \\ 0 & p_2 - 1 & q_3 & 0 \\ 0 & 0 & p_3 - 1 & q_4 \\ q_1 & 0 & 0 & p_4 - 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \tag{4.1}$$

where for $j = 1, 2, 3, 4$,

$$\alpha_j = \frac{\prod_{k=j+1}^{j+3}(p_k - 1) + (p_{j+3} - 1)\prod_{k=j+1}^{j+2} q_k - q_{j+1}\prod_{k=j+2}^{j+3}(p_k - 1) - \prod_{k=j+1}^{j+3} q_k}{\prod_{k=1}^{4} q_k - \prod_{k=1}^{4}(1 - p_k)}.$$

If $0 \leq p_j \leq 1$ and $q_j \geq 0$, it is easy to check that the numerator of α_j is negative for $j = 1, 2, 3, 4$.

Definition 4.1. *We say that the solution (U_h, V_h, W_h, Y_h) of (2.1)–(2.5) blows up in finite time if there exists a finite time $T_h > 0$ such that, for $t \in [0, T_h)$,*

$$\max\{\|U_h(t)\|_\infty, \|V_h(t)\|_\infty, \|W_h(t)\|_\infty, \|Y_h(t)\|_\infty\} < \infty$$

and

$$\lim_{t\to T_h} \sup \{\|U_h(t)\|_\infty + \|V_h(t)\|_\infty + \|W_h(t)\|_\infty + \|Y_h(t)\|_\infty\} = +\infty.$$

The time T_h is called the blow-up time of the solution (U_h, V_h, W_h, Y_h).

Theorem 4.1. *If $p_1, p_2, p_3, p_4 > 1$, then the solution (U_h, V_h, W_h, Y_h) of* (2.1)–(2.5) *blows up in finite time T_h.*

Proof. Assume that $p_1 > 1$ and let us consider

$$U_I'(t) \quad = \quad \frac{2U_{I-1}(t) - 2U_I(t)}{h^2} + \frac{2}{h}U_I^{p_1}(t)V_I^{q_2}(t), \qquad t \in [0, T_h),$$

As $V_I(t) \geq \min_{1\leq i\leq I} \varphi_{2,i} = a > 0$, for $t \in [0, T_h)$, then

$$U_I'(t) \geq \frac{2U_{I-1}(t) - 2U_I(t)}{h^2} + \frac{2a^{q_2}}{h}U_I^{p_1}(t), \qquad t \in [0, T_h),$$

as $U_{I-1}(t) < U_I(t)$ (Lemma 3.4) and $U_I'(t) > 0$ (Lemma 3.5) for all $t \in (0, T_h)$, there exists a constant $C > 0$ such that

$$U_I'(t) \geq \frac{2a^{q_2}}{hC}U_I^{p_1}(t), \qquad t \in [0, T_h).$$

By setting $\beta = \dfrac{hC}{2a^{q_{j+1}}}$, we obtain

$$\forall t \in [0, T_h), \quad U_I'(t) \geq \frac{1}{\beta}U_I^{p_1}(t). \tag{4.2}$$

Thus, U_h blows up in finite time T_h and moreover, integrating (4.2) from 0 to T_h we can show that

$$T_h \leq \frac{\beta\|\varphi_{1,h}\|_\infty^{1-p_1}}{p_1 - 1}.$$

Similarly, we can show:

- if $p_2 > 1$, V_h blows up in finite time T_h with

$$T_h \leq \frac{\beta\|\varphi_{2,h}\|_\infty^{1-p_2}}{p_2 - 1},$$

- if $p_3 > 1$, W_h blows up in finite time T_h with

$$T_h \leq \frac{\beta\|\varphi_{3,h}\|_\infty^{1-p_3}}{p_3 - 1},$$

- if $p_4 > 1$, Y_h blows up in finite time T_h with

$$T_h \leq \frac{\beta \|\varphi_{4,h}\|_\infty^{1-p_4}}{p_4 - 1}.$$

Thus, if $p_1, p_2, p_3, p_4 > 1$, the solution (U_h, V_h, W_h, Y_h) of (2.1)–(2.5) blows up in finite time T_h with

$$T_h \leq \beta \min\left\{ \frac{\|\varphi_{1,h}\|_\infty^{1-p_1}}{p_1 - 1}, \frac{\|\varphi_{2,h}\|_\infty^{1-p_2}}{p_2 - 1}, \frac{\|\varphi_{3,h}\|_\infty^{1-p_3}}{p_3 - 1}, \frac{\|\varphi_{4,h}\|_\infty^{1-p_4}}{p_4 - 1} \right\}.$$

□

Theorem 4.2. *Assume that* $0 \leq p_j \leq 1$, *for* $j = 1, 2, 3, 4$. *Let* $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ *be the solution of* (4.1), *then*

(b) *if* $\min_{1\leq j\leq 4} \alpha_j > 0$, *the solution* (U_h, V_h, W_h, Y_h) *of* (2.1)–(2.5) *is global.*

(c) *if* $\min_{1\leq j\leq 4} \alpha_j \leq 0$, *the solution* (U_h, V_h, W_h, Y_h) *of* (2.1)–(2.5) *blows up in a finite time* T_h.

The following lemma describes the behavior of the positive solutions of

$$\begin{cases} Z_j'(s) = Z_j^{p_j}(s) Z_{j+1}^{q_{j+1}}(s), \\ Z_j(0) = \varphi_{j,i} > 0, \quad i = 1, \ldots, I, \quad j = 1, 2, 3, 4, \\ Z_5 = Z_1, \qquad q_5 = q_1. \end{cases} \tag{4.3}$$

where $0 \leq p_j \leq 1$, $q_j \geq 0$.

Lemma 4.1. *Let* $\{Z_j(s)\}$ *be a positive solution of* (4.3) *with* B *nonsingular and* $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ *the solution of* (4.1).

(i) *If* $\min_{1\leq j\leq 4} \alpha_j > 0$, *then* (4.3) *has a global upper solution of the form* $Z_j(s) = M_j(s + s_0)^{\alpha_j}$, *where* $M_j > 0$ *is a constant.*

(ii) *If* $\min_{1\leq j\leq 4} \alpha_j \leq 0$, *then all positive solution of* (4.3) *blow up.*

Proof. See [9], Theorem 2.1. □

Proof. (Theorem 4.2.) Assume that $0 \leq p_j \leq 1$, for $j = 1, 2, 3, 4$. To prove Theorem 4.2, we construct a lower or upper solution of (2.1)–(2.5). The solution either blows up in finite time or exists globally. This is followed by applying the comparison result (Lemma 3.2).

Let $b : [0, T_h) \to \mathbb{R}$ be a continuous, bounded, strictly increasing, and strictly positive function. We denote $s = b(t)$ for a given value of t and $s_0 = b(0)$ to represent the initial value of b at $t = 0$. This function plays a key role in the following development where it is defined depending on the context.

Case (b): We now demonstrate the global existence of the solution of (2.1)–(2.5) by successively constructing upper solution. Let $(\overline{U_h}, \overline{V_h}, \overline{W_h}, \overline{Y_h})$ be an upper solution of (2.1)–(2.5).

To construct an upper solution, we propose the following forms: $\overline{U_i}(t) = \phi_1(b(t))$, $\overline{V_i}(t) = \phi_2(b(t))$, $\overline{W_i}(t) = \phi_3(b(t))$ and $\overline{Y_i}(t) = \phi_4(b(t))$, for $i = 1, \ldots, I,\ t \in (0, T_h)$, then

$$\begin{aligned} \overline{U_i'}(t) &= b'(t)\phi_1'(b(t)), \quad \overline{V_i'}(t) = b'(t)\phi_2'(b(t)) \quad \overline{W_i'}(t) = b'(t)\phi_3'(b(t)), \\ \overline{Y_i'}(t) &= b'(t)\phi_4'(b(t)). \end{aligned}$$

