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ON THE TOPOLOGICAL INDICES OF WEAKLY ZERO-DIVISOR GRAPH

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ABSTRACT. The Wiener index of a connected graph G is defined as $W(G) = \sum_{u,v} d_G(u,v)$, where $d_G(u,v)$ denotes the distance between the vertices u and v and the sum runs over all unordered pairs of vertices. The Harary index of a connected graph G is the sum of reciprocal of distances between all pairs of vertices i.e. $H(G) = \sum \frac{1}{d_G(u,v)}$, where the summation runs over all unordered pairs u and v of vertices of G. The weakly zero-divisor graph of a commutative ring R ($\Gamma'(R)$) is a simple undirected graph with vertex set as the set of all non zero zero-divisors of R and the distinct vertices x and y are joined by an edge if and only if there exist $r \in ann(x)$ and $s \in ann(y)$ such that $rs = 0$. In this paper, we find the Wiener index and Harary index of $\Gamma'(R)$ where R is the ring of integers modulo n.

1. INTRODUCTION

In this paper, G denotes a simple connected graph and $d_G(u,v)$ denotes the distance between the vertices u and v of G ($d(u,v)$ for short), which is defined as the length of a shortest path between them. Clearly $d_G(u,u) = 0$ Wiener index is a topological index used to characterize molecular graphs. The Wiener index of a

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graph G, defined in [15], is

$$W(G) = \sum_{u,v} d_G(u,v),$$

where $d_G(u,v)$ denotes the distance between the distinct vertices u and v in G and the sum runs over all the unordered pairs of vertices. It was first introduced and studied by Wiener in 1947 for acyclic molecular graphs. The Wiener index is one of the most significant topological indices applied in Combinatorial Chemistry (see [8], [13]).

A topological index is a numerical value associated to an invariant under graph isomorphism. The number of vertices and edges of a given graph are obviously topological indices of G. Topological indices find wide applications in various areas like Mathematical Chemistry and Pharmaceutical Science [15]. The Wiener index is the first distance based topological index defined by the chemist Harold Wiener. Followed by Wiener, too many authors continued introducing new topological indices. Nowadays, there are more than thousand topological indices and most of them have wide applications in Computer Science, Coding Theory, Chemistry, Algebraic Graph Theory, Biochemistry and Nanotechnology. For a list of topological indices of graphs and their application, refer [1].

2. Basic definitions and notations

The order of a graph G, denoted by $O(G)$ is the number of vertices of G. A graph without loops and multiple edges is called a simple graph. A graph which has only one vertex is called a trivial graph. The complement of a simple graph G, denoted by $\overline{G}$, is a simple graph with the same vertex set as that of G and such that two vertices are adjacent in $\overline{G}$ if and only if they are non adjacent in G. A graph G is complete if any two distinct vertices are adjacent. In this paper, K_n denotes a complete graph on n vertices and $\overline{K_n}$ denotes an empty graph (null graph) on n vertices. A complete bipartite graph with $m+n$ vertices and mn edges is a graph in which the vertex set has a bi-partition (X,Y) where $\mid X \mid = m, \mid Y \mid = n$ and each vertex of X is adjacent to every vertex of Y and is denoted by $K_{m,n}$. $G[V']$ is the subgraph of G induced by a subset V' of $V(G)$, with vertex set V' and edge set, the set of all edges of G which have both ends in V'. We say the $G[V']$ is the subgraph of G induced by V'. For $u \in V(G)$, the open neighborhood of u; denoted by $N_G(u)$ is the set of all vertices adjacent to u in G. For a vertex v, the degree of v is the number of vertices adjacent to v. The degree of v is denoted as $deg(v)$. It can be seen that, For a simple graph, $deg(v)$ is the cardinality of $N_G(v)$. An isolated vertex of G is a vertex with zero degree. A graph G is k-regular if degree of every vertex is k.

The disjoint union of two graphs G_1 and G_2, denoted by $G_1 \cup G_2$ is the graph with vertex set containing all vertices of G_1 and G_2 and edge set as $E(G_1) \cup E(G_2)$. That is, the union of two graphs essentially combines both graphs to a single larger graph. The join $G_1 \nabla G_2$ of G_1 and G_2 is the graph obtained by taking the union of G_1 and G_2 and by adding edges from each vertex of G_1 to every vertex of G_2. This operation is generalised to a finite collection of vertex disjoint graphs, as follows.

