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METRIC SPACES WITH BINARY OPERATION

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ABSTRACT. In this paper, we introduce an unusual concept, a metric space with a binary operation. Introducing a binary operation into a standard metric space instead of addition could lead to various interesting and potentially standard and non-standard results, depending on the properties of the operation. This study generalizes the concept of known metric spaces in the literature. We also establish fixed point theorems, each with a specific binary operation.

1. INTRODUCTION

A standard metric space introduced by Frechet in 1906 is simply a metric space where the metric satisfies the usual conditions (non-negativity, identity of indiscernibles, symmetry, and triangle inequality), with a specific operation (Addition). In most mathematical structures, a binary operation is a rule that takes two elements from a set and combines them to form another element from the same set. The typical binary operation in vector spaces is addition, but in a normal metric space, no such algebraic operation is required. When an arbitrary binary operation is introduced into a metric space, the behavior of the space can change in several

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ways depending on how the operation is defined. For further understanding, see ([1 - 11]).

Definition 1.1. *Let Z be a nonempty set, $\otimes$, a binary operation with e as its identity element, and $b : X^2 \to \mathbb{R}^+$. b is called an operational metric if the following axioms are satisfied:*

b_1 : $b(u,v) \geq e$;
b_2 : $b(u,v) = e$ if and only if $u = v$;
b_3 : $b(u,v) = b(v,u)$;
b_4 : $b(u,v) \leq b(u,w) \otimes b(w,v)$ for all $u, v, w \in Z$.

Z together with b is called an operational metric space. Denoted by $(Z, b, \otimes)$

Remark 1.1.

(i) *If the binary operation $\otimes$ is defined by $x \otimes y = x + y$, the Definition 2.1 reduces to metric space introduced by Fretchet (1906).*
(ii) *If the binary operation $\otimes$ is defined by $x \otimes y = x \times y$, the Definition 2.1 reduces to b- metric space introduced by Bakhtin (1989).*

Example 1. *Let $Z = \{x \in \mathbb{N} : 3 \leq x \leq 9\}$ and the binary operation $\otimes$ be defined by $x \otimes y = x + y - 3$. If $b(x,y) = |x - y| + 3$, then b is an operational metric and $(Z, b, \otimes)$ is an operational metric space.*

Verification:

(i) By definition

$$|x - y| = \begin{cases} x - y, & if \quad x - y \geq 0; \\ y - x, & if \quad x - y < 0. \end{cases}$$

So, $|x - y| \geq 0$. Since, $|x - y| \geq 0$, $|x - y| + 3 \geq 3$ for all $x \in Z$. Hence, $b(x,y) = |x - y| + 3 \geq e = 3$.

(ii) If $x \otimes e = x$, then $x + e - 3 = x \implies e = 3$. Then, $b(x,y) = e \implies |x - y| + 3 = e \implies |x - y| = 0 \implies x = y$. Conversely, if $x = y$, then $x - y = 0 \implies |x - y| = 0 \implies |x - y| + 3 = e \implies b(x,y) = e$

(iii) $b(x,y) = |-(x - y)| + 3 = |-x + y| + 3 = |y - x| + 3 = b(y,x)$.

(iv)

$$b(x,y) = |x-y|+3 \tag{1.1}$$
$$= |x-a+a-y|+3 \tag{1.2}$$
$$\leq |x-a|+|a-y|+3 \tag{1.3}$$
$$< |x-a|+3+|a-y|+3 \tag{1.4}$$
$$= b(x,a)+b(a,y). \tag{1.5}$$

Example 2. *Let $Z = \mathbb{R}$ and the binary operation $\otimes$ be defined by $x \otimes y = x + y$. If $b(x,y) = |x-y|$, then b is an operational metric and $(Z, b, \otimes)$ is an operational metric space.*

Remark 1.2. *The operational metric in Example 2.2 is an analogue of the usual metric space.*

