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STUDY OF A VARIANT OF THE HARDY INTEGRAL INEQUALITY

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ABSTRACT. This paper investigates a new variant of the Hardy integral inequality, which is formulated using an integral that quantifies the dispersion of a function around its mean. Explicit upper bounds are derived for a variety of configurations. The main inequality is then extended to include pairs of functions. Detailed proofs accompany all results.

1. INTRODUCTION

The Hardy integral inequality is a well-known result in mathematical analysis. First published by G. H. Hardy in 1920, it has since become a fundamental tool, particularly in the study of Sobolev spaces and partial differential equations. See [1–4] for the references and complete theory. The classical form of the Hardy inequality is presented below. Let $p > 1$ and $f : [0, +\infty) \mapsto [0, +\infty)$ be a (non-negative) function such that

$$\int_0^{+\infty} f^p(t)dt < +\infty.$$

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Then we have

$$\int_0^{+\infty} \left[\frac{1}{x}\int_0^x f(t)dt\right]^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} f^p(t)dt.$$

The constant $[p/(p-1)]^p$ is optimal, meaning that the inequality cannot be satisfied by a smaller constant for any function f satisfying the required assumptions. This inequality has inspired a wide variety of generalizations and extensions, including versions in higher dimensions, weighted inequalities, and discrete analogues. For the purposes of this paper, we will focus on an extension to a modifiable interval, as described below. Let $\alpha > 0$, $p > 1$ and $f : [0, \alpha) \mapsto [0, +\infty)$ be a function such that

$$\int_0^{\alpha} f^p(t)dt < +\infty.$$

Then we have

$$\int_0^{\alpha} \left[\frac{1}{x}\int_0^x f(t)dt\right]^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{\alpha} f^p(t)\left[1-\left(\frac{t}{\alpha}\right)^{1-1/p}\right] dt. \tag{1.1}$$

We will refer to it as the "finite interval version" of the Hardy integral inequality. For detailed discussions and derivations on this version, see [6, 7]. It can also be viewed as a refinement of a result established by N. Levinson in 1964 (see [5]).

In this paper, we investigate a variant of the "finite interval version" of the Hardy integral inequality. More precisely, within a similar framework and subject to additional monotonicity and integrability assumptions on the function f, we derive bounds for the following integral:

$$\int_0^{\alpha} \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]^p dx.$$

In a sense, this integral can be viewed as a global measure of the dispersion of f around its integral mean, i.e., $(1/x)\int_0^x f(t)dt$. Our objective is to quantify this deviation using an inequality approach. Subsequently, we extend our analysis to two natural generalizations involving pairs of functions. More precisely, denoting the second function by g, we investigate bounds for the following two-function expressions:

$$\int_0^{\alpha} \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]\left[g(x) - \frac{1}{x}\int_0^x g(t)dt\right] dx$$

and

$$\int_0^\alpha \left| f(x)g(x) - \frac{1}{x^2}\left[\int_0^x f(t)dt\right]\left[\int_0^x g(t)dt\right]\right|^p dx.$$

These expressions arise naturally when studying the interactions between functions and their respective integral means. The resulting inequalities are called the first and second types for the first and second integrals, respectively. They are of interest due to their potential applications in the theory of operators and integral transforms. Moreover, they enhance our comprehension of the interaction between integral means and multiplication forms.

The rest of the paper is organized as follows: Section 2 contains the inequalities involving a single function. Results involving two functions are presented in Section 3. Section 4 provides a conclusion.

2. Results involving a single function

2.1. Main theorem. The main theorem is presented below, together with the necessary assumptions and technical details. The proof is primarily based on an integration by parts and a thorough application of the "finite interval version" of the Hardy integral inequality.

Theorem 2.1. *Let $\alpha > 0$ or $\alpha \to +\infty$, $p > 1$ and $f : [0, \alpha) \mapsto [0, +\infty)$ be a differentiable non-decreasing function such that*

$$\lim_{t\to 0} tf(t) = 0, \quad \int_0^\alpha t^p \left[f'(t)\right]^p \left[1 - \left(\frac{t}{\alpha}\right)^{1-1/p}\right] dt < +\infty.$$

Then we have

$$\int_0^\alpha \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\alpha t^p \left[f'(t)\right]^p \left[1 - \left(\frac{t}{\alpha}\right)^{1-1/p}\right] dt.$$

Proof. To begin, an integration by parts using $\lim_{t\to 0} tf(t) = 0$ gives

$$\int_0^x f(t)dt = \left[tf(t)\right]_{t\to 0}^{t=x} - \int_0^x tf'(t)dt = xf(x) - 0 - \int_0^x tf'(t)dt$$
$$= xf(x) - \int_0^x tf'(t)dt.$$

From this, we derive

$$f(x) - \frac{1}{x}\int_0^x f(t)dt = \frac{1}{x}\int_0^x tf'(t)dt.$$

We thus can write

$$\begin{aligned}\int_0^\alpha \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]^p dx &= \int_0^\alpha \left[\frac{1}{x}\int_0^x tf'(t)dt\right]^p dx \\ &= \int_0^\alpha \frac{1}{x^p}\left[\int_0^x f_\star(t)dt\right]^p dx, \quad (2.1)\end{aligned}$$

where $f_\star(t) = tf'(t)$. Since f is non-decreasing, $f_\star$ is a non-negative function. Applying the "finite interval version" of the Hardy integral inequality to $f_\star$ as described in Equation (1.1), we get

$$\begin{aligned}\int_0^\alpha \frac{1}{x^p}\left|\int_0^x f_\star(t)dt\right|^p dx &\leq \left(\frac{p}{p-1}\right)^p \int_0^\alpha [f_\star(t)]^p \left[1 - \left(\frac{t}{\alpha}\right)^{1-1/p}\right] dt \\ &= \left(\frac{p}{p-1}\right)^p \int_0^\alpha t^p [f'(t)]^p \left[1 - \left(\frac{t}{\alpha}\right)^{1-1/p}\right] dt. \quad (2.2)\end{aligned}$$