Let H be a finite graph with vertex set $V(H) = \{v_1, v_2, \ldots, v_n\}$ and let $G_1, G_2, \ldots, G_n$ be a family of vertex disjoint graphs. The H- join of $G_1, G_2, \ldots, G_n$ denoted by $H[G_1, G_2, \ldots, G_n]$ is obtained by replacing each vertex v_i of H by the graph G_i and making all vertices of G_i adjacent to every vertex of G_j if and only if v_i and v_j are adjacent in H. ie, the H- join of $G_1, G_2, \ldots, G_n$ is created by taking the union of $G_1, G_2, \ldots, G_n$ and joining each vertex of G_i to all vertices of G_j if and only if $v_i \sim v_j$ in H. It is a convenient method that such complicated discrete structures can be studied and their graph properties may be explored through their induced subgraphs. Let $\{V_1, V_2, \ldots, V_m\}$ be a partition of the vertex set of $V(G)$ of a graph G. It is said to be an equitable partition, if for $1 \leq i < j \leq m$ and for any two vertices u and v in V_i, u and v have the same number of neighbours in V_j.

The Euler totient function of a natural number n, denoted by $\phi(n)$ is the number of positive integer less than n which are relatively prime to n. The basic definitions in graph theory are standard and are from [5].

3. The weakly zero-divisor graph $\Gamma'(\mathbb{Z}_n)$

It has been an interesting research field to explore algebraic structures through graph theory, ever since 1988, when Ivan Beck defined zero divisor graph over commutative rings in pursuit of some colouring problems. The weakly zero-divisor graph over a commutative ring R, denoted by $\Gamma'(R)$ was introduced by M. J. Nikmehr et al [11] and it was further studied analysed in [14]. Various graph parameters like clique number, girth and vertex chromatic number of this graph are explored in [11].

In this paper, we consider the weakly zero divisor graph on the ring of integers modulo n. The number of non zero zero-divisors of $\mathbb{Z}_n$ is $n - \phi(n) - 1$ [10]. By a proper divisor of n, we mean a positive divisor d such that d/n, $1 < d < n$. If $n = q_1^{k_1} q_2^{k_2} \dots q_r^{k_r}$, where $q_1, q_2, \dots, q_r$ are distinct primes, and $k_i \geq 2$, for every $1 \leq i \leq r$, then $\Gamma'(\mathbb{Z}_n)$ is a complete graph. Also, for a prime number n, $\Gamma'(\mathbb{Z}_n)$ is an empty graph. Hence, we choose $n = q_1 q_2 \dots q_s p_1^{k_1} p_2^{k_2} \dots p_t^{k_t}$, where $q_1, q_2, \dots, q_s, p_1, p_2, \dots, p_t$ are distinct primes, and $s \geq 1, k_i \geq 2, t \geq 0$.

Let $\xi(n)$ denote the number of proper divisors of n. Arrange the proper divisors of n as $d_1 = q_1, d_2 = q_2, \dots, d_s = q_s, d_{s+1}, d_{s+2}, \dots, d_{\xi(n)}$ Obviously,

$$\xi(n) = 2^s \prod_{i=1}^{t} (k_i + 1) \quad - 2.$$

The fascinating structure of $\Gamma'(\mathbb{Z}_n)$ is constituted by the equitable partition $\{\mathcal{A}(d_1), \mathcal{A}(d_2), \dots, \mathcal{A}(d_{\xi(n)})\}$ of vertices, where for every proper divisor d of n, $\mathcal{A}(d) = \{k \in \mathbb{Z}_n : \gcd(k, n) = d\}$, such that $\mathcal{A}(d_i)$ and $\mathcal{A}(d_j)$ are disjoint, for $i \neq j$. For any divisor d of n, let $\Gamma'(\mathcal{A}(d))$ denote the subgraph of $\Gamma'(\mathbb{Z}_n)$ induced by $\mathcal{A}(d)$.

