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THEORY ON A MULTI-PARAMETER THREE-DIMENSIONAL HARDY-HILBERT TYPE INTEGRAL INEQUALITY

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ABSTRACT. In 2011, W.T. Sulaiman established an interesting triple integral inequality of the Hardy-Hilbert type, which is expressed using a power-absolute-value-difference kernel function. In this paper, we generalize this inequality by introducing additional parameters. Some of these parameters activate new components and allow for greater adaptability. We provide a rigorous and detailed proof of this generalized inequality, emphasizing the role of each parameter and the beta function in determining the upper bound.

1. INTRODUCTION

The classical Hardy-Hilbert integral inequality is a basic result in mathematical analysis, which allows the study of bivariate integral inequalities. It provides an upper bound for a double integral involving a singular kernel, i.e., $k(x,y) = 1/(x+y)$. The inequality establishes a connection between the integrability properties of two functions and their weighted interaction. Formally, it states that, for any

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$p, q > 1$ such that $1/p + 1/q = 1$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$, i.e., two non-negative functions defined on $[0, +\infty)$, we have

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{x+y} dx dy \leq \delta \left[\int_0^{+\infty} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} g^q(x) dx \right]^{1/q},$$

where the constant δ is given by

$$\delta = \frac{\pi}{\sin(\pi/p)}.$$

This inequality holds under the condition that all the integrals involved are finite (or converge). It is strict unless f and g are both equal to zero. Furthermore, the constant δ is optimal, meaning that it cannot be improved. For detailed proofs and discussions, we refer the reader to [10, 29].

Since its discovery, the Hardy-Hilbert inequality has been the subject of extensive study, inspiring many generalizations and extensions. These include weighted versions, discrete analogues and multidimensional forms. Important contributions to this topic can be found in [1–4, 7, 9, 14, 16, 19–23, 25–28, 30, 32]. For a comprehensive overview of the theory, its refinements, and its wide range of applications, we recommend the survey paper [6].

A relevant example of a modified Hardy-Hilbert-type integral inequality using a power-absolute-value-difference kernel function is given below. For any $p, q > 1$ such that $1/p + 1/q = 1$, $s \in (0, 1)$ and $f, g : [0, +\infty) \mapsto [0, +\infty)$, we have

$$\begin{aligned} & \int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)}{|x-y|^s} dx dy \\ & \leq \iota \left[\int_0^{+\infty} x^{(1-s/2)p-1} f^p(x) dx \right]^{1/p} \left[\int_0^{+\infty} y^{(1-s/2)q-1} g^q(x) dx \right]^{1/q}, \end{aligned} \tag{1.1}$$

where the constant ι is given by

$$\iota = 2\mathfrak{B}\left(\frac{s}{2}, 1-s\right)$$

and $\mathfrak{B}(a, b)$ is the beta function defined by

$$\mathfrak{B}(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt = \int_0^{+\infty} \frac{t^{a-1}}{(1+t)^{a+b}} dt, \tag{1.2}$$

with $a, b > 0$. This inequality holds under the condition that all the integrals involved are finite. See, for example, [16, Corollary 3]. This formulation is original because it combines a singular kernel, $k(x, y) = 1/|x - y|^s$, with weighted integral norms in the upper bound involving power-type weight functions. It thus modifies the structure of the classical Hardy-Hilbert integral inequality and incorporates fractional-type singularities, which are important in potential theory and fractional integral operators. The explicit appearance of the beta function in the sharp constant further illustrates the complexity of the inequality.

In addition, numerous multivariate variants of the Hardy-Hilbert integral inequality have been studied in the literature. We refer to [5, 8, 11–13, 15, 17, 18, 24, 31, 33]. Among these, [17, Theorem 1] is particularly notable as it provides a natural three-dimensional extension of the inequality in Equation (1.1). More precisely, it determine a valuable upper bound for the following triple integral:

$$\int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \frac{f(x)g(y)h(z)}{|x - y - z|^s} dxdydz,$$

where $f, g, h : [0, +\infty) \mapsto [0, +\infty)$ and s is an adjustable parameter. The following three-dimensional power-absolute-value-difference kernel is thus considered: $k(x, y, z) = 1/|x-y-z|^s$. The upper bound is defined as a constant factor resulting from the repeated application of the beta function, multiplied by the product of the three weighted integral norms of the functions f, g and h.