Joining Equations (2.1) and (2.2), we obtain

$$\int_0^\alpha \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\alpha t^p [f'(t)]^p \left[1 - \left(\frac{t}{\alpha}\right)^{1-1/p}\right] dt.$$

This concludes the proof of Theorem 2.1. □

It is worth noting that, since f is non-decreasing, we have

$$\frac{1}{x}\int_0^x f(t)dt \leq f(x)\frac{1}{x}\int_0^x dt = f(x),$$

making the main integral well defined.

Furthermore, since f is non-negative, the following inequality is obviously always true:

$$\int_0^\alpha \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]^p dx \leq \int_0^\alpha f^p(x)dx.$$

In a sense, Theorem 2.1 proposes an alternative bound, optimized by the use of the "finite interval version" of the Hardy integral inequality. The derivative of f plays a key role in this bound.

It is also worth noting that we can switch to the infinite case by simply considering $\alpha \to +\infty$. The main result thus reduces to

$$\int_0^{+\infty} \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} t^p \left[f'(t)\right]^p dt.$$

This can be viewed as a complement to, or a counterpart of, the classical Hardy integral inequality.

As another immediate consequence, for any $\epsilon > 0$, using the Markov inequality, we have

$$\begin{aligned}\int_{\left\{x\in[0,\alpha);\ f(x)-(1/x)\int_0^x f(t)dt\geq\epsilon\right\}} dx &\leq \frac{1}{\epsilon^p}\int_0^{\alpha} \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]^p dx \\ &\leq \frac{1}{\epsilon^p}\left(\frac{p}{p-1}\right)^p \int_0^{\alpha} t^p \left[f'(t)\right]^p \left[1 - \left(\frac{t}{\alpha}\right)^{1-1/p}\right] dt.\end{aligned}$$

Therefore, thanks to our inequality, we can control of the measure of the set

$$\left\{x \in [0,\alpha);\ f(x) - \frac{1}{x}\int_0^x f(t)dt \geq \epsilon\right\}.$$

We would also like to mention that an alternative version of Theorem 2.1 incorporating an auxiliary convex function is presented in the appendix. This version may be of independent interest.

The rest of the section is devoted to complementary results. These are presented in the form of propositions.

2.2. **Complementary propositions.** The result below simplifies the weight in the integral of the upper bound in Theorem 2.1. This simplification comes at the cost of a modified constant factor. Note that the case $\alpha \to +\infty$ is excluded.

Proposition 2.1. *Let $\alpha > 0$, $p > 1$ and $f : [0,\alpha) \mapsto [0,+\infty)$ be a differentiable non-decreasing function such that*

$$\lim_{t\to 0} tf(t) = 0, \quad \int_0^{\alpha} t^{p+1/p-1} \left[f'(t)\right]^p dt < +\infty.$$

Then we have

$$\int_0^{\alpha} \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]^p dx \leq \frac{1}{4}\alpha^{1-1/p}\left(\frac{p}{p-1}\right)^p \int_0^{\alpha} t^{p+1/p-1} \left[f'(t)\right]^p dt.$$

Proof. Applying Theorem 2.1 and rearranging the right-hand side term, we have

$$\int_0^\alpha \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\alpha t^p \left[f'(t)\right]^p \left[1 - \left(\frac{t}{\alpha}\right)^{1-1/p}\right] dt$$

$$= \alpha^{1-1/p}\left(\frac{p}{p-1}\right)^p \int_0^\alpha t^{p+1/p-1}\left[f'(t)\right]^p k(t)dt, \tag{2.3}$$

where

$$k(t) = \left(\frac{t}{\alpha}\right)^{1-1/p}\left[1 - \left(\frac{t}{\alpha}\right)^{1-1/p}\right].$$

Since $t \in [0, \alpha]$ and $p > 1$, we have $(t/\alpha)^{1-1/p} \in [0,1]$, so that

$$0 \le k(t) \le \sup_{u\in[0,1]} u(1-u) = \frac{1}{4}. \tag{2.4}$$

Joining Equations (2.3) and (2.4), we get

$$\int_0^\alpha \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]^p dx \le \frac{1}{4}\alpha^{1-1/p}\left(\frac{p}{p-1}\right)^p \int_0^\alpha t^{p+1/p-1}\left[f'(t)\right]^p dt.$$

This concludes the proof of Proposition 2.1. □

This result is mainly of interest if α is not too large because $\lim_{\alpha\to+\infty}\alpha^{1-1/p} = +\infty$.

The proposition below emphasizes the case $p = 2$ of Theorem 2.1, leading to an original integral inequality.