Lemma 3.1. *[10] The number of vertices in each $\mathcal{A}(d)$ is $\phi(\frac{n}{d})$.*

Lemma 3.2. *[14] The induced subraphs $\Gamma'(\mathcal{A}(d))$ are either complete or null as given by* $\Gamma'(\mathcal{A}(d)) = \begin{cases} \overline{K}_{\phi(\frac{n}{d})} ; d \in \{q_1, q_2, \dots, q_s\} \\ K_{\phi(\frac{n}{d})} ; \textit{otherwise} \end{cases}$

Lemma 3.3. *[14]* $\Gamma'(\mathbb{Z}_n) = K_{\xi(n)} [\Gamma'(\mathcal{A}(q_1)), \dots, \Gamma'(\mathcal{A}(q_s)), \Gamma'(\mathcal{A}(d_{s+1})), \dots, \Gamma'(\mathcal{A}(d_{\xi(n)}))]$

For example, consider $\Gamma'(\mathbb{Z}_{18})$. The four proper divisors of 18 which are precisely $2, 3, 6, 9$, partition the set of non-zero divisors of $\mathbb{Z}_{18}$ is into 4 classes as follows: $\mathcal{A}(2) = \{2, 4, 8, 10, 14, 16\}$, $\mathcal{A}(3) = \{3, 15\}$, $\mathcal{A}(6) = \{6, 12\}$, $\mathcal{A}(9) = \{9\}$. The weakly zero divisor graph $\Gamma'(\mathbb{Z}_{18})$ is the K_4 - join of $\overline{K}_6$, K_2, K_2 and K_1.

4. Wiener index of $\Gamma'(\mathbb{Z}_n)$

The combinatorial structure of $\Gamma'(\mathbb{Z}_n)$ as the generalized join of the subgraphs $\Gamma'(\mathcal{A}(q_1)), \Gamma'(\mathcal{A}(q_2)) \ldots, \Gamma'(\mathcal{A}(q_s)), \Gamma'(\mathcal{A}(d_{s+1})), .., \Gamma'(\mathcal{A}(d_{\xi(n)}))$ by a complete graph on $\xi(n)$ vertices, precisely the proper divisors of n is, analysed in this section to find the Wiener index of this graph.

Theorem 4.1. *For $n = q_1^{k_1} q_2^{k_2} \ldots q_r^{k_r}$, where $q_1, q_2, \ldots, q_r$ are distinct primes, and $k_i \geq 2$, for every $1 \leq i \leq r$, then the Wiener index of $\Gamma'(\mathbb{Z}_n)$ is $\frac{(n-\phi(n)-1)(n-\phi(n)-2)}{2}$.*

Proof. When $n = q_1^{k_1} q_2^{k_2} \ldots q_r^{k_r}$, where $q_1, q_2, \ldots, q_r$ are distinct primes, and $k_i \geq 2$, for every $1 \leq i \leq r$, then $\Gamma'(\mathbb{Z}_n)$ is a complete graph on $n - \phi(n) - 1$ vertices. For a complete graph on n vertices, the Wiener index is exactly the number of edges. $\square$

Corollary 4.1. *The Wiener index of $\Gamma'(\mathbb{Z}_{p^k})$, for any prime $p \geq 2$ and any positive integer $k \geq 2$ is $\frac{(p^{k-1}-1)(p^{k-1}-2)}{2}$.*

Corollary 4.2. *Consider $\Gamma'(\mathbb{Z}_{p^2})$ for any prime p. Then, $W(\Gamma'(\mathbb{Z}_{p^2})) = \frac{(p-1)(p-2)}{2}$.*

Corollary 4.3. *Consider $\Gamma'(\mathbb{Z}_{p^3})$ for any prime p. Then, $W(\Gamma'(\mathbb{Z}_{p^3})) = \frac{(p^2-1)(p^2-2)}{2}$.*

Theorem 4.2. *For $n = pq$, where p and q are distinct primes, then $W(\Gamma'(\mathbb{Z}_{pq})) = p^2 + q^2 + pq - 4p - 4q + 5$.*