Definition 1.2. *Let $(Z, b, \otimes)$ be an operational metric space. An open sphere centered at x with radius r in Z is defined by*

$$S_r(x) = \{a : b(x,a) < r\}.$$

Definition 1.3. *Let $(Z, b, \otimes)$ be an operational metric space. A closed sphere centered at x with radius r in Z is defined by*

$$S_r[x] = \{a : b(x,a) \leq r\}.$$

Definition 1.4. *Let $(Z, b, \otimes)$ be an operational metric space. A sphere centered at x with radius r in Z is defined by*

$$S(r,x) = \{a : b(x,a) = r\}.$$

Definition 1.5. *Let $(Z, b, \otimes)$ be an operational metric space and $\{x_n\}$, a sequence in Z. A sequence, $\{x_n\}$ converge to t if for $n \in \mathbb{N}$, $b(x_n, t) \to e$ as $n \to \infty$.*

Definition 1.6. *Let $(Z, b, \otimes)$ be an operational metric space and $\{x_n\}$, a sequence in Z. A sequence, $\{x_n\}$ in Z is said to be a Cauchy sequence if for $n, m \in \mathbb{N}$ with $n > m$, $b(x_n, x_m) \to e$ as $n, m \to \infty$.*

Definition 1.7. *Let $(Z_1, b_1, \otimes)$ and $(Z_2, b_2, \otimes)$ be two operational metric spaces. A $f : Z_1 \to Z_2$ is said to be continuous at a point $z \in Z_1$ if for all $\epsilon > e$ the exists $\delta > e$*

such that

$$b_1(y,z) < \delta \implies b_2(f(y), f(z)) < \epsilon.$$

The function f is continuous on Z_1 if it is continuous at every point $z \in Z_1$.

2. Main Results

Theorem 2.1. *Let $(Z, b, \otimes)$ be a complete operational metric space with an operation defined by $a \otimes b = a + b$. Suppose $f : Z \to Z$ is a map and there exists a real number k satisfying $0 \leq k < 1$ for each $a, b \in Z$ with*

$$b(fa, fb) \leq k(b(a,b)). \tag{2.1}$$

Then f has a unique fixed point.

Proof. Considering (2.1) with an arbitrary point $x_0 \in X$ and define a sequence x_n by $x_n = f^n x_0$,

$$b(x_n, x_{n+1}) = b(fx_{n-1}, fx_n) \leq k(b(x_{n-1}, x_n)). \tag{2.2}$$

Suppose f satisfies condition (7), then

$$\begin{aligned} b(x_n, x_{n+1}) &= b(Tx_{n-1}, Tx_n) && (2.3)\\ &\leq k(b(x_{n-1}, x_n)) && (2.4)\\ &\leq k^2(b(x_{n-2}, x_{n-1})). && (2.5) \end{aligned}$$

Using this repeatedly, we obtain

$$b(x_n, x_{n+1}) \leq k^n(b(x_0, x_1)). \tag{2.6}$$

By using (b_4) of Definition 2.1 with $n > m$, we have

$$\begin{aligned} b(x_n, x_m) &\leq b(x_n, x_{n-1}) \otimes b(x_{n-1}, x_m) && (2.7)\\ &= b(x_n, x_{n-1}) + b(x_{n-1}, x_m) && (2.8)\\ &= b(x_n, x_{n-1}) + b(x_{n-1}, x_{n-2}) + \ldots + b(x_{m+1}, x_m). && (2.9) \end{aligned}$$

So, we obtain

$$\begin{aligned}
(2.10)\qquad b(x_n, x_m) &\leq b(x_n, x_{n-1}) + b(x_{n-1}, x_{n-2}) + \ldots + b(x_{m+1}, x_m) \\
(2.11)\qquad &\leq k^{n-1}b(x_0, x_1) + k^{n-2}b(x_0, x_1) + \ldots + k^m b(x_0, x_1) \\
(2.12)\qquad &\leq [k^{n-1} + k^{n-2} + \ldots + k^m]b(x_0, x_1) \\
(2.13)\qquad &\leq k^n[k^{-1} + k^{-2} + \ldots + k^{m-n}]b(x_0, x_1) \\
(2.14)\qquad &\leq \frac{k^n}{k-1}b(x_0, x_1).
\end{aligned}$$

Taking the limit of $b(x_n, x_m)$ as $n \to \infty$, we have

$$\lim_{n,m\to\infty} b(x_n, x_m) \to e. \tag{2.15}$$

So, $\{x_n\}$ is a S-Cauchy Sequence.

