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SEARCHING OF INFINITE PERFECT NUMBERS

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ABSTRACT. It is well known that only 51 perfect numbers have been discovered in number system. The present study gives an overview of existence of infinite perfect numbers with the help of a theoretical investigation. In this the investigation, it is found that all infinite perfect numbers belong to a special type of set. Some properties, propositions and important results have also been discussed.

1. INTRODUCTION

Perfect number is a positive integer which is equal to the sum of all proper positive divisors except the number itself. This number can be considered as a sophisticated number in Number Theory only because of the fact that there are only some known perfect numbers. It is found that the form of even perfect numbers is $2^{P-1}(2^P-1)$ for prime $p \geq 3$, when 2^P-1 is primes. There is no odd perfect number in number system. It is not known till now how many perfect numbers there are in number theory. It is found that the total number of perfect numbers, obtained so far, has been found by many scientists of the world applying some computer languages or developing some algorithms, according to their thoughts and idea. There is no general formula or rule to know all perfect numbers. It has been

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found that there are, till today, 51 known perfect number discovered by Mathematicians writing some program or using some techniques without any theoretical proof. There is no any formula to know whether there are infinite perfect numbers or not. To study this type of question, two sets have been conjectured in 2006 by Kalita [2], where one set contains all infinite perfect number and other set does not have any perfect number. These two infinite sets are namely, $A = \{4n + 10/n \geq 4\}$ and $B = \{4n + 12/\text{for some particular values of } n \geq 4\}$ [2]. It is interesting to note that the set of all even number ≥ 26 can be found from the union of these two infinite sets $A = \{4n + 10/n \geq 4\}$ and $B = \{4n + 12/n \geq 4\}$ [1], [2], [3], [4].Some important properties and results have been forwarded by Kalita [2], [8], [10] for the set $= \{4n + 10/n \geq 4\}$. A new direction of finding all consecutive prime number has been conjectured and discussed which was stated that if the number $4n + 10$ for $n \geq 4$ is divided by 2, when the number $4n + 10$ is not the multiple of $3, 5, 7$ and 11, then one can find out the consecutive primes numbers. For example, when we put the values of $n = 4, 6, 7, 9, 12, 13, 16, 18, 19, 21 \ldots\ldots$. in $4n + 10$, then the number $4n + 10$ is not multiple of $3, 5, 7,$ and 11 and therefore the consecutive prime numbers $13, 17, 19, 23, 29, 31, 37, 41, 43, 47 \ldots\ldots$ are obtained. The most important conjecture has been proposed by Kalita [2] that all perfect numbers lie in the set $B = \{4n + 12/\text{for some particular values of } n \geq 4\}$.

The eight different particular values of n when we put in B which are $n = 4$, $n = 121$, $n = 2029$, $n = 8387581$, $n = 2147467261$, $n = 34359672829$, $n = 576460752034988029$, $n = 664613997892457936163673153838460541$, the different eight consecutive perfect numbers [except 6] are obtained which are already known and they are found as 28, 496, 8128, 33550336, 8589869056, 137438691328, 2305843008139952128336, $658455991569831744654692615353842176$. It has been examined and found that all perfect numbers already discovered till now lie in the set $B = \{4n + 12/ \text{ for some particular values of } n \geq 4\}$. Hence it is necessary to find whether there exist infinite perfect numbers or not, and if exist they lie in the set $B = \{4n + 12/ \text{ for some particular values of } n \geq 4\}$. We know that there are some conjectural properties in number theory and some already have been proved. But it is observed that when some procedures or explanations for the proof of number theory results have been considered, then some new problems or relation occur. Hence number theory can be considered as a never ending theory. It is well known to the number theorist that the triple numbers which are related with Pythagorean triples such tat $c^2 = a^2 + b^2$, where a, b, c are integers and the form of these type of numbers is found as (a, b, c). Till now no one discussed about the relationship of triple numbers with the perfect numbers. Hence it is very important to discuss how a perfect numbers are related with the triple numbers. In addition to this, it is found that the application of graph theoretical concept has been discussed by Kalita [5], [6], [7], [8] for the proof of Gold Bach Conjecture which states that every even number numbers larger than 2 can be expressed as a sum of two primes. Kalita used some new definitions such that $PVEEWG$, $BKSTPVEEWG$, $CPVEEWG$, $EEWE$ and forwarded some theoretical explanations in his research findings. Besides, Kalita, B. etal [6] has been forwarded the utilization of consecutive even number finding graph $(CENFG)$ to prove the Gold Bach Conjecture. Kalita, B., [3] has also been discussed some properties of the two sets $A = \{4n + 10/n \geq 4\}$ and $B = \{4n + 12/n \geq 4\}$. He forwarded that the numbers of the form $i^3 + 2i$ for $i \geq 2j$ for $j \geq 2$ and $i^4 - 4i^2$ for $i \geq 2j$ for $j \geq 2$ are always lie in the set $\{4n + 12/n \geq 4\}$ but they are not perfect numbers. Various properties of graphs, obtained from the graphical partition of even numbers have been discussed by Kalita, B. [7] [9]. The total number of simple Hamiltonian graphs of vertex $n \geq 6$ for the graphical partition of the even number k where $2i + 10 \leq k \leq m(m + 1)$ for $i \geq 1$, with simultaneous changes of $m \geq 5$ and $n \geq 6$ has been found as $(n^2 - 3n + 2)/2$.

In this paper, a theoretical explanation for existence of infinite perfect numbers which are lying in the set $B = \{4n + 12/$ for some particular values of $n \geq 4\}$ has been forwarded. The application of $CPVEEWG$ has been considered in theoretical explanation. Besides, some new results, propositions and properties of perfect numbers have also been discussed.

The paper is organized as follows: In section 1, the introduction of different works done on perfect numbers and other relation of even numbers have been explained. Section 2 is considered for some definitions related to perfect numbers. In section 3, some propositions have been proposed. Section 4 includes new results. Section 5 explains the theorem related to perfect numbers and conclusion is included in section 6.