Proof. The only two proper divisors of n are p and q and the $p + q - 2$ non zero zero-divisors of $\mathbb{Z}_{pq}$ partition the vertex set of $\Gamma'(\mathbb{Z}_{pq})$ into two disjoint sets $\mathcal{A}(p)$ and $\mathcal{A}(q)$, the cardinalities of which are $q - 1$ and $p - 1$ respectively, by Lemma 3.1. Also, the subgraphs induced by them are null graphs, by Lemma 3.2. The generalized join structure of the weakly zero divisor graph reveals that $\Gamma'(\mathbb{Z}_{pq})$ is a complete bipartite graph. For each pair of vertices $u, v \in \mathcal{A}(p)$, the distance between them is 2 and the same in $\mathcal{A}(q)$ too. Also the distance between a vertex in $\mathcal{A}(p)$ and a vertex in $\mathcal{A}(q)$ is 1. Thus summing up all the possible distance between the pairs of vertices, it can be seen that, the Wiener index of $\Gamma'(\mathbb{Z}_{pq}) = p^2 + q^2 + pq - 4p - 4q + 5$. $\square$

Theorem 4.3. *For $n = pqr$, where p, q and r are distinct primes, then*

$$\begin{aligned}
W(\Gamma'(\mathbb{Z}_{pqr})) = &(p-1)^2\left[(q-1)^2+(r-1)^2+qr-1\right]+(q-1)^2\left[(r-1)^2+rp-1\right]\\
&+(r-1)^2\left[p+q-2\right]+4(p-1)(q-1)(r-1)\\
&+\binom{p-1}{2}+\binom{q-1}{2}+\binom{r-1}{2}.
\end{aligned}$$

Proof. The proper divisors of $n = pqr$ are $p, q, r, pq, pr,$ and qr. The vertex set of $\Gamma'(\mathbb{Z}_{pqr})$ is partitioned into six mutually disjoint sets $\mathcal{A}(p)$, $\mathcal{A}(q)$, $\mathcal{A}(r)$, $\mathcal{A}(pq)$, $\mathcal{A}(pr)$, and $\mathcal{A}(qr)$, among which the first three subsets induce null subgraphs of cardinality $\phi(qr)$, $\phi(pr)$ and $\phi(pq)$ respectively and the remaining three induce complete subgraphs of cardinality $\phi(r)$, $\phi(q)$, and $\phi(p)$ respectively of $\Gamma'(\mathbb{Z}_{pqr})$. Thus,

$$\Gamma'(\mathbb{Z}_{pqr}) = K_6\left[\overline{K}_{\phi(qr)}, \overline{K}_{\phi(pr)}, \overline{K}_{\phi(pq)}, K_{\phi(r)}, K_{\phi(q)}, K_{\phi(p)}\right].$$

Since the joining graph is complete, We note that the distance between any two distinct vertices in the null subgraph $\overline{K}_{(q-1)(r-1)}$ is 2 and so is in $\overline{K}_{(p-1)(r-1)}$ and $\overline{K}_{(p-1)(q-1)}$. Also the distance between any two distinct vertices in a complete graph is 1. Obviously, if the vertices belong to different partitions subsets, the distance between them is 1 since the joining graph is complete. Thus, summing up the distance between all distinct pairs of vertices,

$$\begin{aligned}
&W(\Gamma'(\mathbb{Z}_{pqr}))\\
= &\ 2\left[\binom{\phi(qr)}{2}+\binom{\phi(pr)}{2}+\binom{\phi(pq)}{2}\right]+\binom{\phi(p)}{2}+\binom{\phi(q)}{2}+\binom{\phi(r)}{2}\\
+ &\ \phi(p)\phi(q)+\phi(p)\phi(r)+\phi(p)\phi(pq)+\phi(p)\phi(pr)+\phi(p)\phi(qr)\\
+ &\ \phi(q)\phi(r)+\phi(q)\phi(pq)+\phi(q)\phi(pr)+\phi(q)\phi(qr)\\
+ &\ \phi(r)\phi(pq)+\phi(r)\phi(pr)+\phi(r)\phi(qr)\\
+ &\ \phi(pq)\phi(pr)+\phi(pq)\phi(qr)\\
+ &\ \phi(pr)\phi(qr)\\
= &\ (p-1)^2\left[(q-1)^2+(r-1)^2+qr-1\right]+(q-1)^2\left[(r-1)^2+rp-1\right]\\
+ &\ (r-1)^2\left[pq-1\right]+3(p-1)(q-1)(r-1)\\
+ &\ \binom{p-1}{2}+\binom{q-1}{2}+\binom{r-1}{2}.
\end{aligned}$$

□

Example 1. *For example, for $n = 30$, the proper divisors are $p = 2, q = 3, r = 5$ and $W(\Gamma'(\mathbb{Z}_{30})) = 245$.*

For $n = pq^2$, where p and q are primes, $p < q$, the proper divisors are p, q, pq, and q^2. Proceeding as in the previous theorem we have the following theorem.