By the completeness of $(Z, b, \otimes)$, there exists $u \in Z$ such that $\{x_n\}$ is convergent to u.

Suppose $fu \neq u$

$$b(x_n, fu) \leq k(b(x_{n-1}, u)). \tag{2.16}$$

Taking the limit as n $\to \infty$ and using the fact that the function is continuous in its variables, we get

$$b(u, fu) \leq k(b(u, u)). \tag{2.17}$$

Hence,

$$b(u, fu) \leq e. \tag{2.18}$$

This is a contradiction. So, $fu = u$.

To show the uniqueness, suppose $v \neq u$ is such that $fv = v$ and $fu = u$, then

$$b(fu, fv) \leq k(b(u, v)). \tag{2.19}$$

Since $fu = u$ and $fv = v$, we have

$$b(u, v) \leq e, \tag{2.20}$$

which implies that $v = u$. □

Theorem 2.2. *Let $(Z, b, \otimes)$ be a complete operational metric space with an operation defined by $a \otimes b = a + b$. Suppose $f : Z \to Z$ is a map and there exists a real number*

k satisfying $0 \leq k < 0.5$ for each $a, b \in Z$ with

$$b(fa, fb) \leq k[b(a, fa) + b(b, fb)]. \tag{2.21}$$

Then f has a unique fixed point.

Proof. Considering (26) with an arbitrary point $x_0 \in X$ and define a sequence x_n by $x_n = f^n x_0$,

$$b(x_n, x_{n+1}) = b(fx_{n-1}, fx_n) \leq k[b(x_{n-1}, x_n) + b(x_{n+1}, x_n)]. \tag{2.22}$$

SO,

$$b(x_n, x_{n+1}) \leq \frac{k}{1-k} b(x_{n-1}, x_n). \tag{2.23}$$

If $q = \frac{k}{1-k}$, then

$$b(x_n, x_{n+1}) \leq q b(x_{n-1}, x_n). \tag{2.24}$$

Suppose f satisfies condition (2.21), then

$$\begin{aligned} b(x_n, x_{n+1}) &\leq q(b(x_{n-1}, x_n)) && (2.25)\\ &\leq q^2(b(x_{n-2}, x_{n-1})). && (2.26) \end{aligned}$$

Using this repeatedly, we obtain

$$b(x_n, x_{n+1}) \leq q^n(b(x_0, x_1)). \tag{2.27}$$

By using (b_4) of Definition 2.1 with $n > m$, we have

$$\begin{aligned} b(x_n, x_m) &\leq b(x_n, x_{n-1}) \otimes b(x_{n-1}, x_m) && (2.28)\\ &= b(x_n, x_{n-1}) + b(x_{n-1}, x_m) && (2.29)\\ &= b(x_n, x_{n-1}) + b(x_{n-1}, x_{n-2}) + \ldots + b(x_{m+1}, x_m). && (2.30) \end{aligned}$$

We obtain

$$\begin{aligned} b(x_n, x_m) &\leq b(x_n, x_{n-1}) + b(x_{n-1}, x_{n-2}) + \ldots + b(x_{m+1}, x_m) && (2.31)\\ &\leq q^{n-1}b(x_0, x_1) + q^{n-2}b(x_0, x_1) + \ldots + q^m b(x_0, x_1) && (2.32)\\ &\leq [q^{n-1} + q^{n-2} + \ldots + q^m]b(x_0, x_1) && (2.33) \end{aligned}$$

$$\leq \quad q^n[k^{-1} + q^{-2} + \ldots + q^{m-n}]b(x_0, x_1) \tag{2.34}$$

$$\leq \quad \frac{q^n}{q-1} b(x_0, x_1). \tag{2.35}$$

Taking the limit of $b(x_n, x_m)$ as $n \to \infty$, we have

$$\lim_{n,m\to\infty} b(x_n, x_m) \to e. \tag{2.36}$$

So, $\{x_n\}$ is a S-Cauchy Sequence.