Proposition 2.2. *Let $\alpha > 0$ or $\alpha \to +\infty$, and $f : [0,\alpha) \mapsto [0,+\infty)$ be a differentiable non-decreasing function such that*

$$\lim_{t\to 0} tf(t) = 0, \quad \int_0^\alpha t^2\left[f'(t)\right]^2\left[1 - \sqrt{\frac{t}{\alpha}}\right] dt < +\infty.$$

Then we have

$$\int_0^\alpha f^2(x)dx + \int_0^\alpha \frac{1}{x^2}\left[\int_0^x f(t)dt\right]^2 dx$$
$$\le 4\int_0^\alpha t^2|f'(t)|^2\left[1 - \sqrt{\frac{t}{\alpha}}\right] dt + 2\int_0^\alpha f(x)\frac{1}{x}\left[\int_0^x f(t)dt\right] dx.$$

Proof. Applying Theorem 2.1 with $p = 2$, we get

$$\int_0^\alpha \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]^2 dx \leq \left(\frac{2}{2-1}\right)^2 \int_0^\alpha t^2 \left[f'(t)\right]^2 \left[1 - \sqrt{\frac{t}{\alpha}}\right] dt.$$

Developing the left-hand side term and determining the constant factor in the right-hand side term, we obtain

$$\int_0^\alpha \left\{f^2(x) - 2f(x)\frac{1}{x}\int_0^x f(t)dt + \frac{1}{x^2}\left[\int_0^x f(t)dt\right]^2\right\} dx$$
$$\leq 4\int_0^\alpha t^2 \left[f'(t)\right]^2 \left[1 - \sqrt{\frac{t}{\alpha}}\right] dt.$$

This can be rearranged as follows:

$$\int_0^\alpha f^2(x)dx + \int_0^\alpha \frac{1}{x^2}\left[\int_0^x f(t)dt\right]^2 dx$$
$$\leq 4\int_0^\alpha t^2 |f'(t)|^2 \left[1 - \sqrt{\frac{t}{\alpha}}\right] dt + 2\int_0^\alpha f(x)\frac{1}{x}\left[\int_0^x f(t)dt\right] dx.$$

This concludes the proof of Proposition 2.2. □

A simplified version of Proposition 2.2 is given below.

Proposition 2.3. *Let* $\alpha > 0$ *and* $f : [0, \alpha) \mapsto [0, +\infty)$ *be a differentiable non-decreasing function such that*

$$\lim_{t\to 0} tf(t) = 0, \qquad \int_0^\alpha t^{3/2}\left[f'(t)\right]^2 dt < +\infty.$$

Then we have

$$\int_0^\alpha f^2(x)dx + \int_0^\alpha \frac{1}{x^2}\left[\int_0^x f(t)dt\right]^2 dx$$
$$\leq \sqrt{\alpha}\int_0^\alpha t^{3/2}|f'(t)|^2 dt + 2\int_0^\alpha f(x)\frac{1}{x}\left[\int_0^x f(t)dt\right] dx.$$

Proof. Applying Proposition 2.2 and rearranging the right-hand side term, we have

$$\int_0^\alpha f^2(x)dx + \int_0^\alpha \frac{1}{x^2}\left[\int_0^x f(t)dt\right]^2 dx$$
$$\leq 4\int_0^\alpha t^2|f'(t)|^2\left[1-\sqrt{\frac{t}{\alpha}}\right]dt + 2\int_0^\alpha f(x)\frac{1}{x}\left[\int_0^x f(t)dt\right]dx$$
$$= 4\sqrt{\alpha}\int_0^\alpha t^{3/2}|f'(t)|^2 k(t)dt + 2\int_0^\alpha f(x)\frac{1}{x}\left[\int_0^x f(t)dt\right]dx, \tag{2.5}$$

where
$$k(t) = \sqrt{\frac{t}{\alpha}}\left[1-\sqrt{\frac{t}{\alpha}}\right].$$

Since $t \in [0, \alpha]$, we have $\sqrt{t/\alpha} \in [0, 1]$, so that
$$0 \leq k(t) \leq \sup_{u\in[0,1]} u(1-u) = \frac{1}{4}. \tag{2.6}$$

Joining Equations (2.5) and (2.6), we get
$$\int_0^\alpha f^2(x)dx + \int_0^\alpha \frac{1}{x^2}\left[\int_0^x f(t)dt\right]^2 dx$$
$$\leq \sqrt{\alpha}\int_0^\alpha t^{3/2}|f'(t)|^2 dt + 2\int_0^\alpha f(x)\frac{1}{x}\left[\int_0^x f(t)dt\right]dx.$$

This concludes the proof of Proposition 2.3. □

The rest of the paper focuses on the two-function version of these results, distinguishing between the first and second types.

3. Results involving two functions

3.1. **Results of the first type.** In line with the approach taken in Theorem 2.1, the proposition below provides a tractable upper bound for an integral involving two functions and their respective mean integrals. We recall that the main integral is of the following form:
$$\int_0^\alpha \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]\left[g(x) - \frac{1}{x}\int_0^x g(t)dt\right]dx.$$

Proposition 3.1. *Let $\alpha > 0$ or $\alpha \to +\infty$, $p > 1$ and $f, g : [0, \alpha) \mapsto [0, +\infty)$ be two differentiable non-decreasing functions such that*

$$\lim_{t\to 0} tf(t) = 0, \quad \int_0^\alpha t^p \left[f'(t)\right]^p \left[1 - \left(\frac{t}{\alpha}\right)^{1-1/p}\right] dt < +\infty$$

and

$$\lim_{t\to 0} tg(t) = 0, \quad \int_0^\alpha t^{p/(p-1)} \left[g'(t)\right]^{p/(p-1)} \left[1 - \left(\frac{t}{\alpha}\right)^{1/p}\right] dt < +\infty.$$