Theorem 4.4. *For $n = pq^2$, where $p < q$ are distinct primes, then $W(\Gamma'(\mathbb{Z}_{pq^2})) = \frac{(q-1)^2}{2}[2q^2 + 2pq + (p-1)^2 + 1] + \frac{(p-1)(q-1)}{2}[4q + 2p - 3] + \frac{(p-1)^2}{2} - \frac{p+q-2}{2} - q(q-1)$*

The graph of $\Gamma'(\mathbb{Z}_{50})$ is shown in figure 1. The proper divisors of 50 are $2, 5, 10, 25$. The equitable partition of the vertex set of this graph is given below:

$$\mathcal{A}(2) = \{2, 4, 6, 8, 12, 14, 16, 18, 22, 24, 26, 28, 32, 34, 36, 38, 42, 44, 46, 48\}$$
$$\mathcal{A}(5) = \{5, 15, 35, 45\}$$
$$\mathcal{A}(10) = \{10, 20, 30, 40\}$$
$$\mathcal{A}(25) = \{25\}$$

The weakly zero divisor graph $\Gamma'(\mathbb{Z}_{50})$ is the K_4 - join of $\overline{K}_{20}$, K_4, K_4 and K_1. The wiener index of $\Gamma'(\mathbb{Z}_{50})$ is 596. Refer figure 1.

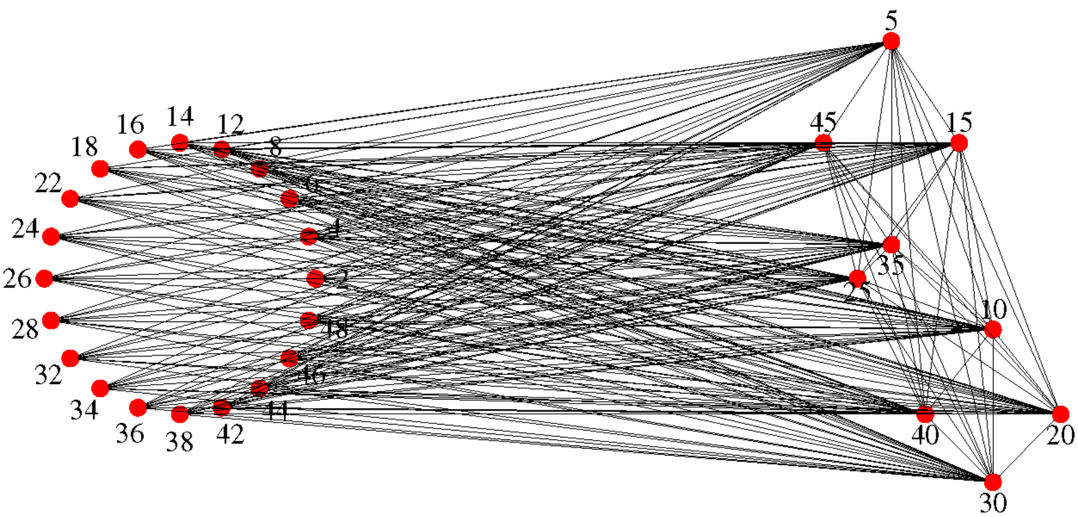

FIGURE 1. The weakly zero divisor graph $\Gamma'(\mathbb{Z}_{50})$

It can be seen that when n is a product of more distinct primes, the weakly zero divisor graph on $\mathbb{Z}_n$ becomes more sparse and the computation of topological indices becomes more tedious and it involves more combinatorial effort. The generalized join structure of this graph plays a vital role to reduce the complexity of this task. In the next theorem, we find the Wiener index of $\Gamma'(\mathbb{Z}_n)$ for any n.