The goal of this paper is to extend the theory to a triple integral of the following form:

$$\int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \frac{x^\alpha y^\beta z^\gamma (y + z)^\sigma |x - y|^\theta |x - z|^\xi f(x)g(y)h(z)}{|x - y - z|^{s+t+u}} dxdydz,$$

where α, β, γ, σ, θ, ξ, s, t, u, ϵ, λ and ω denote adjustable parameters. We thus consider the three-dimensional power-absolute-value-difference kernel $k(x, y, z) = x^\alpha y^\beta z^\gamma (y + z)^\sigma |x - y|^\theta |x - z|^\xi / |x - y - z|^{s+t+u}$. This formulation thus extends the scope of the study in [17] by introducing new weight functions and structural components, such as x^α, y^β, z^γ, $(y + z)^\sigma$, $|x - y|^\theta$ and $|x - z|^\xi$. The resulting inequality is more general and adaptable, and can include a wider range of function interactions. The expression for the constant associated with this inequality is technically intricate and involves multiple applications of the beta function. It has been derived to be as sharp as possible within the given parameter framework. In addition to building on the work of [17], this paper introduces novel proof techniques. These methods are of independent interest and can be applied to a broader class of three-dimensional integral inequalities.

The rest of the paper is organized as follows: Section 2 presents the main theorem together with its proof. Section 3 gives the conclusion.

2. Main theorem

The statement of our main theorem is formulated below, followed by its detailed proof.

Theorem 2.1. *Let* $p, q, r > 1$ *such that* $1/p + 1/q + 1/r = 1$, $f, g, h : [0, +\infty) \mapsto [0, +\infty)$ *be three functions, and* $\alpha, \beta, \gamma, \sigma, \theta, \xi, s, t, u, \epsilon, \lambda, \omega \in \mathbb{R}$ *such that* $\gamma\omega p + 1 > 0$, $1 - sp > 0$, $(s - \gamma\omega)p - 1 > 0$, $1 - sp > 0$, $\beta\lambda p + 1 > 0$, $2 - (s - \theta - \gamma\omega)p > 0$, $(s - \theta - \gamma\omega - \beta\lambda)p - 2 > 0$, $2 - (s - \theta - \gamma\omega)p > 0$, $\alpha\epsilon q + 1 > 0$, $1 - tq > 0$, $(t - \alpha\epsilon)q - 1 > 0$, $1 - tq > 0$, $\gamma(1 - \omega)q + 1 > 0$, $(t - \sigma - \alpha\epsilon - \gamma(1 - \omega))q - 2 > 0$, $\beta(1 - \lambda)r + 1 > 0$, $1 - ur > 0$, $(u - \beta(1 - \lambda))r - 1 > 0$, $1 - ur > 0$, $\alpha(1 - \epsilon)r + 1 > 0$, $2 - (u - \xi - \beta(1 - \lambda))r > 0$, $(u - \xi - \beta(1 - \lambda) - \alpha(1 - \epsilon))r - 2 > 0$ *and* $2 - (u - \xi - \beta(1 - \lambda))r > 0$.

Then the following inequality holds:

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta z^\gamma (y+z)^\sigma |x-y|^\theta |x-z|^\xi f(x)g(y)h(z)}{|x-y-z|^{s+t+u}} dxdydz$$

$$\leq \aleph \left[\int_0^{+\infty} x^{(\theta-s+\gamma\omega+\beta\lambda)p+2} f^p(x)dx\right]^{1/p} \left[\int_0^{+\infty} y^{(\sigma-t+\alpha\epsilon+\gamma(1-\omega))q+2} g^q(y)dy\right]^{1/q}$$

$$\times \left[\int_0^{+\infty} z^{(\xi-u+\beta(1-\lambda)+\alpha(1-\epsilon))r+2} h^r(z)dz\right]^{1/r},$$