By the completeness of $(Z, b, \otimes)$, there exists $u \in Z$ such that $\{x_n\}$ is convergent to u.

Suppose $fu \neq u$

$$b(x_n, fu) \leq k[b(x_{n-1}, x_n) + b(u, fu)]. \tag{2.37}$$

Taking the limit as n $\to \infty$ and using the fact that the function is continuous in its variables, we get

$$b(u, fu) \leq k(b(u, fu)). \tag{2.38}$$

Hence,

$$b(u, fu) \leq e. \tag{2.39}$$

This is a contradiction. So, $fu = u$.

To show the uniqueness, suppose $v \neq u$ is such that $fv = v$ and $fu = u$, then

$$b(fu, fv) \leq 2k(b(u, v)). \tag{2.40}$$

Since $fu = u$ and $fv = v$, we have

$$b(u, v) \leq e. \tag{2.41}$$

which implies that $v = u$. □

Theorem 2.3. *Let $(Z, b, \otimes)$ be a complete operational metric space with an operation defined by $a \otimes b = a + b$. Suppose $f : Z \to Z$ is a map and there exists a real number k satisfying $0 \leq k < 0.5$ for each $a, b \in Z$ with*

$$b(fa, fb) \leq k[b(a, fb) + b(b, fa)]. \tag{2.42}$$

Then f has a unique fixed point.

Proof. Considering (2.42) with an arbitrary point $x_0 \in X$ and define a sequence x_n by $x_n = f^n x_0$,

$$b(x_n, x_{n+1}) = b(fx_{n-1}, fx_n) \leq k[b(x_{n-1}, x_{n+1}) + b(x_n, x_n)]. \tag{2.43}$$

Further,

$$b(x_n, x_{n+1}) \leq \frac{k}{1-k} b(x_{n-1}, x_n). \tag{2.44}$$

If $q = \frac{k}{1-k}$, then

$$b(x_n, x_{n+1}) \leq q b(x_{n-1}, x_n). \tag{2.45}$$

Suppose f satisfies condition (50), then

$$\begin{aligned} b(x_n, x_{n+1}) &\leq q(b(x_{n-1}, x_n)) && (2.46)\\ &\leq q^2(b(x_{n-2}, x_{n-1})). && (2.47) \end{aligned}$$

Using this repeatedly, we obtain

$$b(x_n, x_{n+1}) \leq q^n(b(x_0, x_1)). \tag{2.48}$$

By using (b_4) of Definition 2.1 with $n > m$, we have

$$\begin{aligned} b(x_n, x_m) &\leq b(x_n, x_{n-1}) \otimes b(x_{n-1}, x_m) && (2.49)\\ &= b(x_n, x_{n-1}) + b(x_{n-1}, x_m) && (2.50)\\ &= b(x_n, x_{n-1}) + b(x_{n-1}, x_{n-2}) + \ldots + b(x_{m+1}, x_m). && (2.51) \end{aligned}$$

Next, we obtain

$$\begin{aligned} b(x_n, x_m) &\leq b(x_n, x_{n-1}) + b(x_{n-1}, x_{n-2}) + \ldots + b(x_{m+1}, x_m) && (2.52)\\ &\leq q^{n-1}b(x_0, x_1) + q^{n-2}b(x_0, x_1) + \ldots + q^m b(x_0, x_1) && (2.53)\\ &\leq [q^{n-1} + q^{n-2} + \ldots + q^m] b(x_0, x_1) && (2.54)\\ &\leq q^n[k^{-1} + q^{-2} + \ldots + q^{m-n}] b(x_0, x_1) && (2.55)\\ &\leq \frac{q^n}{q-1} b(x_0, x_1) && (2.56) \end{aligned}$$

Taking the limit of $b(x_n, x_m)$ as $n \to \infty$, we have

$$\lim_{n,m\to\infty} b(x_n, x_m) \to e. \tag{2.57}$$

So, $\{x_n\}$ is a S-Cauchy Sequence.