Let $\phi_j(s)$ $(j = 1, 2, 3, 4)$ a solution of the O.D.E. system

$$\begin{cases} Z_j'(s) = (Z_j(s))^{p_j} (Z_{j+1}(s))^{q_{j+1}} \\ Z_j(0) = \varphi_{j,i} > 0, \quad i = 1, \ldots, I, \quad j = 1, 2, 3, 4, \end{cases} \tag{4.4}$$

where $Z_5 = Z_1$ and $q_5 = q_1$.
Using the results established in Lemma 4.1, it follows that $\phi_j(s) = M_j(s + s_0)^{\alpha_j}$ $(j = 1, 2, 3, 4)$ is a global upper solution of (4.4), because $\min_{1 \le j \le 4} \alpha_j > 0$. Thus

$$\begin{aligned} \phi_1'(b(t)) &\ge (\phi_1(b(t)))^{p_1} (\phi_2(b(t)))^{q_2}, \ t \in (0, T_h), \\ \phi_2'(b(t)) &\ge (\phi_2(b(t)))^{p_2} (\phi_3(b(t)))^{q_3}, \ t \in (0, T_h), \\ \phi_3'(b(t)) &\ge (\phi_3(b(t)))^{p_3} (\phi_4(b(t)))^{q_4}, \ t \in (0, T_h), \\ \phi_4'(b(t)) &\ge (\phi_4(b(t)))^{p_4} (\phi_1(b(t)))^{q_1}, \ t \in (0, T_h), \\ \phi_1(0) &\ge \varphi_{1,i}, \quad \phi_2(0) \ge \varphi_{2,i}, \quad \phi_3(0) \ge \varphi_{3,i}, \quad \phi_4(0) \ge \varphi_{4,i}, \quad i = 1, \ldots, I. \end{aligned}$$

Let $b(t)$ be a function such that $b'(t) \ge \omega_i$, for $i = 1, \ldots, I$, then

$$b'(t)\phi_1'(b(t)) \ge b'(t)(\phi_1(b(t)))^{p_1} (\phi_2(b(t)))^{q_2}, \ t \in (0, T_h), \tag{4.5}$$

$$b'(t)\phi_2'(b(t)) \ge b'(t)(\phi_2(b(t)))^{p_2} (\phi_3(b(t)))^{q_3}, \ t \in (0, T_h), \tag{4.6}$$

$$b'(t)\phi_3'(b(t)) \ge b'(t)(\phi_3(b(t)))^{p_3} (\phi_4(b(t)))^{q_4}, \ t \in (0, T_h), \tag{4.7}$$

$$b'(t)\phi_4'(b(t)) \ge b'(t)(\phi_4(b(t)))^{p_4} (\phi_1(b(t)))^{q_1}, \ t \in (0, T_h), \tag{4.8}$$

$$\phi_1(0) \ge \varphi_{1,i}, \quad \phi_2(0) \ge \varphi_{2,i}, \quad \phi_3(0) \ge \varphi_{3,i}, \quad \phi_4(0) \ge \varphi_{4,i}, \tag{4.9}$$

we can easily observe that

$$\phi_j\left(b(0)\right) \geq \phi_j\left(0\right), \quad j=1,2,3,4, \tag{4.10}$$

using (4.5)–(4.10), we deduce that, for $i=1,\dots,I$,

$$\begin{aligned}
\overline{U_i'}(t) &\geq b'(t)\overline{U_i}^{p_1}(t)\overline{V_i}^{q_2}(t) \geq \omega_i\overline{U_i}^{p_1}(t)\overline{V_i}^{q_2}(t),\ t\in(0,T_h),\\
\overline{V_i'}(t) &\geq b'(t)\overline{V_i}^{p_2}(t)\overline{W_i}^{q_3}(t) \geq \omega_i\overline{V_i}^{p_2}(t)\overline{W_i}^{q_3}(t),\ t\in(0,T_h),\\
\overline{W_i'}(t) &\geq b'(t)\overline{W_i}^{p_3}(t)\overline{Y_i}^{q_4}(t) \geq \omega_i\overline{W_i}^{p_3}(t)\overline{Y_i}^{q_4}(t),\ t\in(0,T_h),\\
\overline{Y_i'}(t) &\geq b'(t)\overline{Y_i}^{p_4}(t)\overline{U_i}^{q_1}(t) \geq \omega_i\overline{Y_i}^{p_4}(t)\overline{U_i}^{q_1}(t),\ t\in(0,T_h),\\
\overline{U_i}(0) &\geq \varphi_{1,i}, \quad \overline{V_i}(0)\geq\varphi_{2,i}, \quad \overline{W_i}(0)\geq\varphi_{3,i}, \quad \overline{Y_i}(0)\geq\varphi_{4,i}.
\end{aligned}$$

So, for $i=1,\dots,I$

$$\begin{aligned}
\overline{U_i'}(t) &\geq \delta^2\overline{U_i}(t)+\omega_i\overline{U_i}^{p_1}(t)\overline{V_i}^{q_2}(t),\ t\in(0,T_h),\\
\overline{V_i'}(t) &\geq \delta^2\overline{V_i}(t)+\omega_i\overline{V_i}^{p_2}(t)\overline{W_i}^{q_3}(t),\ t\in(0,T_h),\\
\overline{W_i'}(t) &\geq \delta^2\overline{W_i}(t)+\omega_i\overline{W_i}^{p_3}(t)\overline{Y_i}^{q_4}(t),\ t\in(0,T_h),\\
\overline{Y_i'}(t) &\geq \delta^2\overline{Y_i}(t)+\omega_i\overline{Y_i}^{p_4}(t)\overline{U_i}^{q_1}(t),\ t\in(0,T_h),\\
\overline{U_i}(0) &\geq U_i(0),\ \overline{V_i}(0)\geq V_i(0),\ \overline{W_i}(0)\geq W_i(0),\ \overline{Y_i}(0)\geq Y_i(0).
\end{aligned}$$

From lemma 3.2, we conclude that $(\overline{U_h},\overline{V_h},\overline{W_h},\overline{Y_h})$ is an upper solution of (2.1)–(2.5), hence the global existence of the solution of (2.1)–(2.5).

Case (c): Let $(\underline{U_h},\underline{V_h},\underline{W_h},\underline{Y_h})$ be a lower solution of (2.1)–(2.5). We propose $\underline{U_i}(t)=\phi_1\left(b(t)\right)$, $\underline{V_i}(t)=\phi_2\left(b(t)\right)$, $\underline{W_i}(t)=\phi_3\left(b(t)\right)$ and $\underline{Y_i}(t)=\phi_4\left(b(t)\right)$, for $i=1,\dots,I,\ t\in(0,T_h)$, then,

$$\begin{aligned}
\underline{U_i'}(t) &= b'(t)\phi_1'\left(b(t)\right), \quad \underline{V_i'}(t)=b'(t)\phi_2'\left(b(t)\right), \quad \underline{W_i'}(t)=b'(t)\phi_3'\left(b(t)\right),\\
\underline{Y_i'}(t) &= b'(t)\phi_4'\left(b(t)\right).
\end{aligned}$$

Denote $s=b(t)=\varepsilon t$ where $0<\varepsilon\leq\dfrac{2}{h}=\omega_i$, for $i=1,\dots,I$. Thus $b'(t)=\varepsilon\leq\omega_i$, for $i=1,\dots,I$.

Let $\phi_j(s)$ $(j=1,2,3,4)$ be a solution of the O.D.E. system (4.4), hence

$$\begin{aligned}
\phi_1'\left(b(t)\right) &= \left(\phi_1\left(b(t)\right)\right)^{p_1}\left(\phi_2\left(b(t)\right)\right)^{q_2},\ t\in(0,T_h),\\
\phi_2'\left(b(t)\right) &= \left(\phi_2\left(b(t)\right)\right)^{p_2}\left(\phi_3\left(b(t)\right)\right)^{q_3},\ t\in(0,T_h),
\end{aligned}$$

$$
\begin{aligned}
\phi_3'\left(b(t)\right) &= \left(\phi_3\left(b(t)\right)\right)^{p_3}\left(\phi_4\left(b(t)\right)\right)^{q_4},\ t\in(0,T_h),\\
\phi_4'\left(b(t)\right) &= \left(\phi_4\left(b(t)\right)\right)^{p_4}\left(\phi_1\left(b(t)\right)\right)^{q_1},\ t\in(0,T_h),\\
\phi_1\left(0\right) &= \varphi_{1,i},\quad \phi_2\left(0\right)=\varphi_{2,i},\quad \phi_3\left(0\right)=\varphi_{3,i},\quad \phi_4\left(0\right)=\varphi_{4,i},\quad i=1,\ldots,I,
\end{aligned}
$$