where the constant $\aleph$ *is given by*

$$\begin{aligned}
\aleph &= [\mathfrak{B}(\gamma\omega p + 1, 1 - sp) + \mathfrak{B}((s - \gamma\omega)p - 1, 1 - sp)]^{1/p} \\
&\times [\mathfrak{B}(\beta\lambda p + 1, 2 - (s - \theta - \gamma\omega)p) + \mathfrak{B}((s - \theta - \gamma\omega - \beta\lambda)p - 2, 2 - (s - \theta - \gamma\omega)p)]^{1/p} \\
&\times [\mathfrak{B}(\alpha\epsilon q + 1, 1 - tq) + \mathfrak{B}((t - \alpha\epsilon)q - 1, 1 - tq)]^{1/q} \\
&\times [\mathfrak{B}(\gamma(1 - \omega)q + 1, (t - \sigma - \alpha\epsilon - \gamma(1 - \omega))q - 2)]^{1/q} \\
&\times [\mathfrak{B}(\beta(1 - \lambda)r + 1, 1 - ur) + \mathfrak{B}((u - \beta(1 - \lambda))r - 1, 1 - ur)]^{1/r} \\
&\times [\mathfrak{B}(\alpha(1 - \epsilon)r + 1, 2 - (u - \xi - \beta(1 - \lambda))r) \\
&+\mathfrak{B}((u - \xi - \beta(1 - \lambda) - \alpha(1 - \epsilon))r - 2, 2 - (u - \xi - \beta(1 - \lambda))r)]^{1/r}.
\end{aligned} \tag{2.1}$$

This inequality holds under the condition that all the integrals involved are finite. We recall that $\mathfrak{B}(a,b)$ *is the standard beta function at* $a,b>0$ *defined in Equation* (1.2).

Proof. An appropriate decomposition of the integrand, followed by the generalized Hölder integral inequality at p, q and r, yields

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{x^{\alpha}y^{\beta}z^{\gamma}(y+z)^{\sigma}|x-y|^{\theta}|x-z|^{\xi}f(x)g(y)h(z)}{|x-y-z|^{s+t+u}}dxdydz\\
&=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{y^{\beta\lambda}z^{\gamma\omega}|x-y|^{\theta}f(x)}{|x-y-z|^{s}}\\
&\times\frac{x^{\alpha\epsilon}z^{\gamma(1-\omega)}(y+z)^{\sigma}g(y)}{|x-y-z|^{t}}\times\frac{x^{\alpha(1-\epsilon)}y^{\beta(1-\lambda)}|x-z|^{\xi}h(z)}{|x-y-z|^{u}}dxdydz\\
&\le I^{1/p}J^{1/q}K^{1/r}, \qquad (2.2)
\end{aligned}$$

where

$$I=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{y^{\beta\lambda p}z^{\gamma\omega p}|x-y|^{\theta p}f^{p}(x)}{|x-y-z|^{sp}}dxdydz,$$

$$J=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{x^{\alpha\epsilon q}z^{\gamma(1-\omega)q}(y+z)^{\sigma q}g^{q}(y)}{|x-y-z|^{tq}}dxdydz$$

and

$$K=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{x^{\alpha(1-\epsilon)r}y^{\beta(1-\lambda)r}|x-z|^{\xi r}h^{r}(z)}{|x-y-z|^{ur}}dxdydz.$$

Let us now majorize the terms I, J and K successively.

We first focus on the term I. It follows from the triangle inequality that, for any $x,y,z>0$,

$$|x-y-z|\ge||x-y|-|z||=||x-y|-z|\,.$$

Therefore, we have

$$I\le I_{\star}, \qquad (2.3)$$

where

$$I_{\star}=\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{y^{\beta\lambda p}z^{\gamma\omega p}|x-y|^{\theta p}f^{p}(x)}{||x-y|-z|\,|^{sp}}dxdydz.$$

Thanks to the Fubini-Tonelli integral theorem, the integrand being non-negative, we can write

$$I_\star =$$
$$= \int_0^{+\infty} f^p(x) \left\{ \int_0^{+\infty} y^{\beta\lambda p} \left[\int_0^{+\infty} \frac{[z/|x-y|]^{\gamma\omega p} |x-y|^{(\theta-s+\gamma\omega)p+1}}{|1-z/|x-y||^{sp}} \frac{1}{|x-y|} dz \right] dy \right\} dx$$
$$= \int_0^{+\infty} x^{(\theta-s+\gamma\omega+\beta\lambda)p+2} f^p(x) \left[\int_0^{+\infty} \frac{(y/x)^{\beta\lambda p}}{|1-y/x|^{(s-\theta-\gamma\omega)p-1}} \frac{1}{x} \Omega(x,y) dy \right] dx,$$

where

$$\Omega(x,y) = \int_0^{+\infty} \frac{|z/|x-y||^{\gamma\omega p}}{|1-z/|x-y||^{sp}} \frac{1}{|x-y|} dz.$$

The expression of $\Omega(x,y)$ can be derived by applying the general lemma below.