By the completeness of $(Z, b, \otimes)$, there exists $u \in Z$ such that $\{x_n\}$ is convergent to u.

Suppose $fu \neq u$

$$b(x_n, fu) \leq k[b(x_{n-1}, fu) + b(u, x_n)]. \tag{2.58}$$

Taking the limit as n $\to \infty$ and using the fact that the function is continuous in its variables, we get

$$b(u, fu) \leq k(b(u, fu)). \tag{2.59}$$

Hence,

$$b(u, fu) \leq e. \tag{2.60}$$

This is a contradiction. So, $fu = u$.

To show the uniqueness, suppose $v \neq u$ is such that $fv = v$ and $fu = u$, then

$$b(fu, fv) \leq 2k(b(u, v)). \tag{2.61}$$

Since $fu = u$ and $fv = v$, we have

$$b(u, v) \leq e, \tag{2.62}$$

which implies that $v = u$. □

Theorem 2.4. *Let $(Z, b, \otimes)$ be a complete operational metric space with an operation defined by $a \otimes b = \max\{a, b\}$. Suppose $f : Z \to Z$ is a map and there exists a real number k satisfying $0 \leq k < 1$ for each $a, b \in Z$ with*

$$b(fa, fb) \leq k(b(a, b)). \tag{2.63}$$

Then f has a unique fixed point.

Proof. Considering (2.63) with an arbitrary point $x_0 \in X$ and define a sequence x_n by $x_n = f^n x_0$,

$$b(x_n, x_{n+1}) = b(fx_{n-1}, fx_n) \leq k(b(x_{n-1}, x_n)). \tag{2.64}$$

Suppose f satisfies condition (2.64), then

$$b(x_n, x_{n+1}) = b(Tx_{n-1}, Tx_n) \tag{2.65}$$

$$\leq k(b(x_{n-1}, x_n)) \tag{2.66}$$

$$\leq k^2(b(x_{n-2}, x_{n-1})). \tag{2.67}$$

Using this repeatedly, we obtain

$$b(x_n, x_{n+1}) \leq k^n(b(x_0, x_1)). \tag{2.68}$$

By using (b_4) of Definition 2.1 with $n > m$, we have

$$b(x_n, x_m) \leq b(x_n, x_{n-1}) \otimes b(x_{n-1}, x_m) \tag{2.69}$$

$$= \max\{b(x_n, x_{n-1}), b(x_{n-1}, x_m)\} \tag{2.70}$$

$$= \max\{b(x_n, x_{n-1}), b(x_{n-1}, x_{n-2}), \dots, b(x_{m+1}, x_m)\}. \tag{2.71}$$

So, we obtain

$$b(x_n, x_m) \leq \max\{b(x_n, x_{n-1}), b(x_{n-1}, x_{n-2}), \dots, b(x_{m+1}, x_m)\} \tag{2.72}$$

$$\leq \max\{k^{n-1}b(x_0, x_1), k^{n-2}b(x_0, x_1), \dots, k^m b(x_0, x_1)\} \tag{2.73}$$

$$\leq \max\{k^{n-1}, k^{n-2}, \dots, k^m\}b(x_0, x_1). \tag{2.74}$$

Taking the limit of $b(x_n, x_m)$ as $n \to \infty$, we have

$$\lim_{n,m\to\infty} b(x_n, x_m) \to e. \tag{2.75}$$

So, $\{x_n\}$ is a S-Cauchy Sequence.

By the completeness of $(Z, b, \otimes)$, there exists $u \in Z$ such that $\{x_n\}$ is convergent to u. Suppose $fu \neq u$

$$b(x_n, fu) \leq k(b(x_{n-1}, u)). \tag{2.76}$$

Taking the limit as n $\to \infty$ and using the fact that the function is continuous in its variables, we get

$$b(u, fu) \leq k(b(u, u)). \tag{2.77}$$

Hence,

$$b(u, fu) \leq e. \tag{2.78}$$

This is a contradiction. So, $fu = u$.