which implies that,

$$
\begin{aligned}
b'(t)\phi_1'\left(b(t)\right) &= b'(t)\left(\phi_1\left(b(t)\right)\right)^{p_1}\left(\phi_2\left(b(t)\right)\right)^{q_2},\ t\in(0,T_h),\\
b'(t)\phi_2'\left(b(t)\right) &= b'(t)\left(\phi_2\left(b(t)\right)\right)^{p_2}\left(\phi_3\left(b(t)\right)\right)^{q_3},\ t\in(0,T_h),\\
b'(t)\phi_3'\left(b(t)\right) &= b'(t)\left(\phi_3\left(b(t)\right)\right)^{p_3}\left(\phi_4\left(b(t)\right)\right)^{q_4},\ t\in(0,T_h),\\
b'(t)\phi_4'\left(b(t)\right) &= b'(t)\left(\phi_4\left(b(t)\right)\right)^{p_4}\left(\phi_1\left(b(t)\right)\right)^{q_1},\ t\in(0,T_h),\\
\phi_1\left(0\right) &= \varphi_{1,i},\quad \phi_2\left(0\right)=\varphi_{2,i},\quad \phi_3\left(0\right)=\varphi_{3,i},\quad \phi_4\left(0\right)=\varphi_{4,i},\quad i=1,\ldots,I,
\end{aligned}
$$

which implies that, for $i=1,\ldots,I$,

$$
\begin{aligned}
\underline{U_i}'(t) &= b'(t)\underline{U_i}^{p_1}(t)\underline{V_i}^{q_2}(t)\le\omega_i\underline{U_i}^{p_1}(t)\underline{V_i}^{q_2}(t),\ t\in(0,T_h),\\
\underline{V_i}'(t) &= b'(t)\underline{V_i}^{p_2}(t)\underline{W_i}^{q_3}(t)\le\omega_i\underline{V_i}^{p_2}(t)\underline{W_i}^{q_3}(t),\ t\in(0,T_h),\\
\underline{W_i}'(t) &= b'(t)\underline{W_i}^{p_3}(t)\underline{Y_i}^{q_4}(t)\le\omega_i\overline{W_i}^{p_3}(t)\overline{Y_i}^{q_4}(t),\ t\in(0,T_h),\\
\underline{Y_i}'(t) &= b'(t)\underline{Y_i}^{p_4}(t)\underline{U_i}^{q_1}(t)\le\omega_i\overline{Y_i}^{p_4}(t)\overline{U_i}^{q_1}(t),\ t\in(0,T_h),\\
\underline{U_i}(0) &= \varphi_{1,i},\quad \underline{V_i}(0)=\varphi_{2,i},\quad \underline{W_i}(0)=\varphi_{3,i},\quad \underline{Y_i}(0)=\varphi_{4,i}.
\end{aligned}
$$

So, for $i=1,\ldots,I$,

$$
\begin{aligned}
\underline{U_i}'(t) &\le \delta^2\underline{U_i}(t)+\omega_i\underline{U_i}^{p_1}(t)\underline{V_i}^{q_2}(t),\ t\in(0,T_h),\\
\underline{V_i}'(t) &\le \delta^2\underline{V_i}(t)+\omega_i\underline{V_i}^{p_2}(t)\underline{W_i}^{q_3}(t),\ t\in(0,T_h),\\
\underline{W_i}'(t) &\le \delta^2\underline{W_i}(t)+\omega_i\underline{W_i}^{p_3}(t)\underline{Y_i}^{q_4}(t),\ t\in(0,T_h),\\
\underline{Y_i}'(t) &\le \delta^2\underline{Y_i}(t)+\omega_i\underline{Y_i}^{p_4}(t)\underline{U_i}^{q_1}(t),\ t\in(0,T_h),\\
\underline{U_i}(0) &\le U_i(0),\ \underline{V_i}(0)\le V_i(0),\ \underline{W_i}(0)\le W_i(0),\ \underline{Y_i}(0)\le Y_i(0).
\end{aligned}
$$

From Lemma 3.2, we conclude that $(\underline{U_h},\underline{V_h},\underline{W_h},\underline{Y_h})$ is a lower solution of (2.1)–(2.5). As $\min_{1\le j\le 4}\alpha_j\le 0$, by the Lemma 4.1, $\phi_j(s)$ $(j=1,2,3,4)$ blows up in finite time, thus $(\underline{U_h},\underline{V_h},\underline{W_h},\underline{Y_h})$ blows up and hence (U_h,V_h,W_h,Y_h) cannot be global. This ends the proof. □

Remark 4.1. *By multiplying the inequality* (4.2) *by the inverse of* $U_I^{p_1}(t)$ *and by integrating the result obtained on* $[t, T_h]$*, we obtain,* $\forall t \in (0, T_h)$*,*

$$\frac{1}{p_1 - 1} \frac{1}{U_I^{p_1-1}(t)} \geq \gamma \left(T_h - t\right),$$

and there exists a constant C_{p_1} *such that*

$$U_I(t) \leq C_{p_1} \left(T_h - t\right)^{-\frac{1}{p_1-1}}, \quad \forall t \in (0, T_h),$$

for $p_1 > 1$*. In the same way we show, if* $p_2 > 1$*, there exists a constant* C_{p_2} *such that*

$$V_I(t) \leq C_{p_2} \left(T_h - t\right)^{-\frac{1}{p_2-1}}, \quad \forall t \in (0, T_h),$$

if $p_3 > 1$*, there exists a constant* C_{p_3} *such that*

$$W_I(t) \leq C_{p_3} \left(T_h - t\right)^{-\frac{1}{p_3-1}}, \quad \forall t \in (0, T_h),$$

and if $p_4 > 1$*, there exists a constant* C_{p_4} *such that*

$$Y_I(t) \leq C_{p_4} \left(T_h - t\right)^{-\dfrac{1}{p_4-1}}, \quad \forall t \in (0, T_h).$$

Theorem 4.1 and Theorem 4.2 imply the following corollaries:

Corollary 4.1. *The solutions of* (2.1)–(2.5) *blow up in finite time* T_h *if*

$$\max \left\{ p_j - 1 \ (j = 1, 2, 3, 4), \ \prod_{j=1}^{4} q_j - \prod_{j=1}^{4} (1 - p_j) \right\} > 0.$$

Corollary 4.2. *If*

$$\max \left\{ p_j - 1 \ (j = 1, 2, 3, 4), \ \prod_{j=1}^{4} q_j - \prod_{j=1}^{4} (1 - p_j) \right\} \leq 0,$$

then, the solutions of (2.1)–(2.5) *exist globally.*

5. Simultaneous blow-up

In this section, we consider (U_h, V_h, W_h, Y_h) as a positive solution of (2.1)–(2.5) with h fixed, and we provide sufficient conditions for the existence of simultaneous blow-up for all initial data.

Definition 5.1. *We say that the solution* (U_h, V_h, W_h, Y_h) *of* (2.1)–(2.5) *blows up simultaneously in a finite time if there exists a finite time* $T_h > 0$ *such that for* $t \in [0, T_h)$*,* $\max\{\|U_h(t)\|_\infty, \|V_h(t)\|_\infty, \|W_h(t)\|_\infty, \|Y_h(t)\|_\infty\} < \infty$ *and*

$$\lim_{t\to T_h} \sup \min\{\|U_h(t)\|_\infty, \|V_h(t)\|_\infty, \|W_h(t)\|_\infty, \|Y_h(t)\|_\infty\} = +\infty.$$

The time T_h *is called the simultaneous blow-up time of the solution* (U_h, V_h, W_h, Y_h)*.*

Theorem 5.1. *Define* $\beta_4 = \dfrac{1}{p_4 - 1}$*. If* $\beta_k = \dfrac{1 - q_{k+1}\beta_{k+1}}{p_k - 1}$, $p_k \leq 1 < p_4$ $(k = 1, 2, 3)$ *and* $\beta_1 \geq 0, \quad \beta_k > 0$, $(k = 2, 3)$, *then* U_h, V_h, W_h, Y_h *blow up simultaneously in a finite time* T_h*.*

Proof. This proof is divided into three steps.