Lemma 2.1. *Let $a \in (-1,0)$ and $b \in (a+1,1)$. Then the following holds:*

$$\int_0^{+\infty} \frac{x^a}{|1-x|^b} dx = \mathfrak{B}(a+1,1-b) + \mathfrak{B}(b-a-1,1-b).$$

Proof. Using the Chasles integral theorem, the definition of the absolute value and the change of variables $x = 1/y$, we get

$$\int_0^{+\infty} \frac{x^a}{|1-x|^b} dx = \int_0^1 \frac{x^a}{|1-x|^b} dx + \int_1^{+\infty} \frac{x^a}{|1-x|^b} dx$$
$$= \int_0^1 \frac{x^a}{(1-x)^b} dx + \int_1^{+\infty} \frac{x^a}{(x-1)^b} dx$$
$$= \int_0^1 \frac{x^a}{(1-x)^b} dx + \int_1^0 \frac{(1/y)^a}{(1/y-1)^b} \left(-\frac{1}{y^2} dy \right)$$
$$= \int_0^1 x^{(a+1)-1} (1-x)^{(1-b)-1} dx + \int_0^1 y^{(b-a-1)-1} (1-y)^{(1-b)-1} dy$$
$$= \mathfrak{B}(a+1,1-b) + \mathfrak{B}(b-a-1,1-b).$$

This concludes the proof of Lemma 2.1. □

Applying the change of variables $m = z/|x-y|$ with respect to z and using Lemma 2.1 with a well-configured parameterization, we obtain

$$\Omega(x,y) = \int_0^{+\infty} \frac{m^{\gamma\omega p}}{|1-m|^{sp}} dm = \mathfrak{B}(\gamma\omega p+1, 1-sp) + \mathfrak{B}((s-\gamma\omega)p-1, 1-sp).$$

We thus have

$$I_\star = [\mathfrak{B}(\gamma\omega p + 1, 1 - sp) + \mathfrak{B}((s - \gamma\omega)p - 1, 1 - sp)] \int_0^{+\infty} x^{(\theta - s + \gamma\omega + \beta\lambda)p + 2} f^p(x) \Xi(x) dx,$$

where

$$\Xi(x) = \int_0^{+\infty} \frac{(y/x)^{\beta\lambda p}}{|1 - y/x|^{(s - \theta - \gamma\omega)p - 1}} \frac{1}{x} dy.$$

Let us determine $\Xi(x)$. Applying the change of variables $n = y/x$ with respect to y and using Lemma 2.1 with a well-configured parameterization, we obtain

$$\begin{aligned}\Xi(x) &= \int_0^{+\infty} \frac{n^{\beta\lambda p}}{|1 - n|^{(s - \theta - \gamma\omega)p - 1}} dn \\ &= \mathfrak{B}(\beta\lambda p + 1, 2 - (s - \theta - \gamma\omega)p) + \mathfrak{B}((s - \theta - \gamma\omega - \beta\lambda)p - 2, 2 - (s - \theta - \gamma\omega)p).\end{aligned}$$

We thus have

$$\begin{aligned}I_\star &= [\mathfrak{B}(\gamma\omega p + 1, 1 - sp) + \mathfrak{B}((s - \gamma\omega)p - 1, 1 - sp)] \\ &\quad \times [\mathfrak{B}(\beta\lambda p + 1, 2 - (s - \theta - \gamma\omega)p) + \mathfrak{B}((s - \theta - \gamma\omega - \beta\lambda)p - 2, \\ &\qquad 2 - (s - \theta - \gamma\omega)p)] \times \int_0^{+\infty} x^{(\theta - s + \gamma\omega + \beta\lambda)p + 2} f^p(x) dx. \end{aligned} \tag{2.4}$$

Putting Equations (2.3) and (2.4) together, we get

$$\begin{aligned}I &\leq [\mathfrak{B}(\gamma\omega p + 1, 1 - sp) + \mathfrak{B}((s - \gamma\omega)p - 1, 1 - sp)] \\ &\quad \times [\mathfrak{B}(\beta\lambda p + 1, 2 - (s - \theta - \gamma\omega)p) + \mathfrak{B}((s - \theta - \gamma\omega - \beta\lambda)p - 2, \\ &\qquad 2 - (s - \theta - \gamma\omega)p)] \times \int_0^{+\infty} x^{(\theta - s + \gamma\omega + \beta\lambda)p + 2} f^p(x) dx. \end{aligned} \tag{2.5}$$