To show the uniqueness, suppose $v \neq u$ is such that $fv = v$ and $fu = u$, then

$$b(fu, fv) \leq k(b(u, v)). \tag{2.79}$$

Since $fu = u$ and $fv = v$, we have

$$b(u, v) \leq e, \tag{2.80}$$

which implies that $v = u$. □

Theorem 2.5. *Let $(Z, b, \otimes)$ be a complete operational metric space with an operation defined by $a \otimes b = \max\{a, b\}$. Suppose $f : Z \to Z$ is a map and there exists a real number k satisfying $0 \leq k < 0.5$ for each $a, b \in Z$ with*

$$b(fa, fb) \leq k[b(a, fa) + b(b, fb)]. \tag{2.81}$$

Then f has a unique fixed point.

Proof. Considering (2.81) with an arbitrary point $x_0 \in X$ and define a sequence x_n by $x_n = f^n x_0$,

$$b(x_n, x_{n+1}) = b(fx_{n-1}, fx_n) \leq k[b(x_{n-1}, x_n) + b(x_{n+1}, x_n)]. \tag{2.82}$$

The above implies

$$b(x_n, x_{n+1}) \leq \frac{k}{1-k} b(x_{n-1}, x_n). \tag{2.83}$$

If $q = \frac{k}{1-k}$, then

$$b(x_n, x_{n+1}) \leq q b(x_{n-1}, x_n). \tag{2.84}$$

Then

$$b(x_n, x_{n+1}) \leq q(b(x_{n-1}, x_n)) \tag{2.85}$$

$$\leq q^2(b(x_{n-2}, x_{n-1})). \tag{2.86}$$

Using this repeatedly, we obtain

$$b(x_n, x_{n+1}) \leq q^n(b(x_0, x_1)). \tag{2.87}$$

By using (b_4) of Definition 2.1 with $n > m$, we have

$$b(x_n, x_m) \leq b(x_n, x_{n-1}) \otimes b(x_{n-1}, x_m) \tag{2.88}$$
$$= \max\{b(x_n, x_{n-1}), b(x_{n-1}, x_m)\} \tag{2.89}$$
$$= \max\{b(x_n, x_{n-1}), b(x_{n-1}, x_{n-2}), \ldots, b(x_{m+1}, x_m)\}. \tag{2.90}$$

Next, we obtain

$$b(x_n, x_m) \leq \max\{b(x_n, x_{n-1}), b(x_{n-1}, x_{n-2}), \ldots, b(x_{m+1}, x_m)\} \tag{2.91}$$
$$\leq \max\{q^{n-1}b(x_0, x_1), q^{n-2}b(x_0, x_1), \ldots, q^m b(x_0, x_1)\} \tag{2.92}$$
$$\leq \max\{q^{n-1}, q^{n-2}, \ldots, q^m\}b(x_0, x_1). \tag{2.93}$$

Taking the limit of $b(x_n, x_m)$ as $n \to \infty$, we have

$$\lim_{n,m\to\infty} b(x_n, x_m) \to e. \tag{2.94}$$

So, $\{x_n\}$ is a S-Cauchy Sequence.

By the completeness of $(Z, b, \otimes)$, there exists $u \in Z$ such that $\{x_n\}$ is convergent to u.

Suppose $fu \neq u$

$$b(x_n, fu) \leq k[b(x_{n-1}, x_n) + b(u, fu)]. \tag{2.95}$$

Taking the limit as n $\to \infty$ and using the fact that the function is continuous in its variables, we get

$$b(u, fu) \leq k(b(u, fu)). \tag{2.96}$$

Hence,

$$b(u, fu) \leq e. \tag{2.97}$$

This is a contradiction. So, $fu = u$.