Then we have

$$\begin{aligned} &\int_0^\alpha \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]\left[g(x) - \frac{1}{x}\int_0^x g(t)dt\right] dx \\ &\leq \frac{p^2}{p-1}\left\{\int_0^\alpha t^p \left[f'(t)\right]^p \left[1 - \left(\frac{t}{\alpha}\right)^{1-1/p}\right] dt\right\}^{1/p} \\ &\times \left\{\int_0^\alpha t^{p/(p-1)} \left[g'(t)\right]^{p/(p-1)} \left[1 - \left(\frac{t}{\alpha}\right)^{1/p}\right] dt\right\}^{1-1/p}. \end{aligned}$$

Proof. Let $q = p/(p-1) > 1$. Considering the main integrand as the product of two terms, one depending on f and the other depending on g, and applying the Hölder integral inequality, we obtain

$$\int_0^\alpha \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]\left[g(x) - \frac{1}{x}\int_0^x g(t)dt\right] dx$$

$$\leq \left\{\int_0^\alpha \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]^p dx\right\}^{1/p} \tag{3.1}$$

$$\times \left\{\int_0^\alpha \left[g(x) - \frac{1}{x}\int_0^x g(t)dt\right]^q dx\right\}^{1/q}. \tag{3.2}$$

It follows from Theorem 2.1 that

$$\int_0^\alpha \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]^p dx$$

$$\leq \left(\frac{p}{p-1}\right)^p \int_0^\alpha t^p \left[f'(t)\right]^p \left[1 - \left(\frac{t}{\alpha}\right)^{1-1/p}\right] dt. \tag{3.3}$$

In a similar way but with the function g instead of f and the parameter q instead of p, we get

$$\text{(3.4)}\quad \int_0^\alpha \left[g(x) - \frac{1}{x}\int_0^x g(t)dt\right]^q dx \le \left(\frac{q}{q-1}\right)^q \int_0^\alpha t^q \left[g'(t)\right]^q \left[1 - \left(\frac{t}{\alpha}\right)^{1-1/q}\right] dt.$$

Joining Equations (3.1), (3.3) and (3.4), and using the definition of q, we obtain

$$\begin{aligned}
&\int_0^\alpha \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]\left[g(x) - \frac{1}{x}\int_0^x g(t)dt\right] dx \\
&\le \left\{\left(\frac{p}{p-1}\right)^p \int_0^\alpha t^p \left[f'(t)\right]^p \left[1 - \left(\frac{t}{\alpha}\right)^{1-1/p}\right] dt\right\}^{1/p} \\
&\quad \times \left\{\left(\frac{q}{q-1}\right)^q \int_0^\alpha t^q \left[g'(t)\right]^q \left[1 - \left(\frac{t}{\alpha}\right)^{1-1/q}\right] dt\right\}^{1/q} \\
&= \left(\frac{p}{p-1}\right)\left(\frac{q}{q-1}\right)\left\{\int_0^\alpha t^p \left[f'(t)\right]^p \left[1 - \left(\frac{t}{\alpha}\right)^{1-1/p}\right] dt\right\}^{1/p} \\
&\quad \times \left\{\int_0^\alpha t^q \left[g'(t)\right]^q \left[1 - \left(\frac{t}{\alpha}\right)^{1-1/q}\right] dt\right\}^{1/q} \\
&= \frac{p^2}{p-1}\left\{\int_0^\alpha t^p \left[f'(t)\right]^p \left[1 - \left(\frac{t}{\alpha}\right)^{1-1/p}\right] dt\right\}^{1/p} \\
&\quad \times \left\{\int_0^\alpha t^{p/(p-1)} \left[g'(t)\right]^{p/(p-1)} \left[1 - \left(\frac{t}{\alpha}\right)^{1/p}\right] dt\right\}^{1-1/p}.
\end{aligned}$$

This ends the proof of Proposition 3.1. □

The proposition below is a simplified version of Proposition 3.1, with a different weight function for the integral of the upper bound. This comes at the cost of a wider constant factor.

Proposition 3.2. *Let $\alpha > 0$, $p > 1$ and $f, g : [0, \alpha) \mapsto [0, +\infty)$ be two differentiable non-decreasing functions such that*

$$\lim_{t\to 0} tf(t) = 0, \quad \int_0^\alpha t^{p+1/p-1} \left[f'(t)\right]^p dt < +\infty$$

and

$$\lim_{t\to 0} tg(t) = 0, \quad \int_0^\alpha t^{p/(1-p)-1/p} \left[g'(t)\right]^{p/(p-1)} dt < +\infty.$$

Then we have

$$\begin{aligned} &\int_0^\alpha \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]\left[g(x) - \frac{1}{x}\int_0^x g(t)dt\right] dx \\ &\le \frac{1}{4}\left(\frac{p^2}{p-1}\right)\alpha^{2(1-1/p)/p}\left\{\int_0^\alpha t^{p+1/p-1}\left[f'(t)\right]^p dt\right\}^{1/p} \\ &\quad \times \left\{\int_0^\alpha t^{p/(p-1)-1/p}\left[g'(t)\right]^{p/(p-1)} dt\right\}^{1-1/p}. \end{aligned}$$