For the term J we proceed in a similar way, paying particular attention to the parameters involved. Using the Fubini-Tonelli integral theorem, the following expression holds:

$$\begin{aligned}J &= \int_0^{+\infty} g^q(y) \left\{ \int_0^{+\infty} z^{\gamma(1-\omega)q} \left[\int_0^{+\infty} \frac{[x/(y+z)]^{\alpha\epsilon q} (y+z)^{(\sigma - t + \alpha\epsilon)q + 1}}{|1 - x/(y+z)|^{tq}} \frac{1}{y+z} dx \right] dz \right\} dy \\ &= \int_0^{+\infty} y^{(\sigma - t + \alpha\epsilon + \gamma(1-\omega))q + 2} g^q(y) \left[\int_0^{+\infty} \frac{(z/y)^{\gamma(1-\omega)q}}{(1 + z/y)^{(t - \sigma - \alpha\epsilon)q - 1}} \frac{1}{y} \Upsilon(y, z) dz \right] dy,\end{aligned}$$

where

$$\Upsilon(y,z)=\int_0^{+\infty}\frac{[x/(y+z)]^{\alpha\epsilon q}}{|1-x/(y+z)|^{tq}}\frac{1}{y+z}dx.$$

Let us determine $\Upsilon(y,z)$. Applying the change of variables $m=x/(y+z)$ with respect to x and using Lemma 2.1 with a well-configured parameterization, we get

$$\Upsilon(y,z)=\int_0^{+\infty}\frac{m^{\alpha\epsilon q}}{|1-m|^{tq}}dm=\mathfrak{B}(\alpha\epsilon q+1,1-tq)+\mathfrak{B}((t-\alpha\epsilon)q-1,1-tq).$$

We thus have

$$J=[\mathfrak{B}(\alpha\epsilon q+1,1-tq)+\mathfrak{B}((t-\alpha\epsilon)q-1,1-tq)]\int_0^{+\infty}y^{(\sigma-t+\alpha\epsilon+\gamma(1-\omega))q+2}g^q(y)\Phi(y)dy,$$

where

$$\Phi(y)=\int_0^{+\infty}\frac{(z/y)^{\gamma(1-\omega)q}}{(1+z/y)^{(t-\sigma-\alpha\epsilon)q-1}}\frac{1}{y}dz.$$

Let us determine $\Phi(y)$. Applying the change of variables $n=z/y$ with respect to z and using Lemma 2.1 with a well-configured parameterization, we obtain

$$\Phi(y)=\int_0^{+\infty}\frac{n^{\gamma(1-\omega)q}}{(1+n)^{(t-\sigma-\alpha\epsilon)q-1}}dn=\mathfrak{B}(\gamma(1-\omega)q+1,(t-\sigma-\alpha\epsilon-\gamma(1-\omega))q-2).$$

We thus have

$$\begin{aligned}J&=[\mathfrak{B}(\alpha\epsilon q+1,1-tq)+\mathfrak{B}((t-\alpha\epsilon)q-1,1-tq)]\\&\quad\times\mathfrak{B}(\gamma(1-\omega)q+1,(t-\sigma-\alpha\epsilon-\gamma(1-\omega))q-2)\\&\quad\times\int_0^{+\infty}y^{(\sigma-t+\alpha\epsilon+\gamma(1-\omega))q+2}g^q(y)dy.\end{aligned}\tag{2.6}$$

Finally, we consider the term K. It follows from the triangle inequality that, for any $x,y,z>0$,

$$|x-y-z|\geq||x-z|-|y||=||x-z|-y|.$$

Therefore, we have

$$K\leq K_\star,\tag{2.7}$$

where

$$K_\star = \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^{\alpha(1-\epsilon)r}y^{\beta(1-\lambda)r}|x-z|^{\xi r}h^r(z)}{||x-z|-y|^{ur}}dxdydz.$$