Theorem 4.5. *let $n = q_1q_2\dots q_sp_1^{k_1}p_2^{k_2}\dots p_t^{k_t}$, where $q_1, q_2, \dots, q_s, p_1, p_2, \dots, p_t$ are distinct primes, and $s \geq 1, k_i \geq 2, t \geq 0$. Let the proper divisors of n be arranged as $d_1 = q_1, d_2 = q_2, \dots, d_s = q_s, d_{s+1}, d_{s+2}, \dots, d_{\xi(n)}$. Then the Wiener index of $\Gamma'(\mathbb{Z}_n)$ is given by*

$$W(\Gamma'(\mathbb{Z}_n)) = 2\sum_{i=1}^{s}\binom{\phi(\frac{n}{q_i})}{2} + \sum_{i=s+1}^{\xi(n)}\binom{\phi(\frac{n}{d_i})}{2} + \sum_{1\leq i<j\leq\xi(n)}\phi(\frac{n}{d_i})\ \ \phi(\frac{n}{d_j}).$$

Proof. Consider $n = q_1q_2\dots q_sp_1^{k_1}p_2^{k_2}\dots p_t^{k_t}$. Let the proper divisors of n be arranged as $d_1 = q_1, d_2 = q_2, \dots, d_s = q_s, d_{s+1}, d_{s+2}, \dots, d_{\xi(n)}$ where $\xi(n)$ denotes the number of proper divisors of n. The subgraphs $\Gamma'(\mathcal{A}(d))$ are null graphs when $d = q_1, q_2, \dots, q_s$ and complete graphs otherwise.

Case (i): Consider vertices $u, v \in \mathcal{A}(d_i)$ for some $1 \leq i \leq s$. Since $\Gamma'(\mathcal{A}(d_i))$, $1 \leq i \leq s$ is a null graph on $\phi(\frac{n}{d_i})$ vertices, from Lemma 3.3, $d(u, v) = 2$. There are $\binom{\phi(\frac{n}{d_i})}{2}$ such distinct pairs of vertices u, v in each $\Gamma'(\mathcal{A}(d_i))$, $1 \leq i \leq s$, by Lemma 3.1. Thus sum of the distances between such distinct pairs of vertices u, v is $2\binom{\phi(\frac{n}{d_i})}{2}$, for every $1 \leq i \leq s$.

Case (ii): Consider vertices $u, v \in \mathcal{A}(d_i)$, for some $s+1 \leq i \leq \xi(n)$. Since, by Lemma 3.2, $\mathcal{A}(d_i)$ induces complete subgraphs, $d(u, v) = 1$. Thus sum of the distances between such vertices in each such $\Gamma'(\mathcal{A}(d_i))$ amounts to $\binom{\phi(\frac{n}{d_i})}{2}$, for every $s+1 \leq i \leq \xi(n)$.

Case (iii): Consider vertices $u \in \mathcal{A}(d_i)$ and $v \in \mathcal{A}(d_j)$, for $i \neq j, 1 \leq i < j \leq \xi(n)$, obviously by Lemma 3.3, $d(u, v) = 1$.

Thus summing up the distances in each of the above cases, the Wiener index of $\Gamma'(\mathbb{Z}_n)$ is found for any n. □

Corollary 4.4. *let $n = q_1q_2\dots q_sp_1^{k_1}p_2^{k_2}\dots p_t^{k_t}$, where $q_1, q_2, \dots, q_s, p_1, p_2, \dots, p_t$ are distinct primes, and $s \geq 1, k_i \geq 2, t \geq 0$. Let the proper divisors of n be arranged as $d_1 = q_1, d_2 = q_2, \dots, d_s = q_s, d_{s+1}, d_{s+2}, \dots, d_{\xi(n)}$, where $\xi(n)$ denotes the number of proper divisors of n. Consider $G = \Gamma'(\mathbb{Z}_n)$. Let $e(G)$ denote the number of edges and $W(G)$ denote the Wiener index of G. Then,*

$$W(G) + e(G) = (n - \phi(n) - 1)(n - \phi(n) - 2).$$

Proof. The number of vertices of $G = \Gamma'(\mathbb{Z}_n)$ is $n - \phi(n) - 1$. For each proper divisor d of n, the induced subgraph $\Gamma'(\mathcal{A}(d))$, is a null graph if and only if $d = d_i$ for some $i \in \{1, 2, 3, \dots, s\}$. Also $\Gamma'(\mathcal{A}(d))$, is a complete graph for $d = d_{s+1}, \dots, d_{\xi(n)}$, from Lemma 3.2. Thus, the number of edges $e(G) = \sum_{i=s+1}^{\xi(n)} \binom{\phi(\frac{n}{d_i})}{2} + \sum_{1 \le i < j \le \xi(n)} \phi(\frac{n}{d_i}) \ \ \phi(\frac{n}{d_j})$ and moreover if each pair of vertices were adjacent in $\Gamma'(\mathcal{A}(d_i))$, for $1 \le i \le s$, the graph G would be complete. Thus, from Theorem 4.5, it is trivial that,