Proof. Applying Proposition 3.1 and rearranging the right-hand side term, we have

$$\begin{aligned} &\int_0^\alpha \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]\left[g(x) - \frac{1}{x}\int_0^x g(t)dt\right] dx \\ &\le \frac{p^2}{p-1}\left\{\int_0^\alpha t^p \left[f'(t)\right]^p \left[1 - \left(\frac{t}{\alpha}\right)^{1-1/p}\right] dt\right\}^{1/p} \\ &\quad \times \left\{\int_0^\alpha t^{p/(p-1)}\left[g'(t)\right]^{p/(p-1)}\left[1 - \left(\frac{t}{\alpha}\right)^{1/p}\right] dt\right\}^{1-1/p} \\ &= \frac{p^2}{p-1}\left\{\alpha^{1-1/p}\int_0^\alpha t^{p+1/p-1}k(t)\left[f'(t)\right]^p dt\right\}^{1/p} \\ &\quad \times \left\{\alpha^{1/p}\int_0^\alpha t^{p/(p-1)-1/p}\ell(t)\left[g'(t)\right]^{p/(p-1)} dt\right\}^{1-1/p}, \end{aligned} \tag{3.5}$$

where

$$k(t) = \left(\frac{t}{\alpha}\right)^{1-1/p}\left[1 - \left(\frac{t}{\alpha}\right)^{1-1/p}\right], \qquad \ell(t) = \left(\frac{t}{\alpha}\right)^{1/p}\left[1 - \left(\frac{t}{\alpha}\right)^{1/p}\right].$$

Since $t \in [0, \alpha]$ and $p > 1$, we have $(t/\alpha)^{1-1/p} \in [0, 1]$ and $(t/\alpha)^{1/p} \in [0, 1]$, so that

$$\begin{aligned} 0 \le k(t) \le \sup_{u\in[0,1]} u(1-u) = \frac{1}{4}, \\ 0 \le \ell(t) \le \sup_{v\in[0,1]} v(1-v) = \frac{1}{4}. \end{aligned} \tag{3.6}$$

Joining Equations (3.5) and (3.6), we obtain

$$\begin{aligned}
&\int_0^\alpha \left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]\left[g(x) - \frac{1}{x}\int_0^x g(t)dt\right]dx \\
&\leq \frac{p^2}{p-1}\left\{\int_0^\alpha t^{p+1/p-1}\alpha^{1-1/p}\frac{1}{4}\left[f'(t)\right]^p dt\right\}^{1/p} \\
&\quad \times \left\{\int_0^\alpha t^{p/(p-1)-1/p}\alpha^{1/p}\frac{1}{4}\left[g'(t)\right]^{p/(p-1)} dt\right\}^{1-1/p} \\
&= \frac{1}{4}\left(\frac{p^2}{p-1}\right)\alpha^{2(1-1/p)/p}\left\{\int_0^\alpha t^{p+1/p-1}\left[f'(t)\right]^p dt\right\}^{1/p} \\
&\quad \times \left\{\int_0^\alpha t^{p/(p-1)-1/p}\left[g'(t)\right]^{p/(p-1)} dt\right\}^{1-1/p}.
\end{aligned}$$

This ends the proof of Proposition 3.2. □

3.2. **Results of the second type.** The main result of the second type is developed below, based on an integral of the following form:

$$\int_0^\alpha \left|f(x)g(x) - \frac{1}{x^2}\left[\int_0^x f(t)dt\right]\left[\int_0^x g(t)dt\right]\right|^p dx.$$

We emphasize the assumptions made on f and g, and the originality of the upper bound obtained.

Proposition 3.3. *Let $\alpha > 0$ or $\alpha \to +\infty$, $p > 1$ and $f, g : [0, \alpha) \mapsto [0, +\infty)$ be two differentiable non-decreasing functions such that*

$$\lim_{t\to 0} tf(t) = 0, \quad f(\alpha) < +\infty,$$
$$\int_0^\alpha t^p\left[f'(t)\right]^p\left[1 - \left(\frac{t}{\alpha}\right)^{1-1/p}\right]dt < +\infty$$

and

$$\lim_{t\to 0} tg(t) = 0, \quad g(\alpha) < +\infty,$$
$$\int_0^\alpha t^{p/(p-1)}\left[g'(t)\right]^{p/(p-1)}\left[1 - \left(\frac{t}{\alpha}\right)^{1/p}\right]dt < +\infty.$$

Then we have

$$\int_0^\alpha \left| f(x)g(x) - \frac{1}{x^2}\left[\int_0^x f(t)dt\right]\left[\int_0^x g(t)dt\right]\right|^p dx$$
$$\leq 2^{p-1}\left(\frac{p}{p-1}\right)^p \max\left[f^p(\alpha), g^p(\alpha)\right]$$
$$\times \left\{\int_0^\alpha t^p\left\{[f'(t)]^p + [g'(t)]^p\right\}\left[1 - \left(\frac{t}{\alpha}\right)^{1-1/p}\right]dt\right\}.$$

Proof. The following decomposition holds:

$$f(x)g(x) - \frac{1}{x^2}\left[\int_0^x f(t)dt\right]\left[\int_0^x g(t)dt\right]$$
$$= f(x)g(x) - g(x)\frac{1}{x}\int_0^x f(t)dt + g(x)\frac{1}{x}\int_0^x f(t)dt$$
$$- \frac{1}{x^2}\left[\int_0^x f(t)dt\right]\left[\int_0^x g(t)dt\right]$$
$$= g(x)\left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right] + \frac{1}{x}\left[\int_0^x f(t)dt\right]\left[g(x) - \frac{1}{x}\int_0^x g(t)dt\right].$$