Using the Fubini-Tonelli integral theorem, we can write

$$K_\star = \int_0^{+\infty} h^r(z)\left\{\int_0^{+\infty} x^{\alpha(1-\epsilon)r}\cdot\right.$$

$$\left.\cdot\left[\int_0^{+\infty}\frac{[y/|x-z|]^{\beta(1-\lambda)r}\,|x-z|^{(\xi-u+\beta(1-\lambda))r+1}}{|1-y/|x-z||^{ur}}\frac{1}{|x-z|}dy\right]dx\right\}dz$$

$$= \int_0^{+\infty} z^{(\xi-u+\beta(1-\lambda)+\alpha(1-\epsilon))r+2}h^r(z)\left[\int_0^{+\infty}\frac{(x/z)^{\alpha(1-\epsilon)r}}{|1-x/z|^{(u-\xi-\beta(1-\lambda))r-1}}\frac{1}{z}\Psi(x,z)dx\right]dz,$$

where

$$\Psi(x,z) = \int_0^{+\infty}\frac{[y/|x-z|]^{\beta(1-\lambda)r}}{|1-y/|x-z||^{ur}}\frac{1}{|x-z|}dy.$$

Let us determine $\Psi(x,z)$. Applying the change of variables $m = y/|x-z|$ with respect to y and using Lemma 2.1 with a well-configured parameterization, we obtain

$$\Psi(x,z) = \int_0^{+\infty}\frac{m^{\beta(1-\lambda)r}}{|1-m|^{ur}}dm = \mathfrak{B}(\beta(1-\lambda)r+1, 1-ur)$$
$$+\,\mathfrak{B}((u-\beta(1-\lambda))r-1, 1-ur).$$

We thus have

$$K_\star = [\mathfrak{B}(\beta(1-\lambda)r+1, 1-ur) + \mathfrak{B}((u-\beta(1-\lambda))r-1, 1-ur)]$$
$$\times\int_0^{+\infty} z^{(\xi-u+\beta(1-\lambda)+\alpha(1-\epsilon))r+2}h^r(z)\Theta(z)dz,$$

where

$$\Theta(z) = \int_0^{+\infty}\frac{(x/z)^{\alpha(1-\epsilon)r}}{|1-x/z|^{(u-\xi-\beta(1-\lambda))r-1}}\frac{1}{z}dx.$$

Let us determine $\Theta(z)$. Applying the change of variables $n = x/z$ with respect to x and using Lemma 2.1 with a well-configured parameterization, we find that

$$\Theta(z) = \int_0^{+\infty} \frac{n^{\alpha(1-\epsilon)r}}{|1-n|^{(u-\xi-\beta(1-\lambda))r-1}} dn$$
$$= \mathfrak{B}(\alpha(1-\epsilon)r+1, 2-(u-\xi-\beta(1-\lambda))r)$$
$$+ \mathfrak{B}((u-\xi-\beta(1-\lambda)-\alpha(1-\epsilon))r-2, 2-(u-\xi-\beta(1-\lambda))r).$$

We thus have

$$K_\star = [\mathfrak{B}(\beta(1-\lambda)r+1, 1-ur) + \mathfrak{B}((u-\beta(1-\lambda))r-1, 1-ur)]$$
$$\times [\mathfrak{B}(\alpha(1-\epsilon)r+1, 2-(u-\xi-\beta(1-\lambda))r)$$
$$+\mathfrak{B}((u-\xi-\beta(1-\lambda)-\alpha(1-\epsilon))r-2, 2-(u-\xi-\beta(1-\lambda))r)] \cdot$$
$$\cdot \int_0^{+\infty} z^{(\xi-u+\beta(1-\lambda)+\alpha(1-\epsilon))r+2} h^r(z) dz. \tag{2.8}$$

Putting Equations (2.7) and (2.8) together, we get

$$K \leq [\mathfrak{B}(\beta(1-\lambda)r+1, 1-ur) + \mathfrak{B}((u-\beta(1-\lambda))r-1, 1-ur)]$$
$$\times [\mathfrak{B}(\alpha(1-\epsilon)r+1, 2-(u-\xi-\beta(1-\lambda))r)$$
$$+\mathfrak{B}((u-\xi-\beta(1-\lambda)-\alpha(1-\epsilon))r-2, 2-(u-\xi-\beta(1-\lambda))r)] \cdot$$
$$\cdot \int_0^{+\infty} z^{(\xi-u+\beta(1-\lambda)+\alpha(1-\epsilon))r+2} h^r(z) dz. \tag{2.9}$$