$$\begin{aligned} W(\Gamma'(\mathbb{Z}_n)) + e(G) &= 2\sum_{i=1}^{s} \binom{\phi(\frac{n}{p_i})}{2} + 2\sum_{i=s+1}^{\xi(n)} \binom{\phi(\frac{n}{d_i})}{2} + 2\sum_{1 \le i < j \le \xi(n)} \phi(\frac{n}{d_i}) \ \ \phi(\frac{n}{d_j}) \\ &= 2e(K_{n-\phi(n)-1}) \\ &= (n - \phi(n) - 1)(n - \phi(n) - 2). \end{aligned}$$

□

5. Harary index of weakly zero divisor graph

Now we consider the following well-known distance-based topological index, the Harary index of a graph which was introduced by Plavšić et al. [12] and by Ivanciuc et al. [9], in 1993. The Harary index of G is defined as $H(G) = \sum \frac{1}{d_G(u,v)}$, where the summation runs over all unordered pairs u and v of vertices of G. The Harary index has been studied extensively in [6], [7], [16].

In this section we study the Harary index of $\Gamma'(\mathbb{Z}_n)$. Clearly, for $n = p^k$ for some prime p and positive integer $k \ge 2$, $\Gamma'(\mathbb{Z}_n)$ is a complete graph and the Harary index and Wiener index make no difference. Due to the fascinating combinatorial structure of $\Gamma'(\mathbb{Z}_n)$, it is easy to calculate the Harary index, since the reciprocal of the distance between distinct pairs of vertices u and v is $\frac{1}{2}$ if and only if both u and v belong to the same $\mathcal{A}(d_i)$, for some $1 \le i \le s$ and other wise 1. Thus summing up the reciprocal distances between all unordered pairs of vertices in $\Gamma'(\mathbb{Z}_n)$, we arrive at the following theorem.

Theorem 5.1. *let $n = q_1 q_2 \dots q_s p_1^{k_1} p_2^{k_2} \dots p_t^{k_t}$, where $q_1, q_2, \dots, q_s, p_1, p_2, \dots, p_t$ are distinct primes, and $s \ge 1, k_i \ge 2, t \ge 0$. Let the proper divisors of n be arranged as $d_1 = q_1, d_2 = q_2, \dots, d_s = q_s, d_{s+1}, d_{s+2}, \dots, d_{\xi(n)}$, where $\xi(n)$ denotes the number of* *proper divisors of n. Then the Harary index of $\Gamma'(\mathbb{Z}_n)$ is given by*

$$H(\Gamma'(\mathbb{Z}_n)) = \frac{1}{2}\sum_{i=1}^{s}\binom{\phi(\frac{n}{p_i})}{2} + \sum_{i=s+1}^{\xi(n)}\binom{\phi(\frac{n}{d_i})}{2} + \sum_{1\le i<j\le \xi(n)} \phi(\frac{n}{d_i}) \quad \phi(\frac{n}{d_j}).$$

Adding edges between each pair of vertices in $\mathcal{A}(d_i)$, for each $1 \le i \le s$, the graph $\Gamma'(\mathbb{Z}_n)$ would be a complete graph on $n - \phi(n) - 1$ vertices. Hence from the above theorem, we have the following corollary.

Corollary 5.1. *Let $G = \Gamma'(\mathbb{Z}_n)$. Then the number of edges $e(G)$ and the Harary index of G are connected by,*

$$2H(G) + e(G) = \binom{n - \phi(n) - 1}{2}.$$

Conclusion

The exploration into the generalised join structure of the weakly zero divisor graph on the ring of integers modulo n helps to find the topological indices of this graph and the inter connection between these indices and its graph parameters.

Conflict of interest

The authors declare that they have no conflict of interest.

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