Using this and the convexity inequality $|a+b|^p \leq 2^{p-1}[|a|^p + |b|^p]$, with $a, b \in \mathbb{R}$, we get

$$\left|f(x)g(x) - \frac{1}{x^2}\left[\int_0^x f(t)dt\right]\left[\int_0^x g(t)dt\right]\right|^p$$
$$= \left|g(x)\left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]\right.$$
$$\left. + \frac{1}{x}\left[\int_0^x f(t)dt\right]\left[g(x) - \frac{1}{x}\int_0^x g(t)dt\right]\right|^p \tag{3.7}$$
$$\leq 2^{p-1}\left\{g^p(x)\left[f(x) - \frac{1}{x}\int_0^x f(t)dt\right]^p\right.$$
$$\left. + \frac{1}{x^p}\left[\int_0^x f(t)dt\right]^p\left[g(x) - \frac{1}{x}\int_0^x g(t)dt\right]^p\right\}.$$

Since f and g are non-decreasing and $f(\alpha) < +\infty$ and $g(\alpha) < +\infty$, for any $x \in [0, \alpha)$, we have $g(x) \leq g(\alpha)$ and

$$\frac{1}{x}\int_0^x f(t)dt \leq f(x)\frac{1}{x}\int_0^x dt = f(x) \leq f(\alpha).$$

We thus derive

$$
\begin{aligned}
&2^{p-1}\left\{g^p(x)\left[f(x)-\frac{1}{x}\int_0^x f(t)dt\right]^p\right.\\
&+\left.\frac{1}{x^p}\left[\int_0^x f(t)dt\right]^p\left[g(x)-\frac{1}{x}\int_0^x g(t)dt\right]^p\right\}\\
\leq\ & 2^{p-1}\max\left[f^p(\alpha),g^p(\alpha)\right]\\
&\times\left\{\left[f(x)-\frac{1}{x}\int_0^x f(t)dt\right]^p+\left[g(x)-\frac{1}{x}\int_0^x g(t)dt\right]^p\right\}.
\end{aligned}
\tag{3.8}
$$

Joining Equations (3.7) and (3.8), we obtain

$$
\begin{aligned}
&\left|f(x)g(x)-\frac{1}{x^2}\left[\int_0^x f(t)dt\right]\left[\int_0^x g(t)dt\right]\right|^p\\
\leq\ & 2^{p-1}\max\left[f^p(\alpha),g^p(\alpha)\right]\\
&\times\left\{\left[f(x)-\frac{1}{x}\int_0^x f(t)dt\right]^p+\left[g(x)-\frac{1}{x}\int_0^x g(t)dt\right]^p\right\}.
\end{aligned}
$$

Integrating both sides with respect to x with $x\in[0,\alpha]$, we obtain

$$
\begin{aligned}
&\int_0^\alpha\left|f(x)g(x)-\frac{1}{x^2}\left[\int_0^x f(t)dt\right]\left[\int_0^x g(t)dt\right]\right|^p dx\\
\leq\ & 2^{p-1}\max\left[f^p(\alpha),g^p(\alpha)\right]\\
&\times\left\{\int_0^\alpha\left[f(x)-\frac{1}{x}\int_0^x f(t)dt\right]^p dx+\int_0^\alpha\left[g(x)-\frac{1}{x}\int_0^x g(t)dt\right]^p dx\right\}.
\end{aligned}
\tag{3.9}
$$

Applying Theorem 2.1 to f and g, we have

$$
\begin{aligned}
&\left\{\int_0^\alpha\left[f(x)-\frac{1}{x}\int_0^x f(t)dt\right]^p dx+\int_0^\alpha\left[g(x)-\frac{1}{x}\int_0^x g(t)dt\right]^p dx\right\}\\
\leq\ & \left(\frac{p}{p-1}\right)^p\int_0^\alpha t^p\left[f'(t)\right]^p\left[1-\left(\frac{t}{\alpha}\right)^{1-1/p}\right]dt\\
&+\left(\frac{p}{p-1}\right)^p\int_0^\alpha t^p\left[g'(t)\right]^p\left[1-\left(\frac{t}{\alpha}\right)^{1-1/p}\right]dt\\
=\ & \left(\frac{p}{p-1}\right)^p\int_0^\alpha t^p\left\{\left[f'(t)\right]^p+\left[g'(t)\right]^p\right\}\left[1-\left(\frac{t}{\alpha}\right)^{1-1/p}\right]dt.
\end{aligned}
\tag{3.10}
$$

Joining Equations (3.9) and (3.10), we obtain

$$\int_0^\alpha \left| f(x)g(x) - \frac{1}{x^2}\left[\int_0^x f(t)dt\right]\left[\int_0^x g(t)dt\right]\right|^p dx$$
$$\leq 2^{p-1}\left(\frac{p}{p-1}\right)^p \max\left[f^p(\alpha), g^p(\alpha)\right] \times$$
$$\left\{\int_0^\alpha t^p \left\{[f'(t)]^p + [g'(t)]^p\right\}\left[1-\left(\frac{t}{\alpha}\right)^{1-1/p}\right] dt\right\}.$$

This ends the proof of Proposition 3.3. □

The proposition below is a simplified version of Proposition 3.3, with modifications to the weight function in the upper-bound integral and another constant factor.