To show the uniqueness, suppose $v \neq u$ is such that $fv = v$ and $fu = u$, then

$$b(fu, fv) \leq 2k(b(u, v)). \tag{2.98}$$

Since $fu = u$ and $fv = v$, we have $b(u, v) \leq e$, which implies that $v = u$. □

Theorem 2.6. *Let $(Z, b, \otimes)$ be a complete operational metric space with an operation defined by $a \otimes b = \max\{a, b\}$. Suppose $f : Z \to Z$ is a map and there exists a real* *number k satisfying $0 \leq k < 0.5$ for each $a, b \in Z$ with*

$$b(fa, fb) \leq k[b(a, fb) + b(b, fa)]. \tag{2.99}$$

Then f has a unique fixed point.

Proof. Considering (2.100) with an arbitrary point $x_0 \in X$ and define a sequence x_n by $x_n = f^n x_0$,

$$b(x_n, x_{n+1}) = b(fx_{n-1}, fx_n) \leq k[b(x_{n-1}, x_{n+1}) + b(x_n, x_n)]. \tag{2.100}$$

The above implies

$$b(x_n, x_{n+1}) \leq \frac{k}{1-k} b(x_{n-1}, x_n). \tag{2.101}$$

If $q = \frac{k}{1-k}$, then

$$b(x_n, x_{n+1}) \leq q b(x_{n-1}, x_n). \tag{2.102}$$

Then

$$\begin{aligned} b(x_n, x_{n+1}) &\leq q(b(x_{n-1}, x_n)) && (2.103) \\ &\leq q^2(b(x_{n-2}, x_{n-1})). && (2.104) \end{aligned}$$

Using this repeatedly, we obtain

$$b(x_n, x_{n+1}) \leq q^n(b(x_0, x_1)). \tag{2.105}$$

By using (b_4) of Definition 2.1 with $n > m$, we have

$$\begin{aligned} b(x_n, x_m) &\leq b(x_n, x_{n-1}) \otimes b(x_{n-1}, x_m) && (2.106) \\ &= \max\{b(x_n, x_{n-1}), b(x_{n-1}, x_m)\} && (2.107) \\ &= \max\{b(x_n, x_{n-1}), b(x_{n-1}, x_{n-2}), \ldots, b(x_{m+1}, x_m)\}. && (2.108) \end{aligned}$$

So, we obtain

$$\begin{aligned} b(x_n, x_m) &\leq \max\{b(x_n, x_{n-1}), b(x_{n-1}, x_{n-2}), \ldots, b(x_{m+1}, x_m)\} && (2.109) \\ &\leq \max\{q^{n-1}b(x_0, x_1), q^{n-2}b(x_0, x_1), \ldots, q^m b(x_0, x_1)\} && (2.110) \\ &\leq \max\{q^{n-1}, q^{n-2}, \ldots, q^m\} b(x_0, x_1). && (2.111) \end{aligned}$$

Taking the limit of $b(x_n, x_m)$ as $n \to \infty$, we have

$$\lim_{n,m\to\infty} b(x_n, x_m) \to e. \tag{2.112}$$

So, $\{x_n\}$ is a S-Cauchy Sequence.

By the completeness of $(Z, b, \otimes)$, there exists $u \in Z$ such that $\{x_n\}$ is convergent to u.

Suppose $fu \neq u$

$$b(x_n, fu) \leq k[b(x_{n-1}, fu) + b(u, x_n)]. \tag{2.113}$$

Taking the limit as n $\to \infty$ and using the fact that the function is continuous in its variables, we get

$$b(u, fu) \leq k(b(u, fu)). \tag{2.114}$$

Hence, $b(u, fu) \leq e$. This is a contradiction. So, $fu = u$.

To show the uniqueness, suppose $v \neq u$ is such that $fv = v$ and $fu = u$, then $b(fu, fv) \leq 2k(b(u, v))$. Since $fu = u$ and $fv = v$, we have $b(u, v) \leq e$, which implies that $v = u$. □

CONCLUSION

In conclusion, a new abstract space is introduced in this research work and some contractive mappings are established and used to prove some fixed point results on the newly introduced space. Examples are given to validate the originality and applicability of our results.

CONFLICT OF INTEREST

There is no conflict of interest of any kind, financial or non-financial type.

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