2. Some Definitions

Definition 2.1. *The Divisor function $\sigma(n) = 2n$ where n is any perfect number.*

Definition 2.2. *The square root of any perfect numbers is not an integer value.*

Definition 2.3. *The numbers of the form $i^3 + 2i$ for $i \geq 2j$ and $j \geq 2$ and $i^4 - 4i^2$ for $i \geq 2j$ and $j \geq 2$ lie in the set $\{4n + 12/$ for some particular values of $n \geq 4$ but they are not perfect numbers.*

Definition 2.4. *Sum of proper divisor of a perfect number except the number it is equal to the number.*

Definition 2.5. *The set $H = \{2n + 27/n \geq 1\}$ is infinite set.*

Definition 2.6. *The set $K = \{4n + 26/n \geq 1\}$ is infinite set. These two sets are disjoint set.*

The infinite sets 2.5 and 2.6 will be considered when the existence of infinite perfect numbers has been discussed in theoretical nature in section 5.

3. Propositions

The following propositions 3.1 and 3.2 have been considered for perfect numbers and non-perfect numbers and other propositions 3.3, 3.4, 3.5 and 3.6 have been considered only for perfect numbers.

Proposition 3.1. $\sum(1/d_i) = (M-1)/M$*, where d_i's are divisors of sufficiently large perfect number M.*

Proposition 3.2. $\sum(1/d_i) \neq (M-1)/M$ *if M is not perfect number.*

Proposition 3.3. *In every sufficiently large perfect number M there is only one odd divisor except 1.It is observed that if d is an odd divisor of M , then nd is also divisor of M greater than d where n =2, 4, 8, 16, 32, 64. . . .*

Few verification of proposition 3.3: For the perfect number M = **28**, there is only one odd divisor of 28 which is equal to d=7 and other divisor greater than d=7 is $2d$=14 [for n=2].

For the perfect number $M = \mathbf{496}$, there is only one odd divisor of 496 which is equal to d=31 and other divisors greater than d=31 are $2d$=62, $4d$=124, $8d$=248 [for n=2, 4, 8].

For the perfect number $M = \mathbf{8128}$, there is only one odd divisor of 8128 which is equal to d=127 and other divisors greater than d=127 are $2d$=254, $4d$=508, $8d$=1016, $16d$=2032, $32d$=4064 [for n=2, 4, 8, 16, 32].

For the perfect number $M = \mathbf{33550336}$, there is only one odd divisor of 33550336, which is equal to d=8191 and hence other divisors greater than d=8191 are found as $2d$=16382, $4d$=32764, $8d$=65528, $16d$=131056, $32d$=262112, $64d$=524224, $128d$=1048448, $256d$=2096896, $512d$=4193792, $1024d$=8387584, $2048d$=16775168 [for n =2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048].

For the perfect number $M = \mathbf{8589869056}$, there is only one odd divisor of 8589869056 which is equal to d=131071 and other divisors greater than d = 131071 are $2d$ = 262142, $4d$ = 524284, $8d$ = 1048568, $16d$ = 2097136, $32d$ = 4194272, $64d$ = 8388544, $128d$ = 16777088, $256d$ = 33554176, $512d$ = 67108352, $1024d$ = 134216704, $2048d$ = 268433408, $4096d$ = 536866816, $8192d$ = 1073733632, $16384d$ = 2147467264, $32768d$ = 4294934528, $65535d$ = 8589869056 [for n=2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, 65535].

Proposition 3.4. *In every sufficiently large perfect number M, the other divisor before d = odd divisor are all even divisor (ed) of M which start from $ed = 2$ and other divisors $n(ed)$ are obtained for the values of n=2, 4, 8, 16, 32..... before d = odd divisor.*

Few verification of proposition 3.4: For the perfect number $M = \mathbf{28}$, the divisor of 28 before d = 7 are ed = 2 and $2ed$ = 4 [n = 2]

For the perfect number $M = \mathbf{496}$, there is only one odd divisor of 496 which is equal to d=31 and other divisors greater than d=31 are $2d$=62, $4d$=124, $8d$=248 [for n=2, 4, 8].

For the perfect number $M = \mathbf{8128}$, the divisors of 8128 before d=127 are ed=2, $2ed$=4, $4ed$=8, $8ed$=16, $16ed$=32, $32ed$=64 [for n=2, 4, 8, 16, 32].

For the perfect number $M = \mathbf{33550336}$, the divisors of 33550336 before d=8191 are ed=2, $2ed$=4, $4ed$=8, $8ed$=16, $16ed$=32, $32ed$=64, $64ed$=128,

$128ed$=256, $256ed$=512, $512ed$=1024, $1024ed$=2048, $2048ed$= 4096 [for n=2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048].

For the perfect number $M = \mathbf{8589869056}$, the divisors of 8589869056 before ed=131071 are ed=2, $2ed$=4, $4ed$=8, $8ed$=16, $16ed$=32, $32ed$=64, $64ed$=128, $128ed$=256, $256ed$=512, $512ed$=1024, $1024ed$=2048, $2048ed$=4096, $4096ed$=8192, $8192ed$=16384, $16384ed$=32768, $32768ed$=65536[for n=2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768].

Hence from the proposition 3.3 and 3.4, all the divisors of the perfect numbers 28, 496, 8128, 33550336 and 8589869056 are obtained and it is applicable for other known perfect numbers till now.

Proposition 3.5. *The numbers, obtained from the infinite sets whose forms are of the type* $\{3n/n \geq 1\}$, $\{5n/n \geq 1\}$, $\{11n/n \geq 1\}$, $\{13n/ \geq 1\}$, $\{17n/ \geq 1\}$, $\{19n/n \geq 1\}$, $\{23n/n \geq 1\}$, $\{29n/n \geq 1\}$, $\{31n/ \geq 1\}$, $\{37n/n \geq 1\}$, $\{41n/n \geq 1\}$, $\{43n/n \geq 1\}$, *. . . are not proper divisors of any perfect number and these sets will be used later in the* **Theorem 5.**

Remark 1: There is only one infinite set $\{31n/n \geq 1\}$ in Proposition 3.5 which contains only one divisor of the perfect number 496 when n=1.