Step 1. Y_h blows up in a finite time T_h and $Y_I(t) \geq c_{p_4}(T_h - t)^{-\beta_4}$. We claim that Y_h is the blow-up component. If not, the other components would remain bounded for $p_k \leq 1 \quad (k = 1, 2, 3)$, which leads to a contradiction, since $p_4 > 1$. Consequently Y_h blows up in T_h. From (2.4), we have

$$Y_I'(t) = \frac{2Y_{I-1}(t) - 2Y_I(t)}{h^2} + \frac{2}{h} Y_I^{p_4}(t) U_I^{q_1}(t), \quad t \in (0, T_h).$$

Since $Y_{I-1}(t) < Y_I(t), \quad \forall t \in (0, T_h)$ (Lemma 3.4), we have

$$Y_I'(t) \leq \frac{2}{h} Y_I^{p_4}(t) U_I^{q_1}(t), \quad \forall t \in (0, T_h),$$

since $p_1 \leq 1$, there exists a constant $C > 0$ such that $U_I(t) \leq C$, $\forall t \in (0, T_h)$. Therefore $Y_I(t)$ satisfies

$$Y_I'(t) \leq \frac{2}{h} C^{q_1} Y_I^{p_4}(t), \quad \forall t \in (0, T_h),$$

which implies that

$$\frac{Y_I'(t)}{Y_I^{p_4}(t)} \leq \frac{2}{h} C^{q_1}, \quad \forall t \in (0, T_h),$$

integrating this inequality from t to T_h, we obtain

$$Y_I(t) \geq c_{p_4}(T_h - t)^{-\beta_4}, \quad \forall t \in [0, T_h), \tag{5.1}$$

where $c_{p_4} = \left(\frac{2}{h} C^{q_1}(p_4 - 1)\right)^{-\frac{1}{p_4 - 1}}$.

Step 2. W_h blows up in a finite time T_h and $W_I(t) \geq c_{p_3}\left(T_h - t\right)^{-\beta_3}$. Assume that W_h remains bounded up to time T_h. From (2.3) and (5.1), we have

$$W_I'(t) \geq \frac{2W_{I-1}(t) - 2W_I(t)}{h^2} + \frac{2}{h} W_I^{p_3}(t) c_{p_4}^{q_4} \left(T_h - t\right)^{-\beta_4 q_4}, \quad t \in (0, T_h).$$

As $W_{I-1}(t) < W_I(t), \quad \forall t \in (0, T_h)$ (Lemma 3.4) and W_I is bounded, then, there exists a constant $B > 0$ and a constant C dependent of h such that

$$W_I'(t) \;\; \geq \;\; \frac{2B}{h} C^{p_3} c_{p_4}^{q_4} \left(T_h - t\right)^{-\beta_4 q_4}, \quad \forall t \in (t_0, T_h),$$

thus

$$W_I'(t) \;\; \geq \;\; C_1 \left(T_h - t\right)^{-\beta_4 q_4}, \quad \forall t \in (t_0, T_h),$$

with $C_1 = \frac{2B}{h} C^{p_3} c_{p_4}^{q_4}$. Integrating this inequality from t_0 to T_h, we have

$$W_I(T_h) \;\; \geq \;\; W_I(t_0) + C_1 \int_{t_0}^{T_h} \left(T_h - t\right)^{-\beta_4 q_4} dt.$$

For W_h to remain bounded, we must have $q_4 \beta_4 < 1$, which implies $\beta_3 < 0$. This contradicts the assumption $\beta_3 > 0$.

From (2.3) and (5.1), we have

$$W_I'(t) \geq \frac{2W_{I-1}(t) - 2W_I(t)}{h^2} + \frac{2}{h} W_I^{p_3}(t) c_{p_4}^{q_4} \left(T_h - t\right)^{-\beta_4 q_4}, \quad t \in (0, T_h).$$

As $W_{I-1}(t) < W_I(t), \quad \forall t \in (0, T_h)$ (Lemma 3.3), then, there exists a constant $C > 0$ such that

$$W_I'(t) \;\; \geq \;\; \frac{2C}{h} W_I^{p_3}(t) c_{p_4}^{q_4} \left(T_h - t\right)^{-\beta_4 q_4}, \quad \forall t \in (t_0, T_h).$$

Integrating the above inequality from z to t, we obtain

$$W_I(t) \;\; \geq \;\; \frac{2C}{h} c_{p_4}^{q_4} \int_z^t W_I^{p_3}(\tau) \left(T_h - \tau\right)^{-\beta_4 q_4} d\tau, \quad \forall t \in (z, T_h).$$

Define $H(t) = \int_z^t W_I^{p_3}(\tau) \left(T_h - \tau\right)^{-\beta_4 q_4} d\tau$, then $W_I^{-p_3}(t) H'(t) = \left(T_h - t\right)^{-\beta_4 q_4}$.

Since

$$W_I(t) \;\; \geq \;\; \frac{2C}{h} c_{p_4}^{q_4} H(t), \quad \forall t \in (z, T_h), \tag{5.2}$$

then

$$H^{-p_4}(t) H'(t) \;\; \geq \;\; Q \left(T_h - t\right)^{-\beta_4 q_4}, \quad \forall t \in (z, T_h),$$

where $Q = \left(\frac{2C}{h} c_{p_4}^{q_4}\right)^{p_3}$. Integrating the above inequality from z to t and taking $z = 2t - T_h$, we have

$$H^{1-p_3}(t) \geq (1-p_3)\, K\, (T_h - t)^{1-q_4\beta_4}, \quad \forall t \in (0, T_h), \tag{5.3}$$

where $K = \frac{2^{1-q_4\beta_4} Q}{(1-q_4\beta_4)} - \frac{Q}{(1-q_4\beta_4)}$. From (5.2), we deduce that

$$W_I^{1-p_3}(t) \geq \left(\frac{2C}{h} c_{p_4}^{q_4}\right)^{1-p_3} H^{1-p_3}(t), \quad \forall t \in (0, T_h). \tag{5.4}$$

Using (5.3) et (5.4), we obtain

$$W_I^{1-p_3}(t) \geq K \left(\frac{2C}{h} c_{p_4}^{q_4}\right)^{1-p_3} (1-p_3)\, (T_h - t)^{1-q_4\beta_4}, \quad \forall t \in (0, T_h),$$

and hence

$$W_I(t) \geq c_{p_3} (T_h - t)^{-\beta_3}, \quad \forall t \in (0, T_h),$$

where $c_{p_3} = \left(K \left(\frac{2C}{h} c_{p_4}^{q_4}\right)^{1-p_3} (1-p_3)\right)^{\frac{1}{1-p_3}}$.

Step 3. Similar to Step 2, we obtain that V_h blows up in a finite time T_h and $V_I(t) \geq c_{p_2} (T_h - t)^{-\beta_2}$. For $\beta_1 \geq 0$, $U_{1,h}$ also blows up at time T_h, similarly. That means U_h, V_h, W_h, Y_h blow up simultaneously in a finite time T_h and the proof is completed. □

Theorem 5.2. *If* $p_1,\ p_2,\ p_3,\ p_4 \leq 1$ *and* $\prod_{n=1}^{4} q_n > \prod_{n=1}^{4} (1-p_n)$*, then* U_h*,* V_h*,* W_h*, and* Y_h *blow up simultaneously in a finite time* T_h*.*

Proof. Assume that Y_h remains bounded up to the blow-up time T_h. Then, U_h, V_h, W_h would also remain bounded for $p_1,\ p_2,\ p_3 \leq 1$. However since $\prod_{n=1}^{4} q_n > \prod_{n=1}^{4} (1-p_n)$, this contradicts corollary 4.1. Therefore U_h, V_h, W_h, Y_h must blow up simultaneously, and the proof is completed. □

6. Convergence of semidiscrete blow-up time

In this section, we study the convergence of the semidiscrete blow-up time. We now show that, for each time interval $[0, T^*]$ where (u, v, w, y) is defined, the solution (U_h, V_h, W_h, Y_h) of (2.1)–(2.5) converges to (u, v, w, y) as the mesh size h

tends to zero. We denote

$$u_h(t) = (u(x_1,t),\dots,u(x_I,t))^T, \quad v_h(t) = (v(x_1,t),\dots,v(x_I,t))^T,$$
$$w_h(t) = (w(x_1,t),\dots,w(x_I,t))^T, \quad y_h(t) = (y(x_1,t),\dots,y(x_I,t))^T.$$

Theorem 6.1. *Assume that the problem* (1.1) *has solution* $(u,v,w,y) \in (C^{4,1}([0,1]\times[0,T^*]))^4$ *and the initial data* $(\varphi_{1,h},\varphi_{2,h},\varphi_{3,h},\varphi_{4,h})$ *of* (2.1)–(2.5) *satisfies*

(6.1) $$\|\varphi_{1,h} - u_h(0)\|_\infty = o(1), \quad \|\varphi_{2,h} - v_h(0)\|_\infty = o(1),$$

$$\|\varphi_{3,h} - w_h(0)\|_\infty = o(1), \quad \|\varphi_{4,h} - y_h(0)\|_\infty = o(1), \quad h \to 0.$$