Combining Equations (2.2), (2.5), (2.6) and (2.9), we finally obtain

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \frac{x^\alpha y^\beta z^\gamma (y+z)^\sigma |x-y|^\theta |x-z|^\xi f(x)g(y)h(z)}{|x-y-z|^{s+t+u}} dxdydz$$
$$\leq [\mathfrak{B}(\gamma\omega p+1, 1-sp) + \mathfrak{B}((s-\gamma\omega)p-1, 1-sp)]^{1/p}$$
$$\times [\mathfrak{B}(\beta\lambda p+1, 2-(s-\theta-\gamma\omega)p) + \mathfrak{B}((s-\theta-\gamma\omega-\beta\lambda)p-2,$$
$$2-(s-\theta-\gamma\omega)p)]^{1/p}$$
$$\times [\mathfrak{B}(\alpha\epsilon q+1, 1-tq) + \mathfrak{B}((t-\alpha\epsilon)q-1, 1-tq)]^{1/q}$$
$$\times [\mathfrak{B}(\gamma(1-\omega)q+1, (t-\sigma-\alpha\epsilon-\gamma(1-\omega))q-2)]^{1/q}$$
$$\times [\mathfrak{B}(\beta(1-\lambda)r+1, 1-ur) + \mathfrak{B}((u-\beta(1-\lambda))r-1, 1-ur)]^{1/r}$$
$$\times [\mathfrak{B}(\alpha(1-\epsilon)r+1, 2-(u-\xi-\beta(1-\lambda))r)$$

$$+\mathfrak{B}((u-\xi-\beta(1-\lambda)-\alpha(1-\epsilon))r-2,2-(u-\xi-\beta(1-\lambda))r)]^{1/r}$$
$$\times\left[\int_0^{+\infty} x^{(\theta-s+\gamma\omega+\beta\lambda)p+2}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{(\sigma-t+\alpha\epsilon+\gamma(1-\omega))q+2}g^q(y)dy\right]^{1/q}$$
$$\times\left[\int_0^{+\infty} z^{(\xi-u+\beta(1-\lambda)+\alpha(1-\epsilon))r+2}h^r(z)dz\right]^{1/r},$$

so

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\frac{x^\alpha y^\beta z^\gamma(y+z)^\sigma|x-y|^\theta|x-z|^\xi f(x)g(y)h(z)}{|x-y-z|^{s+t+u}}dxdydz$$
$$\leq \aleph\left[\int_0^{+\infty} x^{(\theta-s+\gamma\omega+\beta\lambda)p+2}f^p(x)dx\right]^{1/p}\left[\int_0^{+\infty} y^{(\sigma-t+\alpha\epsilon+\gamma(1-\omega))q+2}g^q(y)dy\right]^{1/q}$$
$$\times\left[\int_0^{+\infty} z^{(\xi-u+\beta(1-\lambda)+\alpha(1-\epsilon))r+2}h^r(z)dz\right]^{1/r}.$$

This ends the proof of Theorem 2.1. □

In addition to its originality, the proof is innovative in its use of an appropriate decomposition of the integrand, careful handling of the parameters, and multiple applications of Lemma 2.1. Each step is designed to isolate the contributions of the individual components of the kernel function, rendering the analysis tractable despite the complexity of the final expression.

This theorem thus significantly generalizes both Equation (1.1) and [17, Theorem 1], extending their scope through the introduction of several new parameters.

3. Conclusion and perspectives

In this paper, we extend a key triple integral inequality of the Hardy-Hilbert type, which was originally introduced by W. T. Sulaiman in 2011. By incorporating additional parameters into the kernel function, we obtained a more flexible framework. These parameters introduce new structural components, such as weighted powers and absolute differences, which make the inequality applicable to a wider range of problems. A rigorous and detailed proof is provided, paying particular attention to the role of each parameter and the repeated use of the beta function in deriving the sharp constant.

This work opens up several avenues for future research. One obvious area for further exploration would be analogous inequalities in higher dimensions or within different domains, such as finite intervals or the entire real line. Another promising area of study would be weighted versions of the inequality, where additional weight functions are applied to each variable. Furthermore, the developed techniques could be adapted to establish discrete analogues or fractional integral counterparts of the triple inequality. Finally, potential applications in analysis, such as estimating solutions to certain integral equations, warrant further investigation.

We hope that the general framework and methods presented here will serve as a useful foundation for ongoing developments in the theory of triple integral inequalities.

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