Proposition 3.6. *This proposition is related with the triple numbers. We know that the form of triple number is found as* (a, b, c) *where the values of* a,b *and* c *are integer, where* $c^2 = a^2 + b^2$. *The triple number of the form* $(3n, 4n, 5n)$ *where* $a = 3$, $b = 4$ *and* $c = 5$ *gives infinite triple numbers for* $n \geq 1$. *But it is interesting to note that this type of triple numbers contain perfect number for some particular values of* $n \geq 1$. *That is, the part* $4n$ *of the triple number* $(3n, 4n, 5n)$ *is a perfect number for some particular values of* $n \geq 1$. *For example the following triple numbers contain perfect numbers for different values of* $n \geq 7$. *The triple number (21, 28, 35) for* n*=7, (372, 496, 620) for* n*=124, (6096, 8128, 10160) for* n*=2032, (25162752, 33550336, 41937920) for* n*=8387584, (6442401792, 8589869056, 10737336320) for* n*=214767264 have been found from the part* $4n$ *as stated above. Hence if* M *is perfect number, then the form of the triple number is* $(3M/4, M, 5M/4)$. *But for other triple numbers obtained from any triple number there does not have perfect number of the above form. Here we see that the three recurrence relations of the triple number of the form* (a_n, b_n, c_n) *exist, where* $a_n = a_{n-1} + 3$, $b_n = b_{n-1} + 4$ *and* $c_n = c_{n-1} + 5$ *are recurrence* *relation with initial conditions* $a_0 = 21$*,* $b_0 = 28$ *and* $c_0 = 35$ *have been existed. Hence forming characteristic equation and solving the recurrence relation an, one can have a homogeneous solution as* $a_n = A\{(1+(13)^{1/2})/2\}^n + B\{(1-(13)^{1/2})/2\}^n$ *and particular solution for initial condition* $a_0 = 21$ *as* $a_n = [\{21(13)^{1/2}+27\}/2(13)^{1/2}][\{1+(13)^{1/2}/2\}]^n + [\{21(13)^{1/2}-27]\}/2(13)^{1/2}][\{(1-(13)^{1/2}\}/2]^n$ *. Similarly we can find the particular solution of linear recurrence relation of* b_n *where* $b_n = [\{14(17)^{1/2}+18\}/(17)^{1/2}][\{1+(17)^{1/2}\}/2]^n + [\{14(17)^{1/2}-18\}/(17)^{1/2}][\{1-(17)^{1/2}\}/2]^n$ *and similarly for* c_n *. Hence for* $n \geq 1$*, the recurrence relation of the above form always has homogeneous and particular solution with the initial condition as stated. Putting the particular values of* $n \geq 1$ *one can find values of* a_n*,* b_n*,* c_n *and similarly can find a perfect number.*

4. Some New Results for perfect numbers

4.1. **Result.** It is true that if M is sufficiently large perfect number [discussed later in **Theorem 5**] and M is even, then we have,

$$M/p_i = k_i \text{ for } i \geq 1$$

[For i=1, p_1=2, then k_1=14 and p_i's and k_i's are divisors of M]

$$\begin{aligned}
\text{Hence, } \sum p_i &= \sum M/k_i \text{ for } 1 \leq p_i \leq \alpha \text{ and} \\
2\sum p_i &= 2\sum M/k_i \\
2\sum p_i &= 2M\sum(1/k_i) = \sigma M/k_i \\
&= \sigma M\{1/k_1 + 1/k_2 + 1/k_3 + \ldots + 1/k_n\} \\
(2\sum p_i)/\sigma M &= \{1/k_1 + 1/k_2 + 1/k_3 + \ldots + 1/k_n\} \\
2(M-1)/\sigma M &= \{(k_2.k_3.k_4 \ldots k_n) + (k_1.k_3.k_4 \ldots k_n) \\
&\quad + \ldots (k_1.k_2.k_3 \ldots k_{n-1})\}/(k_1.k_2.k_3 \ldots k_n)\} \\
\lim_{M\to\alpha}\{2(M-1)/\sigma M\} &= \lim_{k_i\to\alpha}\{(k_2.k_3.k_4 \ldots k_n) + (k_1.k_3.k_4 \ldots k_n) \\
&\quad + \ldots (k_1.k_2.k_3 \ldots k_{n-1})\}/(k_1.k_2.k_3 \ldots k_n)\} \\
1 &= X, \text{ Where,}
\end{aligned}$$

$$X = \lim_{k_i \to \alpha} \{(k_2.k_3.k_4 \dots k_n) + (k_1.k_3.k_4 \dots k_n) \\ + \dots (k_1.k_2.k_3 \dots k_{n-1})\}/(k_1.k_2.k_3 \dots k_n)\}$$