Then, for h sufficiently small, the problem (2.1)–(2.5) *has a unique solution* $(U_h, V_h, W_h, Y_h) \in (C^1([0,T^*],\mathbb{R}^I))^4$ *such that,*

$$\max_{t\in[0,T^*]} \|U_h(t) - u_h(t)\|_\infty = O\Big(\|\varphi_{1,h} - u_h(0)\|_\infty + \|\varphi_{2,h} - v_h(0)\|_\infty + \|\varphi_{3,h} - w_h(0)\|_\infty + \|\varphi_{4,h} - y_h(0)\|_\infty + h^2\Big),$$

$$\max_{t\in[0,T^*]} \|V_h(t) - v_h(t)\|_\infty = O\Big(\|\varphi_{1,h} - u_h(0)\|_\infty + \|\varphi_{2,h} - v_h(0)\|_\infty + \|\varphi_{3,h} - w_h(0)\|_\infty + \|\varphi_{4,h} - y_h(0)\|_\infty + h^2\Big),$$

$$\max_{t\in[0,T^*]} \|W_h(t) - w_h(t)\|_\infty = O\Big(\|\varphi_{1,h} - u_h(0)\|_\infty + \|\varphi_{2,h} - v_h(0)\|_\infty + \|\varphi_{3,h} - w_h(0)\|_\infty + \|\varphi_{4,h} - y_h(0)\|_\infty + h^2\Big),$$

$$\max_{t\in[0,T^*]} \|Y_h(t) - y_h(t)\|_\infty = O\Big(\|\varphi_{1,h} - u_h(0)\|_\infty + \|\varphi_{2,h} - v_h(0)\|_\infty + \|\varphi_{3,h} - w_h(0)\|_\infty + \|\varphi_{4,h} - y_h(0)\|_\infty + h^2\Big).$$

Proof. Let $\nu > 0$ be such that

(6.2) $$(\|u\|_\infty, \|v\|_\infty, \|w\|_\infty, \|y\|_\infty) < \nu, \ t \in [0,T^*].$$

Let $t(h) \le T^*$ be the greatest value of $t > 0$ such that for $t \in (0, t(h))$

(6.3) $$\max\Big\{ \|U_h(t) - u_h(t)\|_\infty, \ \|V_h(t) - vh(t)\|_\infty, \ \|W_h(t) - w_h(t)\|_\infty, \ \|Y_h(t) - yh(t)\|_\infty \Big\} < 1.$$

The relation (6.1) implies $t(h) > 0,$ for h small enough. Using the triangle inequality, we obtain

(6.4)
$$\|U_h(t)\|_\infty \le 1+\nu, \quad \|V_h(t)\|_\infty \le 1+\nu,$$

$$\|W_h(t)\|_\infty \le 1+\nu \text{ and } \|Y_h(t)\|_\infty \le 1+\nu, \text{ for } t \in (0, t(h)).$$

Let $(e_{1,i}, e_{2,i}, e_{3,i}, e_{4,i})(t) = (U_i - u_i, V_i - v_i, W_i - w_i, Y_i - y_i)(t)$, for $i = 1, \dots, I$, $\forall t \in [0, T^*]$ be the discretization error. These error functions verify

$$\begin{aligned}
e'_{1,i}(t) &= \delta^2 e_{1,i}(t) + p_1\omega_i(\alpha_i(t))^{p_1-1}V_i^{q_2}(t)e_{1,i}(t) + q_2\omega_i(\beta_i(t))^{q_2-1}u_i^{p_1}(t)e_{2,i}(t) + O(h^2),\\
e'_{2,i}(t) &= \delta^2 e_{2,i}(t) + p_2\omega_i(\beta_i(t))^{p_2-1}W_i^{q_3}(t)e_{2,i}(t) + q_3\omega_i(\lambda_i(t))^{q_3-1}v_i^{p_2}(t)e_{3,i}(t) + O(h^2),\\
e'_{3,i}(t) &= \delta^2 e_{3,i}(t) + p_3\omega_i(\lambda_i(t))^{p_3-1}Y_i^{q_4}(t)e_{3,i}(t) + q_4\omega_i(\theta_i(t))^{q_4-1}w_i^{p_3}(t)e_{4,i}(t) + O(h^2),\\
e'_{4,i}(t) &= \delta^2 e_{4,i}(t) + p_4\omega_i(\theta_i(t))^{p_4-1}U_i^{q_1}(t)e_{4,i}(t) + q_1\omega_i(\alpha_i(t))^{q_1-1}y_i^{p_4}(t)e_{1,i}(t) + O(h^2),
\end{aligned}$$

where $\alpha_i(t)$, $\beta_i(t)$, $\lambda_i(t)$ and $\theta_i(t)$ lie, respectively, between $U_i(t)$ and $u_i(t)$, between $V_i(t)$ and $v_i(t)$, between $W_i(t)$ and $w_i(t)$ and between $Y_i(t)$ and $y_i(t)$, for $i = 1, \dots, I$. Using (6.2) and (6.4), there exist J and K positive constants such that

$$\begin{aligned}
e'_{1,i}(t) &\le \delta^2 e_{1,i}(t) + \omega_i J|e_{1,i}(t)| + \omega_i J|e_{2,i}(t)| + Kh^2, \quad i = 1, \dots, I, \quad t \in [0, T^*],\\
e'_{2,i}(t) &\le \delta^2 e_{2,i}(t) + \omega_i J|e_{2,i}(t)| + \omega_i J|e_{3,i}(t)| + Kh^2, \quad i = 1, \dots, I, \quad t \in [0, T^*],\\
e'_{3,i}(t) &\le \delta^2 e_{3,i}(t) + \omega_i J|e_{3,i}(t)| + \omega_i J|e_{4,i}(t)| + Kh^2, \quad i = 1, \dots, I, \quad t \in [0, T^*],\\
e'_{4,i}(t) &\le \delta^2 e_{4,i}(t) + \omega_i J|e_{4,i}(t)| + \omega_i J|e_{1,i}(t)| + Kh^2, \quad i = 1, \dots, I, \quad t \in [0, T^*].
\end{aligned}$$

Let $(M, N, Q, H) \in \left(C^{4,1}([0, L], [0, T^*])\right)^4$ be such that

$$\begin{aligned}
M(x,t) &= (\|\varphi_{1,h} - u_h(0)\|_\infty + \|\varphi_{2,h} - v_h(0)\|_\infty + \|\varphi_{3,h} - w_h(0)\|_\infty\\
&+ \|\varphi_{4,h} - y_h(0)\|_\infty + Kh^2)\mathrm{e}^{(F+3)t+Dx}
\end{aligned}$$

and $M = N = Q = H$, $\forall (x,t) \in [0,1] \times [0, T^*]$, with K, F and D positive constants. By the Lemma 3.2, we can prove that

$$(|e_{1,i}(t)|, |e_{2,i}(t)|, |e_{3,i}(t)|, |e_{4,i}(t)|) < (M(x_i,t), N(x_i,t), Q(x_i,t), H(x_i,t)),$$

with $1 \le i \le I$, for $t \in (0, t(h))$. Thus we get

$$\begin{aligned}
\|U_h(t)-u_h(t)\|_\infty &\le \Big(\|\varphi_{1,h}-u_h(0)\|_\infty+\|\varphi_{2,h}-v_h(0)\|_\infty \\
&\quad +\|\varphi_{3,h}-w_h(0)\|_\infty+\|\varphi_{4,h}-y_h(0)\|_\infty+Kh^2\Big)\,\mathrm{e}^{(F+3)t+D},\\
\|V_h(t)-v_h(t)\|_\infty &\le \Big(\|\varphi_{1,h}-u_h(0)\|_\infty+\|\varphi_{2,h}-v_h(0)\|_\infty \\
&\quad +\|\varphi_{3,h}-w_h(0)\|_\infty+\|\varphi_{4,h}-y_h(0)\|_\infty+Kh^2\Big)\,\mathrm{e}^{(F+3)t+D},\\
\|W_h(t)-w_h(t)\|_\infty &\le \Big(\|\varphi_{1,h}-u_h(0)\|_\infty+\|\varphi_{2,h}-v_h(0)\|_\infty \\
&\quad +\|\varphi_{3,h}-w_h(0)\|_\infty+\|\varphi_{4,h}-y_h(0)\|_\infty+Kh^2\Big)\,\mathrm{e}^{(F+3)t+D},\\
\|Y_h(t)-y_h(t)\|_\infty &\le \Big(\|\varphi_{1,h}-u_h(0)\|_\infty+\|\varphi_{2,h}-v_h(0)\|_\infty \\
&\quad +\|\varphi_{3,h}-w_h(0)\|_\infty+\|\varphi_{4,h}-y_h(0)\|_\infty+Kh^2\Big)\,\mathrm{e}^{(F+3)t+D}.
\end{aligned}$$

where $t \in (0, t(h))$. Suppose that $T^* > t(h)$. From (6.3), we obtain

$$\begin{aligned}
1 &= \|U_h(h)-u_h(t(h))\|_\infty \le \big(\|\varphi_{1,h}-u_h(0)\|_\infty+\|\varphi_{2,h}-v_h(0)\|_\infty \\
&+ \|\varphi_{3,h}-w_h(0)\|_\infty+\|\varphi_{4,h}-y_h(0)\|_\infty+Kh^2\big)\mathrm{e}^{(F+3)T^*+D}.
\end{aligned}$$