Proposition 3.4. *Let $\alpha > 0$, $p > 1$ and $f, g : [0,\alpha) \mapsto [0,+\infty)$ be two differentiable non-decreasing functions such that*

$$\lim_{t\to 0} tf(t) = 0, \quad f(\alpha) < +\infty, \quad \int_0^\alpha t^{p+1/p-1}\left[f'(t)\right]^p dt < +\infty$$

and

$$\lim_{t\to 0} tg(t) = 0, \quad g(\alpha) < +\infty, \quad \int_0^\alpha t^{p/(p-1)-1/p}\left[g'(t)\right]^{p/(p-1)} dt < +\infty.$$

Then we have

$$\int_0^\alpha \left| f(x)g(x) - \frac{1}{x^2}\left[\int_0^x f(t)dt\right]\left[\int_0^x g(t)dt\right]\right|^p dx$$
$$\leq 2^{p-3}\left(\frac{p}{p-1}\right)^p \alpha^{1-1/p} \max\left[f^p(\alpha), g^p(\alpha)\right] \times$$
$$\left\{\int_0^\alpha t^{p+1/p-1}\left\{[f'(t)]^p + [g'(t)]^p\right\} dt\right\}.$$

Proof. Applying Proposition 3.3 and rearranging the right-hand side term, we have

$$\int_0^\alpha \left| f(x)g(x) - \frac{1}{x^2}\left[\int_0^x f(t)dt\right]\left[\int_0^x g(t)dt\right]\right|^p dx$$

$$\leq 2^{p-1}\left(\frac{p}{p-1}\right)^p \max\left[f^p(\alpha), g^p(\alpha)\right]$$
$$\times \left\{\int_0^\alpha t^p \left\{[f'(t)]^p + [g'(t)]^p\right\}\left[1-\left(\frac{t}{\alpha}\right)^{1-1/p}\right] dt\right\}$$
$$= 2^{p-1}\left(\frac{p}{p-1}\right)^p \max\left[f^p(\alpha), g^p(\alpha)\right]$$
$$\times \left\{\alpha^{1-1/p}\int_0^\alpha t^{p+1/p-1}\left[[f'(t)]^p + [g'(t)]^p\right] k(t)dt\right\}, \tag{3.11}$$

where

$$k(t) = \left(\frac{t}{\alpha}\right)^{1-1/p}\left[1-\left(\frac{t}{\alpha}\right)^{1-1/p}\right].$$

Since $t \in [0,\alpha]$, we have $(t/\alpha)^{1-1/p} \in [0,1]$, so that

$$0 \leq k(t) \leq \sup_{u\in[0,1]} u(1-u) = \frac{1}{4}. \tag{3.12}$$

Joining Equations (3.11) and (3.12), we get

$$\int_0^\alpha \left| f(x)g(x) - \frac{1}{x^2}\left[\int_0^x f(t)dt\right]\left[\int_0^x g(t)dt\right]\right|^p dx$$
$$\leq 2^{p-1}\left(\frac{p}{p-1}\right)^p \max\left[f^p(\alpha), g^p(\alpha)\right]$$
$$\times \left\{\alpha^{1-1/p}\int_0^\alpha t^{p+1/p-1}\left[[f'(t)]^p + [g'(t)]^p\right]\frac{1}{4}dt\right\}$$
$$= 2^{p-3}\left(\frac{p}{p-1}\right)^p \alpha^{1-1/p}\max\left[f^p(\alpha), g^p(\alpha)\right]$$
$$\times \left\{\int_0^\alpha t^{p+1/p-1}\left\{[f'(t)]^p + [g'(t)]^p\right\} dt\right\}.$$

This ends the proof of Proposition 3.4. □

4. Conclusion

In this paper, we have established new variants of the Hardy-type inequality for an integral that measure the dispersion of a function around its integral mean. By imposing additional monotonicity and integrability conditions, we derived sharp upper bounds for this integral. We also introduced two-function generalizations that address additive and multiplicative interactions between the functions and their integral means. The resulting inequalities, which we termed the first and second types, have applications in operator theory, functional analysis, and the study of integral transforms.

Future work may explore optimality conditions and possible extensions to higher dimensions, as well as applications to partial differential equations and harmonic analysis. This research has the potential to enhance the theoretical foundation and practical application of Hardy-type integral inequalities in contemporary mathematical analysis.

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APPENDIX

This appendix presents a general integral inequality inspired by Theorem 2.1. The main novelty lies in the use of a convex function that goes beyond the power function type.

Proposition 4.1. *Let $\alpha > 0$ or $\alpha \to +\infty$, $f : [0, \alpha) \mapsto \mathbb{R}$ be a differentiable function such that*

$$\lim_{t \to 0} t f(t) = 0$$

and $\phi : \mathbb{R} \mapsto [0, +\infty)$ be a convex function with $\phi(0) = 0$. We assume that

$$\int_0^\alpha \phi\left(\alpha f'(t)\right)\left(1 - \frac{t}{\alpha}\right) dt < +\infty.$$

Then we have

$$\int_0^\alpha \phi\left(f(x) - \frac{1}{x}\int_0^x f(t)dt\right)\frac{1}{x}dx \leq \frac{1}{\alpha}\int_0^\alpha \phi\left(\alpha f'(t)\right)\left(1 - \frac{t}{\alpha}\right) dt.$$