4.2. **Result (New Rule for finding odd Divisor).** The divisors of the perfect number M can be divided into two parts [Proposition 3.3 and 3.4]. One part starts from the even number = even divisor = ed = 2 and other part starts from an odd numbers = odd divisor = d. That is if ed = 2 is a divisor of M then $2ed$, $4ed$, $8ed$, $16ed$,... are also even divisors of M before odd divisor = d obtained from the proposition 3.4 and from proposition 3.3 if d = odd divisor of M other than ed = 2, then $2d$, $4d$, $8d$, $16d$, $32d$, $64d$,... are also divisors of M. Hence for a perfect number M the divisors of M can be evaluated in two ways. We can find first the odd divisor of M and there after even divisor ed before odd divisor as explained above and all together all divisors will be calculated. Hence if M is a sufficiently large perfect number then from the definition of perfect number we have,

$$M = 1 + ed + 2ed + 4ed + 8ed + 16ed + \dots + d + 2d + 4d + 8d + 16d + \dots$$

Hence,

$$\begin{aligned} M - 1 &= ed + 2ed + 4ed + 8ed + 16ed + \dots + d + 2d + 4d + 8d + 16d + \dots \\ &= ed(1 + 2 + 4 + 8 + 16 + \dots) + d(1 + 2 + 4 + 8 + 16 + \dots) \\ &= (ed + d)(1 + 2 + 4 + 8 + 16 + \dots) \\ &= (2 + d)(1 + 2 + 4 + 8 + 16 + \dots), ed = 2 \\ &= (2 + d)(2^n - 1) \\ \implies \frac{(M-1)}{(2^n - 1)} &= 2 + d \end{aligned}$$

Let denote the above relation with (A). Hence, from the relation (A), for sufficiently large perfect number M and for some particular values of $n \geq 2$, immediately one can find out the odd divisors d of M. After obtaining the odd divisor, other divisors can be obtained for any perfect number as defined above.

Verification of relation (A) considering some known perfect numbers:

(1) When n=2, then from A, we have d=7 as, $\frac{(28-1)}{3} = 9 = 2 + d \implies d = 7$ for $M = 28$.

(2) When n=4, then from A, we have d=31 as, $\frac{(496-1)}{15} = 33 = 2 + d \implies d = 31$ for $M = 496$.
(3) When n=6, then from A, we have d=127 as, $\frac{(8128-1)}{63} = 129 = 2 + d \implies d = 127$ for $M = 8128$.
(4) When n=12, then from A ,we have d=8191 as, $\frac{(33550336-1)}{4095} = 8193 = 2 + d \implies d = 8191$ for $M = 33550336$.
(5) When n=16, then from A , we have d=131071 as, $\frac{(8589869056-1)}{65535} = 131073 = 2 + d \implies d = 131071$ for $M = 8589869056$.

5. The Theorem

There are sufficiently large perfect numbers in number system and they lie in the set $B = \{4n + 12/$ for some particular values of $n \geq 4\}$

Proof of Theorem: Let M be any sufficiently large perfect number $\leq \infty$ obtained from a particular large values of n from the set $\{4n + 12/n \geq 4\}$.That is $28 \leq M \leq \infty$ [our perfect number starts from 28 and 6 is not considered here]. Let d_1, d_2, d_3, d_4, $d_5, \ldots, d_n$ be the positive divisors of M less than M. From the definition of perfect number we know that $1+d_1+d_2+d_3+\cdots+d_n = M$. As M is a sufficiently large perfect number and according to definition of perfect number we have,

$$M = 1 + d_1 + d_2 + d_3 + \ldots + d_n$$

$$1 + M/d_1 + M/d_2 + M/d_3 + \ldots M/d_n = M$$

$$M/d_1 + M/d_2 + M/d_3 + \ldots M/d_n = M - 1$$

$$M(1/d_1 + 1/d_2 + 1/d_3 + \ldots 1/d_n) = M - 1$$

$$(1/d_1 + 1/d_2 + 1/d_3 + \ldots 1/d_n) = (M - 1)/M.$$

Let denote the above relation with (B).

But from the relation (B), it is interesting to see that the right hand side of (B) that is $(M - 1)/M$, and left hand side will have equal value when M is perfect number and will not have equal value when M is not perfect number. [For one example, if we take M = 28 ,then $28 - 1/28 = 27/28$ and $1/2 + 1/4 + 1/7 + 1/14 = 54/56$ and the both sides give the same/equal value]. Hence for large values of M, the quantity $M - 1/M$ is true for the values of divisors di for $i \geq 1$. We are now going to search other relation of a perfect number.

It is true that d_1 divides $(d_2.d_3.d_4\ldots d_n)$ and similarly we have, d_2 divides $(d_1.d_3.d_4\ldots d_n)$, d_3 divides $(d_1.d_2.d_4\ldots d_n)$, d_4 divides $(d_1.d_2.d_3\ldots d_n),\ldots,d_{n-1}$ divides $(d_1.d_2.d_3\ldots d_n)$, d_n divides $(d_1.d_2.d_3\ldots d_{n-1})$.

Hence, $(d_1 + d_2 + d_3 + \ldots + d_n)$ divides $(d_2.d_3.d_4\ldots d_n)$ + $(d_1.d_3.d_4\ldots d_n)$ + $(d_1.d_2.d_4\ldots d_n)$ + $(d_1.d_2.d_3\ldots d_n)$ + …+ $(d_1.d_2.d_3\ldots d_{n-1})\ldots$ designates that $M-1$ divides $\{(d_2.d_3.d_4\ldots d_n)$ + $(d_1.d_3.d_4\ldots d_n)$ + $(d_1.d_2.d_4\ldots d_n)$ + $(d_1.d_2.d_3\ldots d_n)$ +… + $(d_1.d_2.d_3\ldots d_{n-1})\}$, which is true.

Again from the relation (B), we have,

$$(1/d_1 + 1/d_2 + 1/d_3 + \ldots 1/d_n) = (M-1)/M$$

$$\Longrightarrow \{(d_2.d_3\ldots d_n) + (d_1.d_3.d_4\ldots d_n) + (d_1.d_2.d_4\ldots d_n) + \ldots + (d_1.d_2.d_3\ldots d_n) + (d_1.d_2.d_3\ldots d_{n-1})\}/(d_1.d_2.d_3\ldots d_n) = M - 1/M$$

$$\Longrightarrow (M-1)\{(d_1.d_2.d_3\ldots d_n)\} = M\{(d_2.d_3\ldots d_n) + (d_1.d_3.d_4\ldots d_n) + (d_1.d_2.d_4\ldots d_n) + (d_1.d_2.d_3\ldots d_n) + (d_1.d_2.d_3\ldots d_{n-1})\}$$

$$\Longrightarrow M\{(d_1.d_2.d_3\ldots d_n) - (d_2.d_3\ldots d_n) - (d_1.d_3.d_4\ldots d_n) - (d_1.d_2.d_3\ldots d_n) - (d_1.d_2.d_3\ldots d_{n-1})\} = (d_1.d_2.d_3\ldots d_n)$$

$$\Longrightarrow M = (d_1.d_2.d_3\ldots d_n)/\{(d_1.d_3\ldots d_n) - (d_2.d_3\ldots d_n) - (d_1.d_3.d_4\ldots d_n) - (d_1.d_2.d_3\ldots d_n) - (d_1.d_2.d_3\ldots d_{n-1})\}$$

Let denote the above relation with (C).