Since the term on the right hand side of the above inequality goes to zero as h tends to zero, we deduce that $1 \le 0$, which is impossible. Consequently $t(h) = T^*$ and we conclude the proof. □

Theorem 6.2. *Suppose that the solution (u, v, w, y) of problem* (1.1) *blows up in a finite time T such that $(u, v, w, y) \in (C^{4,1}([0,1] \times [0,T)))^4$ and the initial data at* (2.1)–(2.5) *satisfies*

$$\begin{aligned}
\|\varphi_{1,h}-u_h(0)\|_\infty = o(1), \quad \|\varphi_{2,h}-v_h(0)\|_\infty = o(1),\\
\|\varphi_{3,h}-w_h(0)\|_\infty = o(1), \quad \|\varphi_{4,h}-y_h(0)\|_\infty = o(1), \quad h \to 0.
\end{aligned}$$

Under the assumptions of corollary 4.1, the solution (U_h, V_h, W_h, Y_h) of problem (2.1)–(2.5) *blows up in finite time T_h and we have*

$$\lim_{h\to 0} T_h = T.$$

Proof. Set $\mu > 0$, there exists a constant $\kappa > 0$ such that

$$\frac{y^{1-p_1}}{\gamma\,(p_1-1)} \leq \frac{\mu}{2}, \quad \kappa \leq y. \tag{6.5}$$

Since u blows up in a finite time T, there exists a time $T_0 \in (T-\mu/2;T)$ such that $\|u(.,t)\|_\infty \geq 2\kappa$, for $t \in [T_0,T)$. Denote $T_1 = \frac{T_0+T}{2}$, we see easily that $\sup_{t\in[0,T_1]} \|u(.,t)\|_\infty < \infty$. It follows from Theorem 6.1 that for h sufficiently small

$$\sup_{t\in[0,T_1]} \|U_h(t) - u_h(t)\|_\infty \leq \kappa.$$

Applying the triangle inequality, we get

$$\|U_h(T_1)\|_\infty \geq \|u_h(T_1)\|_\infty - \|U_h(T_1) - u_h(T_1)\|_\infty \geq \kappa.$$

From corollary 4.1, U_h blows up at the time T_h. We deduce from Remark 4.1 and (6.5) that

$$|T_h - T| \leq |T_h - T_1| + |T_1 - T| \leq \frac{\|U_h(T_1)\|_\infty^{1-p_1}}{\gamma(p_1-1)} + \frac{\mu}{2} \leq \mu.$$

The cases where V_h, W_h and Y_h blow up are analogous. □

7. NUMERICAL EXPERIMENTS

In this section, we provide numerical approximations of the simultaneous blow-up time of (1.1) using the initial data $u_0(x) = v_0(x) = w_0(x) = y_0(x) = -\frac{1}{2}x^4 + \frac{3}{2}x^2$, with varying values of p_j and q_j, where $j = 1,2,3,4$. The explicit scheme is defined as follows:

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n)} + \omega_i \left(U_i^{(n)}\right)^{p_1} \left(V_i^{(n)}\right)^{q_2}, \quad i = 1,\dots,I,\ n \geq 0,$$

$$\frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t_n} = \delta^2 V_i^{(n)} + \omega_i \left(V_i^{(n)}\right)^{p_2} \left(W_i^{(n)}\right)^{q_3}, \quad i = 1,\dots,I,\ n \geq 0,$$

$$\frac{W_i^{(n+1)} - W_i^{(n)}}{\Delta t_n} = \delta^2 W_i^{(n)} + \omega_i \left(W_i^{(n)}\right)^{p_3} \left(Y_i^{(n)}\right)^{q_4}, \quad i = 1,\dots,I, \quad n \geq 0,$$

$$\frac{Y_i^{(n+1)} - Y_i^{(n)}}{\Delta t_n} = \delta^2 Y_i^{(n)} + \omega_i \left(Y_i^{(n)}\right)^{p_4} \left(U_i^{(n)}\right)^{q_1}, \quad i = 1,\dots,I, \quad n \geq 0,$$

$$U_i^{(0)} = \varphi_{1,i}, \quad V_i^{(0)} = \varphi_{2,i}, \quad W_i^{(0)} = \varphi_{3,i}, \quad Y_i^{(0)} = \varphi_{4,i}, \quad i = 1, \ldots, I,$$

where

$$\begin{aligned}
\delta^2 U_i^{(n)} &= \frac{U_{i-1}^{(n)} - 2U_i^{(n)} + U_{i+1}^{(n)}}{h^2}, \ 2 \le i \le I-1, \\
\delta^2 U_1^{(n)} &= \frac{2U_2^{(n)} - 2U_1^{(n)}}{h^2}, \ \delta^2 U_I^{(n)} = \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2}, \\
\omega_1 &= \frac{2}{h}, \ \omega_I = \frac{2}{h}, \ \omega_i = 0, \ i = 2, \ldots, I-1,
\end{aligned}$$

and

$$\Delta t_n = \frac{\tau h}{2} \min \Big\{ \frac{h}{\tau}, \|U_h^{(n)}\|_\infty^{1-p_1} \|V_h^{(n)}\|_\infty^{1-q_2}, \|V_h^{(n)}\|_\infty^{1-p_2} \|W_h^{(n)}\|_\infty^{1-q_3},$$
$$\|W_h^{(n)}\|_\infty^{1-p_3} \|Y_h^{(n)}\|_\infty^{1-q_4}, \|Y_h^{(n)}\|_\infty^{1-p_4} \|U_h^{(n)}\|_\infty^{1-q_1} \Big\},$$

where τ satisfies $0 < \tau < 1$. The implicit scheme is also defined as follows:

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \delta^2 U_i^{(n+1)} + \omega_i \left(U_i^{(n)}\right)^{p_1} \left(V_i^{(n)}\right)^{q_2}, \quad i = 1, \ldots, I, \ n \ge 0,$$

$$\frac{V_i^{(n+1)} - V_i^{(n)}}{\Delta t_n} = \delta^2 V_i^{(n+1)} + \omega_i \left(V_i^{(n)}\right)^{p_2} \left(W_i^{(n)}\right)^{q_3}, \quad i = 1, \ldots, I, \ n \ge 0,$$

$$\frac{W_i^{(n+1)} - W_i^{(n)}}{\Delta t_n} = \delta^2 W_i^{(n+1)} + \omega_i \left(W_i^{(n)}\right)^{p_3} \left(Y_i^{(n)}\right)^{q_4}, \quad i = 1, \ldots, I, \quad n \ge 0,$$

$$\frac{Y_i^{(n+1)} - Y_i^{(n)}}{\Delta t_n} = \delta^2 Y_i^{(n+1)} + \omega_i \left(Y_i^{(n)}\right)^{p_4} \left(U_i^{(n)}\right)^{q_1}, \quad i = 1, \ldots, I, \quad n \ge 0,$$

$$U_i^{(0)} = \varphi_{1,i}, \quad V_i^{(0)} = \varphi_{2,i}, \quad W_i^{(0)} = \varphi_{3,i}, \quad Y_i^{(0)} = \varphi_{4,i}, \quad i = 1, \ldots, I,$$

where

$$\begin{aligned}
\delta^2 U_i^{(n+1)} &= \frac{U_{i-1}^{(n+1)} - 2U_i^{(n+1)} + U_{i+1}^{(n+1)}}{h^2}, \ 2 \le i \le I-1, \\
\delta^2 U_1^{(n+1)} &= \frac{2U_2^{(n+1)} - 2U_1^{(n+1)}}{h^2}, \ \delta^2 U_I^{(n+1)} = \frac{2U_{I-1}^{(n+1)} - 2U_I^{(n+1)}}{h^2}, \\
\omega_1 &= \frac{2}{h}, \ \omega_I = \frac{2}{h}, \ \omega_i = 0, \ i = 2, \ldots, I-1,
\end{aligned}$$

and

$$\Delta t_n = \frac{\tau h}{2} \min \Big\{ \|U_h^{(n)}\|_\infty^{1-p_1} \|V_h^{(n)}\|_\infty^{1-q_2}, \|V_h^{(n)}\|_\infty^{1-p_2} \|W_h^{(n)}\|_\infty^{1-q_3},$$
$$\|W_h^{(n)}\|_\infty^{1-p_3} \|Y_h^{(n)}\|_\infty^{1-q_4}, \|Y_h^{(n)}\|_\infty^{1-p_4} \|U_h^{(n)}\|_\infty^{1-q_1} \Big\},$$

where τ satisfies $0 < \tau < 1$.