Furthermore, if $\alpha \in (0, 1]$, assuming that

$$\int_0^\alpha \phi\left(f'(t)\right)\left(1 - \frac{t}{\alpha}\right) dt < +\infty,$$

then we have

$$\int_0^{\alpha} \phi\left(f(x) - \frac{1}{x}\int_0^x f(t)dt\right)\frac{1}{x}dx \le \int_0^{\alpha} \phi\left(f'(t)\right)\left(1 - \frac{t}{\alpha}\right)dt.$$

Proof. Using an integration by parts and $\lim_{t\to 0} tf(t) = 0$, we obtain

$$\begin{aligned}\int_0^x f(t)dt &= [tf(t)]_{t\to 0}^{t=x} - \int_0^x tf'(t)dt \\ &= xf(x) - 0 - \int_0^x tf'(t)dt \\ &= xf(x) - \int_0^x tf'(t)dt.\end{aligned}$$

We therefore have

$$f(x) - \frac{1}{x}\int_0^x f(t)dt = \frac{1}{x}\int_0^x tf'(t)dt$$

and

$$\int_0^{\alpha} \phi\left(f(x) - \frac{1}{x}\int_0^x f(t)dt\right)\frac{1}{x}dx = \int_0^{\alpha} \phi\left(\frac{1}{x}\int_0^x tf'(t)dt\right)\frac{1}{x}dx. \tag{4.1}$$

Applying the Jensen integral inequality, we obtain

$$\begin{aligned}\int_0^{\alpha} \phi\left(\frac{1}{x}\int_0^x tf'(t)dt\right)\frac{1}{x}dx &\le \int_0^{\alpha}\left(\frac{1}{x}\int_0^x \phi\left(tf'(t)\right)dt\right)\frac{1}{x}dx \\ &= \int_0^{\alpha}\int_0^x \phi\left(tf'(t)\right)\frac{1}{x^2}dtdx. \end{aligned}\tag{4.2}$$

Changing the order of integration by the Fubini-Tonelli integral theorem, the integrand being non-negative so of constant sign, we get

$$\begin{aligned}&\int_0^{\alpha}\int_0^x \phi\left(tf'(t)\right)\frac{1}{x^2}dtdx = \int_0^{\alpha}\int_t^{\alpha} \phi\left(tf'(t)\right)\frac{1}{x^2}dxdt \\ &= \int_0^{\alpha}\phi\left(tf'(t)\right)\left(\int_t^{\alpha}\frac{1}{x^2}dx\right)dt = \int_0^{\alpha}\phi\left(tf'(t)\right)\left(\frac{1}{t} - \frac{1}{\alpha}\right)dt \\ &= \int_0^{\alpha}\phi\left(tf'(t)\right)\left(1 - \frac{t}{\alpha}\right)\frac{1}{t}dt.\end{aligned}\tag{4.3}$$

Since $t \in [0, \alpha]$, we have $t/\alpha \in [0, 1]$, so that the inequality of convexity satisfied by ϕ and $\phi(0) = 0$ give

$$\begin{aligned}
&\int_0^\alpha \phi\left(tf'(t)\right)\left(1-\frac{t}{\alpha}\right)\frac{1}{t}dt\\
&= \int_0^\alpha \phi\left(\frac{t}{\alpha}\alpha f'(t)+\left(1-\frac{t}{\alpha}\right)\times 0\right)\left(1-\frac{t}{\alpha}\right)\frac{1}{t}dt\\
&\le \int_0^\alpha \left(\frac{t}{\alpha}\phi\left(\alpha f'(t)\right)+\left(1-\frac{t}{\alpha}\right)\phi(0)\right)\left(1-\frac{t}{\alpha}\right)\frac{1}{t}dt\\
&= \frac{1}{\alpha}\int_0^\alpha \phi\left(\alpha f'(t)\right)\left(1-\frac{t}{\alpha}\right)dt. \qquad (4.4)
\end{aligned}$$

Joining Equations (4.1), (4.2), (4.3) and (4.4), we obtain

$$\int_0^\alpha \phi\left(f(x)-\frac{1}{x}\int_0^x f(t)dt\right)\frac{1}{x}dx \le \frac{1}{\alpha}\int_0^\alpha \phi\left(\alpha f'(t)\right)\left(1-\frac{t}{\alpha}\right)dt.$$

The main result is established.

Furthermore, if $\alpha \in (0,1]$, based on the previous result, and using the inequality of convexity satisfied by ϕ and $\phi(0)=0$, we have

$$\begin{aligned}
&\int_0^\alpha \phi\left(f(x)-\frac{1}{x}\int_0^x f(t)dt\right)\frac{1}{x}dx\\
&\le \frac{1}{\alpha}\int_0^\alpha \phi\left(\alpha f'(t)\right)\left(1-\frac{t}{\alpha}\right)dt\\
&= \frac{1}{\alpha}\int_0^\alpha \phi\left(\alpha f'(t)+(1-\alpha)\times 0\right)\left(1-\frac{t}{\alpha}\right)dt\\
&\le \frac{1}{\alpha}\int_0^\alpha \left(\alpha\phi\left(f'(t)\right)+(1-\alpha)\phi(0)\right)\left(1-\frac{t}{\alpha}\right)dt\\
&= \int_0^\alpha \phi\left(f'(t)\right)\left(1-\frac{t}{\alpha}\right)dt.
\end{aligned}$$

The secondary result is obtained, ending the proof. □

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