Hence this relation (C) gives the values of M which is sufficiently large, which shows that the expression of M expressed in (C) can be considered as other new expression of perfect number. Hence when M is a perfect number then we have the expression (C) is true and such type of expression does not exist when M is not perfect numbers. Some illustrative instances are cited below which shows that the expression (C) is true for perfect number M less than 40000000 and is not true if M is not perfect number.

Instance 1: The divisors of perfect number M = 28 are d_1 = 2, d_2 = 4, d_3 = 7, d_4 = 14. Hence the expression (C) is true.

Instance 2: The divisors of perfect number M = 496 are d_1 = 2, d_2 = 4, d_3 = 8, d_4 = 16, d_5 = 31, d_6 = 62, d_7 = 124, d_8 = 248, then the expression (C) is true.

Instance 3: The divisors of perfect number M = 8128 are d_1 = 2, d_2 = 4, d_3 = 8, d_4 = 16, d_5 = 32, d_6 = 64, d_7 = 127, d_8 = 254, d_9 = 508, d_{10} = 1016, d_{11} = 2032, d_{12} = 4064, then the expression (C) is true.

Instance 4: The divisors of perfect number M = 33550336 are d_1 = 2, d_2 = 4, d_3 = 8, d_4 = 32, d_5 = 64, d_6 = 128, d_7 = 254, d_8 = 512, d_9 = 1024, d_{10} = 2048, d_{11} = 4096, d_{12} = 8191, d_{13} =16382, d_{14} = 32764, d_{15} = 65528, d_{16} = 131056, d_{17} = 262112, d_{18} = 524224, d_{19} = 1048448, d_{20} = 2096896, d_{21} = 4193792, d_{22} = 8387584, d_{23} = 16775168, then the expression (C) is true.

Instance 5: If we consider say, d_1 = 1, d_2 = 2, d_3 = 3, d_4 = 4, d_5 = 5, d_6 = 6, d_7 = 7 and add them then definitely one have value of M = 28 ,but then the expression gives negative number as the number 3,5,6 are not divisor of M = 28 ,which violates the definition of perfect numbers. Hence the expression (C) is not true [Discussed later in case 1]

Instance 6: If we considered d_1 = 1, d_2 = 3, d_3 = 5, d_4 = 7, d_5 = 9, d_6 = 11, d_7 = 13, then from the relation (C) we have value of M is negative. Hence the expression (C) is not true. Thus we can comment that if M is not a perfect number then,

$$M \neq (d_1.d_2.d_3 \ldots d_n)/\{(d_1.d_3 \ldots d_n) - (d_2.d_3 \ldots d_n) - (d_1.d_3.d_4 \ldots d_n) \\ - (d_1.d_2.d_3 \ldots d_n) - (d_1.d_2.d_3 \ldots d_{n-1})\}$$

which will be explained later in different cases.

Now, without loss of generality we proceed to find the large value of n from the set $\{4n + 12/n \geq 4\}$ and for that large values of n, M is a perfect number and thereafter we search that M will belong to the set $\{4n + 12/$ for some particular large values of $n \geq 4\}$. Let the large value of n be equal to $m_1.m_2.m_3 \ldots m_i$ for $1 < i < \infty$. That is $n = m_1.m_2.m_3.m_4 \ldots m_i$ for $1 < i < \infty$. That is $n = \prod m_i$ for $i = 1$ to ∞ and this product of m_i is a large particular value of n. Hence when we put the product $n = \prod m_i$ for $i = 1$ to ∞ in the set $\{4n + 12/n \geq 4\}$, then we must have perfect number M that is $4\{(\prod m_i$ for $i = 1$ to $\infty)\} + 12 = M$. and we have a relation for the product $\prod m_i$ for i=1 to ∞ = $(M - 12)/4$. Let denote this expression with (D). Hence our target is to find the values of m_i to satisfy the relation D and this will show that for the perfect number M otherwise false.

Now we proceed to find the value of $\prod m_i$ for $i = 1$ to ∞ under different cases for which the relation (D) will give different numbers not perfect and different cases for which the relation (D) will give only perfect numbers.

Case 1: If we consider the values of m_i are all consecutive natural numbers, then we have $\prod m_i$ for $i = 1$ to ∞ = $n!$. Then $n!$ is equal to $(M - 12)/4$, but M is not perfect number. For example if $n = 10$ say, then $(10)! = 3628800 = (M - 12)/4$ implies $M = 14515212$ less than 40000000 but it is seen that there are only three perfect numbers less 40000000 and they are 28,8128 and 33550336 not obtained from the relation (D). Hence we see that $\prod m_i$ for $i = 1$ to $\infty \neq n! \neq (M - 12)/4$ which indicate that M is not perfect number.

Case 2: If we consider the values of m_i are odd numbers, then $\prod m_i$ for $i = 1$ to ∞ = $\prod(2n - 1)$ for $n \leq 1$, then also we will not get the perfect number M but get other value of M. For example if = $\prod(2n - 1)$ for $1 < n \leq 5$,then the value of M=3792 less than 4000 but it is seen that there are only two perfect number less than 4000 and they are 28,496. This simple examples shows that $\prod m_i$ for $i = 1$ to $\infty \neq \prod(2n - 1)$ for $n \geq 1 \neq (M - 12)/4$ which shows that M is not perfect number.