Tables and Graphics: The initial conditions are given as follows:

$$\varphi_{1,i} = \varphi_{2,i} = \varphi_{3,i} = \varphi_{4,i} = -\frac{1}{2}(ih-h)^4 + \frac{3}{2}(ih-h)^2, \quad i = 1, \dots, I.$$

The following tables present the simultaneous numerical blow-up times T_h, the number of iterations n, the CPU times, and the order s of the approximations for meshes of $I = 16,\ 32,\ 64,\ 128,\ 256,\ 512$. The simultaneous numerical blow-up time is defined as: $T_h = \sum_{n=0}^{\infty} \Delta t_n < \infty$. The order s of the method is computed as:

$$s = \frac{\log\left((T_{4h} - T_{2h})/(T_{2h} - T_h)\right)}{\log(2)}.$$

First case: Theorem 5.1 with $p_1 = 0.4,\ p_2 = 0.5,\ p_3 = 0.5,\ p_4 = 2,\ q_1 = 1,\ q_2 = 2.1,\ q_3 = 1.5,\ q_4 = 2$.

TABLE 1. Explicit Euler method: Results

I	T_h	n	$CPUt$	s
16	0.61764384	383	0.20	-
32	0.61223563	1447	0.09	-
64	0.61058746	5658	0.78	1.71
128	0.61010181	22435	1.88	1.76
256	0.60996220	89436	3.95	1.80
512	0.60992279	357265	14.046875	1.82

TABLE 2. Implicit Euler method: Results

I	T_h	n	$CPUt$	s
16	0.61397791	377	0.52	-
32	0.61144769	1419	0.63	-
64	0.61041245	5542	7.02	1.29
128	0.61006145	21960	54.25	1.56
256	0.60995262	87522	8.27e+02	1.69
512	0.60992047	349593	2.78e+04	1.76

Second case: Theorem 5.2 with $p_1 = 0.11,\ p_2 = 0.51,\ p_3 = 0.1,\ p_4 = 0.9,\ q_1 = 1,\ q_2 = 4,\ q_3 = 2,\ q_4 = 3.$

TABLE 3. Explicit Euler method: Results

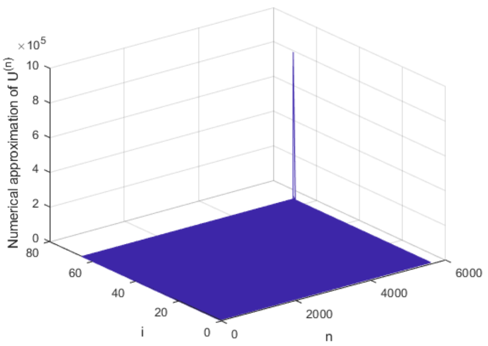
I	T_h	n	$CPUt$	s
16	0.616442447	398	0.06	-
32	0.611211351	1522	0.14	-
64	0.609601879	5997	0.81	1.70
128	0.609125261	23867	1.78	1.76
256	0.608987812	95304	3.88	1.79
512	0.608948925	380991	17.28125	1.82

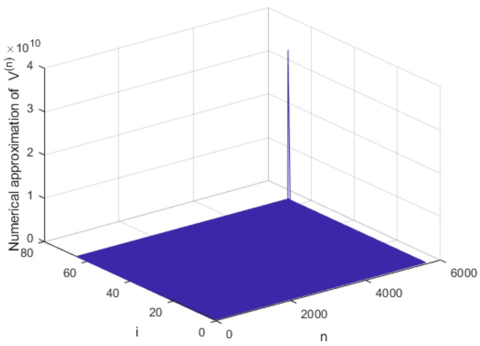
TABLE 4. Implicit Euler method: Results

I	T_h	n	$CPUt$	s
16	0.612395007	398	0.19	-
32	0.610341231	1524	0.47	-
64	0.609412625	6001	6.33	1.15
128	0.609083065	23872	66.43	1.49
256	0.608978161	95310	1.02e+03	1.65
512	0.608946668	380998	3.47e+04	1.74

The following plots compare the explicit and implicit schemes, demonstrating the simultaneous blow-up behavior of U_h, V_h, W_h and Y_h in the case where $I = 64$.

First case: Theorem 5.1 with $p_1 = 0.4,\ p_2 = 0.5;\ p_3 = 0.5,\ p_4 = 2;\ q_1 = 1,\ q_2 = 2.1,\ q_3 = 1.5,\ q_4 = 2.$





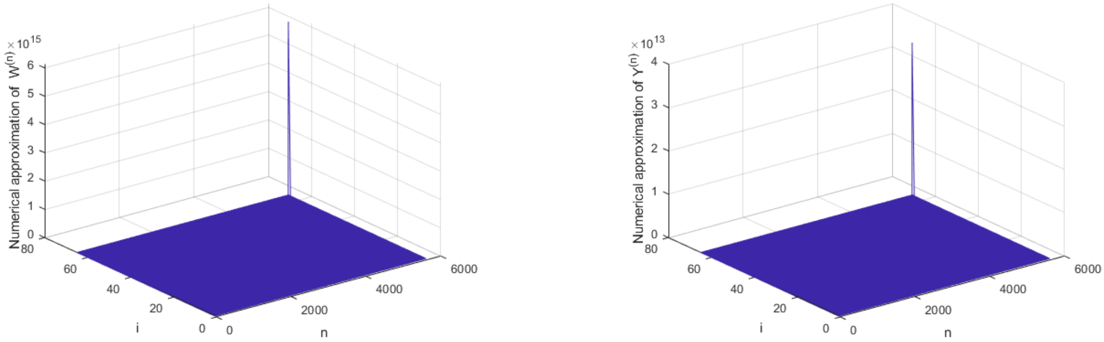


FIGURE 1. (Explicit scheme): U_h, V_h, W_h and Y_h blow up simultaneously.