Case 3: If we consider the values of $\prod m_i$ for $i = 1$ to ∞ = $\prod 2n$ for $n \geq 1$, Then also $\prod m_i$ for $i = 1$ to $\prod \neq 2n$ for $n \geq 1 \neq (M - 12)/4$. Continuing the process, the following cases have been found.

Case 4: If $\prod m_i$ for $i = 1$ to ∞ = $\prod(3n+1)$ for $n \geq 1$, Then $\prod m_i$ for $i = 1$ to ∞ = $\prod(3n + 1)$ for $n \geq 1 \neq (M - 12)/4$.

Case 5: If $\prod m_i$ for $i = 1$ to ∞ = $\prod n(n + 1)$ for $n \geq 1$, Then $\prod m_i$ for $i = 1$ to ∞ = $\prod n(n + 1)$ for $n \geq 1 \neq (M - 12)/4$.

Case 6: If $\prod m_i$ for $i = 1$ to ∞ = $\prod 3n$ for $n \geq 1$, Then $\prod m_i$ for $i = 1$ to ∞ = $\prod 3n$ for $n \geq 1 \neq (M - 12)/4$.

Case 7: If $\prod m_i$ for $i = 1$ to ∞ = $\prod 5n$ for $n \geq 1$, Then $\prod m_i$ for $i = 1$ to ∞ = $\prod 5n$ for $n \geq 1 \neq (M - 12)/4$.

Case 8: If $\prod m_i$ for $i = 1$ to ∞ = $\prod 11n$ for $n \geq 1$, Then $\prod m_i$ for $i = 1$ to ∞ = $\prod 11n$ for $n \geq 1 \neq (M - 12)/4$.

Case 9: If $\prod m_i$ for $i = 1$ to ∞ = $\prod 13n$ for $n \geq 1$, Then $\prod m_i$ for $i = 1$ to ∞ = $\prod 13n$ for $n \geq 1 \neq (M - 12)/4$.

Case 10: If $\prod m_i$ for $i=1$ to ∞ = $\prod 17n$ for $n \geq 1$, Then $\prod m_i$ for $i=1$ to ∞ = $\prod 17n$ for $n \geq 1 \neq (M-12)/4$.

Case 11: If $\prod m_i$ for $i=1$ to ∞ = $\prod 19n$ for $n \geq 1$, Then $\prod m_i$ for $i=1$ to ∞ = $\prod 19n$ for $n \geq 1 \neq (M-12)/4$.

In this way we can see that for all prime $p \geq 23$, the $\prod m_i$ for $i=1$ to $\infty = \prod pn$ for $n \geq 1 \neq (M-12)/4$.

Again if the product $\prod m_i$ for $i=1$ to ∞ is prime number $= p$, then there are only two values of m_i where $m_1 = 1$ and $m_2 = p$ and then $\prod m_i$ for $i=1$ to ∞ = $(M-12)/4$. Let us prove it with some experimental verification as follows from cases 12 to 15 and thereafter prove it with the help of graph theoretical approach.

Case 12: If $m_1 = 1$ and $m_2 = 2029$ = prime. Then $\prod m_i$ for $i = 1, 2 = (M-12)/4$ $\implies M$=8128.

Case 13: If $m_1 = 1$ and $m_2 = 8387581$ = prime. Then $\prod m_i$ for $i = 1, 2 = (M-12)/4 \implies M$=33550366.

Case 14: If $m_1 = 1$ and $m_2 = 2147467261$ = prime. Then $\prod m_i$ for $i = 1, 2 = (M-12)/4 \implies M$=8589869056.

Case 15: If $m_1 = 1$ and $m_2 = 34359672829$ = prime. Then $\prod m_i$ for $i = 1, 2 = (M-12)/4 \implies M$=137438691328.

We now again see for the equal values of m_i for which the product $\prod m_i$ for $i = 1, 2 = (M-12)/4$,when M is a perfect number.

Case 16: If $m_1 = 2 = m_2$, then the product $\prod m_i$ for $i = 1, 2 = (M-12)/4 \implies$ M=28.

Case 17: If $m_1 = 11 = m_2$, then the product $\prod m_i$ for $i = 1, 2 = (M-12)/4$ $\implies M$=496.

From the observations stated above in different cases we have different values relating to the product $\prod m_i$ for $i=1$ to ∞. The cases 1 to 11 are the cases where from no perfect number can be obtained, and the cases 12 to 17 are the cases where from perfect number can be obtained. Now we show that if the product $\prod m_i$ for $i=1$ to ∞ is a prime number p then the relation (D) will give a perfect number. We precede it with the help of graph theoretical results related with complete graph. Here we have from the relation (D) as $4p + 12 = M$. Again we know that M is even. Hence $4p + 12$ is also even. Now we show that M is expressible as a sum of two sufficiently large primes, say p_i and p_j, where p_i, p_j

less than and equal to ∞, with the help of definition 2.4 discussed in [5] which expressed that every even number can be expressed as a sum of two prime. Let us discuss how the structure of complete graph has been applied in theoretical explanation already discussed in theorem 3.1 of [5].

It is known that the number of edges of a complete graph K_n for $n \geq 3$ is $n(n-1)/2$. We take this concept and show that M can be express as a sum of two sufficiently large prime p_i and p_j. It is found in definition 2.4 of [5] that there is a complete prime vertex and even edge waited graph CPVEEWG (V,E), where V is the set of all consecutive prime numbers and E, the numbers of waited edges which means even numbers and this graph was constructed attaching consecutive prime numbers with the vertex of the complete graph and the wait of the edges was defined as the sum of two primes. If we consider the 5 consecutive primes, then the structure of CPVEEWG(V,E) is the graph CPVEEWG (5,10) obtained from K_5 for five consecutive prime numbers attached with the vertices of K_5 and there are 10 number of waited edges (even numbers) which are obtained as a sum of two primes as per construction of CPVEEWG. The wait of the edges are 12 = 5 + 7, 16 = 5 + 11, 18 = 5 + 13, 22 = 5 + 17, 24 = 7 + 17, 20 = 7 + 13, 18 = 7 + 11, 24 = 11 + 13, 28 = 11 + 17, 30 = 13 + 17 and they satisfy the property of complete graph K_5 that is $n(n-1)/2 = 5(5-1)/2 = 10$. It is known that the degree of the vertices of K_n is $n-1$. Now we extending the definition 2.4 and taking the help of theorem 3.1 in [5] and considering infinite consecutive prime numbers for the graph CPVEEWG(P,E), where the set of consecutive primes $P = \{p_1, p_2, p_3, p_4, p_5, \ldots p_i, \ldots p_j, \ldots p_\infty, p_\infty + 1 \ldots\}$ and E is the set of all even numbers $p_1 + p_2, p_1 + p_3, \ldots p_1 + p_4, \ldots\ldots p_i + p_j, \ldots p_\infty + p_\infty + 1, \ldots p_\infty + p_i, \ldots$ when $P \to \infty$ and $E \to \infty$ which is shown in Figure-1, and it is known that every vertex of the complete graph is connected to all vertices of the complete graph.

Let us consider the consecutive prime numbers p_i and p_j less than ∞ as shown in Figure-1.

The graph shown in Figure-1 is an extended version of CPVEEWG (V,E) to CPVEEWG (P,E) and the degrees of each vertex satisfy the condition $n-1$ considered for the complete graph K_n when $n \to \infty$.

That is $p_i + p_j$ = even = M, where P_i and P_j are sufficiently large prime which shows that M is a perfect number which is sufficiently large and it satisfy the

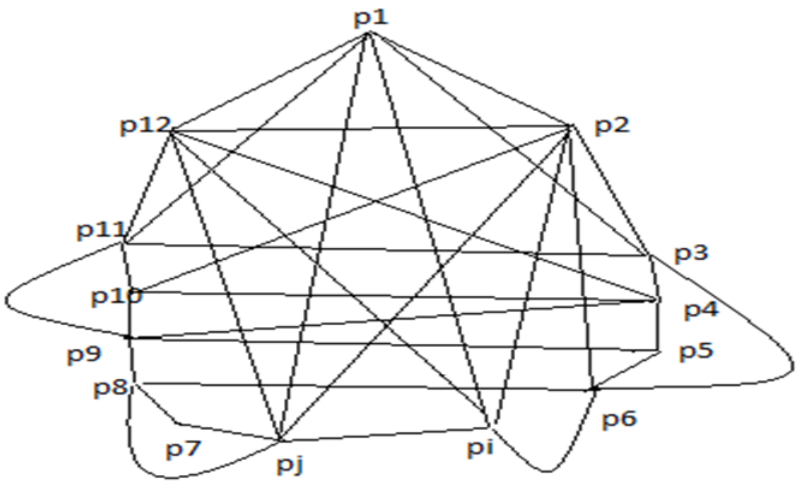

FIGURE 1. Complete Graph

condition (D). In addition to this we definitely can consider $4P + 12 = M$, where the product of $\prod m_i$ for $i = 1$ to $\infty = P$. Hence $4P + 12 = M = P_i + P_j$. That is for any value of the prime number P, $4P + 12 = P_i + P_j$, which is true for sufficiently large prime numbers P_i and P_j less than ∞. We now show that M belongs to the set $\{4n + 12/\text{some particular values of } n \geq 4\}$.

If possible we suppose that M does not belong to the set $B = \{4n + 12/$ for some large values of $n \geq 4\}$. Then $M \in B^C = \{4n + 12/$ for some large values of $n \geq 4\}^C$, that is Complement of $B = \{4n + 12/$ for some large values of $n \geq 4\}$.

But it is seen that $M \in B^C = \{4n + 12/$ for some large values of $n \geq 4\}^C = H \cup K$, where H and K are infinite sets whose structures are already found as $H = \{2n + 27/n \geq 1\}$ [definition 2.6] and $K = \{4n + 26/n \geq 1\}$ [definition 2.7], and they are disjoint sets. Here the universal set can be considered as $B \cup H \cup K$, where $B = \{4n + 12/$ for $n \geq 4\}$ is also infinite set. It is seen that this union give a set of the form $B \cup H \cup K = \{n/n \geq 28\}$, which is also infinite set. Hence,

$\implies M \in H \cup K \implies M \in H$ or $M \in K$

$\implies M \notin H^C$ or $M \notin K^C$

$\implies M \notin H^C \cup K^C$

$\implies M \notin (H \cap K)^C$ by De Morgan's law

$\implies M \in H \cap K$

$\implies M \in \phi =$ null set, as $H \cap K = \phi$

Which is a contradiction, as M is a sufficiently large perfect number and it is not empty. Hence we can comment that there exist a sufficiently large perfect number M which lies in the set $\{4n + 12/$ for some large values of $n \geq 4\}$ and hence M is expressible as the sum of all divisors less then M. Again as $M \in \{4n + 12/$ for some large values of $n \geq 4\}$. Hence $1 + d_1 + d_2 + d_3 + \ldots + d_n \in \{4n + 12/$ for some large values of $n \geq 4\}$. That is $d_1, d_2, d_3, \ldots, d_n$ are the positive divisors of M. Hence M is a perfect number.