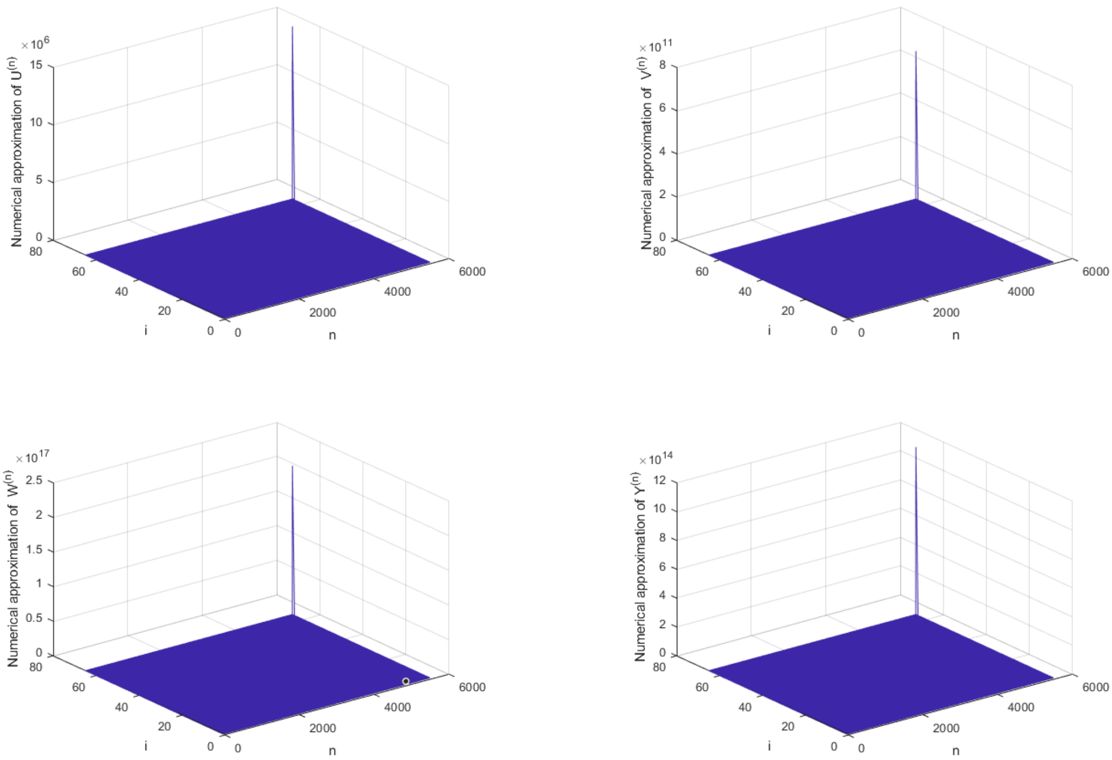


FIGURE 2. (Implicit scheme): U_h, V_h, W_h and Y_h blow up simultaneously.

Second case: Theorem 5.2 with $p_1 = 0.11,\ p_2 = 0.51,\ p_3 = 0.1,\ p_4 = 0.9,\ q_1 = 1,\ q_2 = 4,\ q_3 = 2,\ q_4 = 3.$

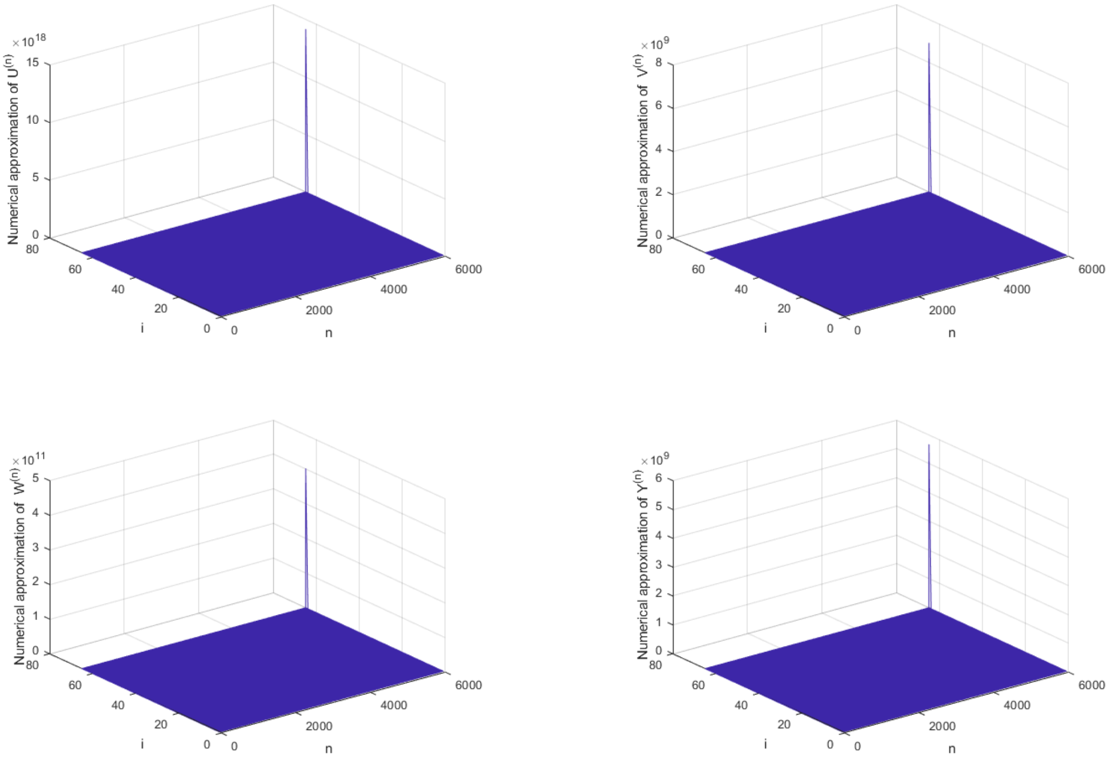
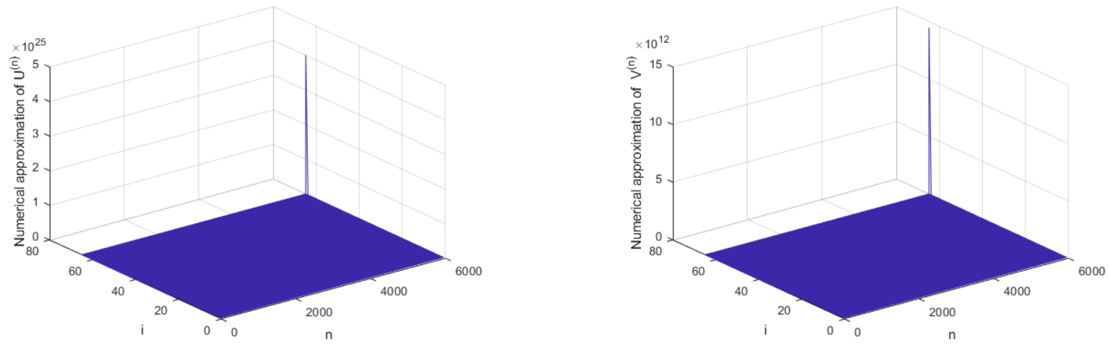


FIGURE 3. (Explicit scheme): U_h, V_h, W_h and Y_h blow up simultaneously.



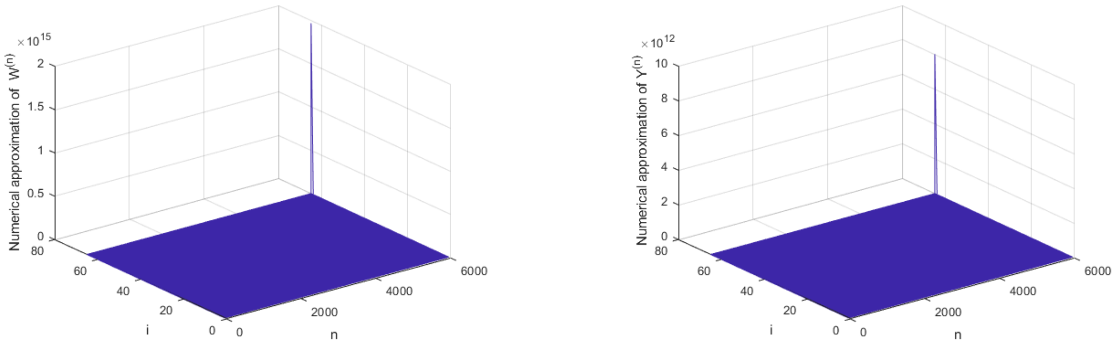


FIGURE 4. (Implicit scheme): U_h, V_h, W_h and Y_h blow up simultaneously.

Remark 7.1. *By considering problem* (2.1)–(2.5) *with initial data given by*

$$\varphi_{1,i} = \varphi_{2,i} = \varphi_{3,i} = \varphi_{4,i} = -\frac{1}{2}(ih-h)^4 + \frac{3}{2}(ih-h)^2, \quad i = 1, \dots, I,$$

- for $p_1 = 0.4, p_2 = 0.5, p_3 = 0.5, p_4 = 2, q_1 = 1, q_2 = 2.1, q_3 = 1.5, q_4 = 2$*, satisfying the conditions of Theorem 5.1, and*

- for $p_1 = 0.11, p_2 = 0.51, p_3 = 0.1, p_4 = 0.9, q_1 = 1, q_2 = 4, q_3 = 2, q_4 = 3$*, satisfying the conditions of Theorem 5.2, we obtain in Tables 1, 2, 3, and 4 an approximate value of the simultaneous blow-up time, which tends to 0.6, and a convergence order* s *that approaches 2 as the mesh size tends to zero. Figures 1 to 4 illustrate the simultaneous finite-time blow-up of the solution* (U_h, V_h, W_h, Y_h) *of problem* (2.1)–(2.5)*, thus validating the theoretical results.*

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