Some Remarks:

Remark 1: If $m_1 = m_2 = m_3 = \ldots = m_i$ for $3 \leq m_i \leq \infty) = 2n + 1$ for $n \geq 1$. Then we have from relation (D),

$(2n + 1)(2n + 1)(2n + 1)(2n + 1) \ldots = (M - 12)/4$. Then for different values of n, we have,

4(3.3.3.3.3.3.3.. . .) + 12 = M for n = 1, but M is not perfect number

4(5.5.5.5.5.. . .) + 12 = M for n = 2, but M is not perfect number

4(7.7.7.7.7. . .) + 12 = M for n = 3, but M is not perfect number

4(9.9.9.9.9. . .) + 12 = M for n = 4 ,but M is not perfect number

$\vdots$

Continuing the process we have $4(2n+1) \ldots + 12 = M$ for large values of n, but M is not perfect number.

Remark 2: If $m_1 = m_2 = m_3 = \ldots = m_i$ for $2 \leq m_i \leq \infty) = 2n$ for $n \geq 1$ then $4\{(2n)(2n)(2n)(2n) \ldots (2n)\} + 12 = M$ [from D]. Then also we see that M is not perfect numbers, as we see that,

4(2.2.2.2.2.2. . .) + 12 = M for n = 1, but M is not perfect number

4(4.4.4.4.4.. . .) + 12 = M for n = 2, but M is not perfect number

4(6.6.6.6.. . .) + 12 = M for n = 3, but M is not perfect number

4(8.8.8.8.8. . .) + 12 = M for n = 4, but M is not perfect number

$\vdots$

Continuing the process we have $4(2n) \ldots) + 12 = M$ for large values of n, but M is not perfect number.

Remark 3: If $m_1 = m_2 = m_3 = \ldots = m_i$ for $1 \leq m_i \leq \infty) = 2n - 1$ for $n \geq 1$. Then we have $4\{(2n-1)(2n-1)(2n-1)\ldots(2n-1)\} + 12 = M$ [from D], which shows that M is not perfect numbers as we see that,

4(1.1.1.1.1.. . .) + 12 = M for n = 1, but M is not perfect number
4(3.3.3.3.3.3. . .) + 12 = M for n = 2, but M is not perfect number
4(5.5.5.5.5.. . .) + 12 = M for n = 3, but M is not perfect number
4(7.7.7.7.7.. . .) + 12 = M for n = 4, but M is not perfect number
4(9.9.9.9.9.. . .) + 12 = M for n = 5, but M is not perfect number
$\vdots$

Continuing the process we have $4(2n-1)\ldots + 12 = M$ for large values of n, but M is not perfect number. [Some results are same with Example 1].

Remark 4: If $m_1 = m_2 = m_3 = \ldots = m_i$ for $4 \leq m_i \leq \infty) = 3n - 1$ for $n \geq 1$. Then we have $4\{(3n+1)(3n+1)(3n+1)\ldots(3n+1)\} + 12 = M$ [from D], which shows that M is not perfect numbers as we see that,

4(4.4.4.4.4. . .) + 12 = M for n = 1, but M is not perfect number
4(7.7.7.7.7. . .) + 12 = M for n = 2, but M is not perfect number
4(10.10.10.10. . .) + 12 = M for n = 3, but M is not perfect number
4(13.13.13.13.13. . .) + 12 = M for n = 4, but M is not perfect number
$\vdots$

Continuing the process we have $4(3n+1)\ldots + 12 = M$ for large values of n, but M is not perfect number.

Remark 5: If $m_1 = m_2 = m_3 = \ldots = m_i$ for $2 \leq m_i \leq \infty) = n(n+1)$ for $n \geq 1$. Then we have $4\{(n+1)(n+1)(n+1)\ldots(n+1)\} + 12 = M$ [from D], which shows that M is not perfect numbers, which explain as follows,

4(2.2.2.2.2. . .) + 12 = M for n = 1, but M is not perfect number
4(6.6.6.6.6. . .) + 12 = M for n = 2, but M is not perfect number
4(12.12.12.12. . .) + 12 = M for n = 3, but M is not perfect number
$\vdots$

Continuing the process we have $4(n(3n+1)\ldots) + 12 = M$ for large values of n, but M is not perfect number.

Remark 6: There may have two consecutive natural numbers as a positive divisor in any perfect number but not more than two.

Remark 7: There may have two consecutive even numbers as divisors of the perfect number M but there does not exists more than two consecutive even numbers as proper divisors. For the perfect number 496 we have 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 = 496. Here 2, 4 are two consecutive even numbers and they are divisor of 496 but there does not have more than two consecutive even numbers except 2 and 4.

Note 1: From Case 1, though we have the value of M is equal to 28 when $m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7) = 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$, then also M is not perfect number according to definition of perfect numbers, as there are numbers which are not proper divisors of 28. Hence at least three consecutive natural numbers are not proper divisors of M.

Note 2: For example if $m_1 = 1, m_2 = 3, m_3 = 5$, then M=72, M is not perfect number or if we consider m_1=9, m_2=11, m_3=13, then M=5160, M is not perfect number etc. [Case 2].

Note 3: For example if $m_1 = 2, m_2 = 4, m_3 = 6$, then M=204, but M is not perfect number.[Case 3]

Note 4: If $m_1 = 4, m_2 = 7, m_3 = 10$, then M=1132 and it is not perfect number. [Case 4]

Note 5: If $m_1 = 2, m_2 = 6, m_3 = 12$, then M=588 it is not perfect.[Case 5]

Note 6: The values of m_i is equal only for i = 1, 2 that is when $m_1 = m_2 = 2$, then the value of M = 28, and similarly $m_1 = m_2 = 11$, then the value of M = 496 etc., [Case 16 and 17]. Similar notes exist for different cases.

6. Conclusion

The paper is meant for the mathematician in general and to common public in particular. The discussion of existence of infinite perfect numbers has been considered in a theoretical nature which are lying in the set $\{4n + 12$ for some large values of $n \geq 4\}$.Some new results and propositions and properties have also been discussed with some experimental results. Hence we find that there are infinite perfect numbers.

Declaration

Ethics approval and consent to participate: This study is conducted in accordance with ethical guidelines and regulation.

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