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TOPOLOZICAL DERIVATIVE OF THE FRACTIONAL p-LAPLACIAN OPERATORS

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ABSTRACT. The objective of this article is the study of topological optimization problems with p-Laplacian operators, i.e. $(-\Delta)^s_p$ where $0 < s < 1$ and $p \geq 2.$ In [22], we began studying this problem to determine the shape derivative. In the same paper, we studied existence results using s-gamma convergence. In this paper, we work with the open class checking the $\epsilon-$ cône property to obtain that the existence of an optimal shape. And finally we found the topological derivative of the functional through the minmax method.

1. INTRODUCTION

This introduction was inspired by our earlier work on fractional p-Laplacian. For more information, the reader can also consult the paper by [22]. In mathematics, the fractional Laplacian is an operator, which extends the concept of Laplacian spatial derivatives to fractional powers. This operator is often used to generalize certain types of partial differential equation. There exist various definitions of fractional Laplacian but most of them are equivalents.

The fractional Laplacian can be defined by Fourier transform [17], singular operator [34], or generate C_0 semi-group [26]. Some of the remarkable physical

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phenomena where fractional Laplacian has found use include anomalous diffusion and quasi geostropic flows, turbulence and water waves, molecular dynamics, anomalous diffusion in plasma. For more informations, we refer the reader to [2] and references therein.

Naturally, due to the various definitions and applications of fractional Laplacian, several methods have been proposed for solving problems involving the fractional Laplacian. In particular, Cafarelli and Silvestre [13] constructed fractional Laplacian from an extension problem to the upper half space for a specific elliptic partial differential equation. In [23], M. Fall et. al. studied the regularity of solutions. The study of fractional p-Laplacian was well investigated in [24,25] where authors described abnormal diffusion by fractional dynamics and derived various fractional partial differential equations from the walking models. Recently, a numerical method for fractional Laplacian based on the singular integral representation operator was presented in [2, 5]. Caffarelli et al. [13, 16] proved characteristics for the general fractional powers of the Laplacian and other integro-differential operators, and from these characterizations they provided some properties from integro-differential equations of purely local argument in extension problems. It is also worth noting that in [6, 7, 29, 31, 32], Ros-oton, Xavier and Serra proposed a new way to investigate the regularity of the solution u to the boundary of the considered domain. Based on the upon works, Fall et al. in [23] established some extended results when u is the non-local schrodinguer solution. Furthermore, we note that some research studies have been devoted to shape optimization problems. For instance, in [14, 15] Dalibard et al. discussed the existence of optimal shape for a functional when $s = \frac{1}{2}$ whereas M. Fall et al. [21] studied the case $0 < s < 1$ by using the variational method and shape optimization.

We recall that these types of problems were studied by [22], but with the objective of studying the form derivative of the associated functional using vector fields. This is what gives us the idea of wanting to look at the topological derivative, but this time using the recent work of [8,9,27]. So we will study the topological derivative using the minmax method. for more information on this method the reader can consult the work of [10]. And for more practical cases the reader can also consult the paper by [22], where the author calculates the topological derivative of a functional linked to a linear thermoplastic problem. on the other hand in the paper by [27] the author established a practical case of the topological derivative linked to Helmholtz problems.

The main objective in this article is to determine the topological derivative of the functional $F(\Omega_t) = F(\Omega_t, u_t)$, where the perturbed domain Ω_t of Ω is defined by $\Omega_t = T_t(\Omega)$ or $\Omega_t = \Omega \setminus E_t$ depending on the derivative to be calculated.

Through in-depth analysis of the above mentioned works, we found that the investigation of shape optimization problem with nonlocal operators like the fractional Laplacian as a constraint are rarely available in the literature, which motivates this present study. Non-local operators such as Δ_p^s appear naturally in the continuum mechanics, phase transition phenomena, dynamics population and games theory. In this paper, we look at following shape optimization problem:

$$\min_{\Omega \in \mathcal{O}} F(\Omega), \tag{1.1}$$

where F is a given cost functional, $\mathcal{O}$ a class of admissible domains, Ω is an open bounded subset of $\mathbb{R}^N$, $N \geq 2$ and such that

$$F(\Omega) = j(\Omega, u_\Omega) = \frac{C(N,s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{| u_\Omega(x) - u_\Omega(y) |^{p-2} (u(x) - u(y))}{| x - y |^{N+ps}} dxdy, \tag{1.2}$$

with u_Ω the solution to the following p-Laplacian operator

$$\begin{cases} (-\Delta)_p^s u = f \text{ in } \Omega, \\ u = 0 \text{ on } \mathbb{R}^N \setminus \Omega, \\ p \geq 2. \end{cases} \tag{1.3}$$

This work is a continuation and a generalization of [14, 21, 22] to the p-Laplacian operator $0 < s < 1$ and $p \geq 2$. But this time, we're looking at things a little more from a different angle, i.e. in the direction of the topological derivative. We first prove the existence of results in the class of open sets verifying the property of the epsilon cône. We consider the established results of M. Fall et al. [21]. This allows us to provide how the topological derivative of the considered functional can be determined in these spaces.

The work program is as follows. Section 2 gives some preliminary results concerning the fractionary problem and some Sobolev inequalities. The existence theorem for a weak solution of the constraint equation is presented in Section 3. Section 4 establishes the existence result for an optimal form in the class of openings verifying the $\epsilon-$ cône property. In Section 5, the topological derivative of the functional is computed and Section 6 provides a conclusion and some possible extensions.

2. Preliminaries on the fractional operators

We recall some fundamental definitions and results on shape optimization problem. Particullary, We focus only on the cases of the laplacian and the p-laplacian operators, for $p \geq 2$. We have the following definitions.

Definition 2.1. *Let $0 < s < 1$ and $p \in [2, +\infty)$, $N \geq sp$ and Ω a bounded open set of $\mathbb{R}^N$, with Lipchitz boundary, Let*

$$[u]_{s,p} = \left(\int_\Omega \int_\Omega \frac{| u(x) - u(y) |^p}{| x - y |^{N+ps}} dxdy \right)^{\frac{1}{p}}$$

be the Gagliardo semi norm of a measurable function u.

1. *We define $W^{s,p}(\Omega)$ as follows*

$$W^{s,p}(\Omega) = \{u \in L^p(\Omega) \ such \ that \ [u]_{s,p} < +\infty\}$$

endowed with the usual norm

$$\| u \|_{W^{s,p}(\Omega)} = \left(\int_\Omega |u|^p_{L^p} + [u]_{s,p} \right)^{\frac{1}{p}}.$$

2. *Consider the closed linear subspace $W_0^{s,p}(\Omega)$ by*

$$W_0^{s,p}(\Omega) = \left\{ u \in W^{s,p}(\mathbb{R}^N) : \ u = 0; \ a.e \text{ in } \mathbb{R}^N \setminus \Omega \right\}.$$

equivalently renormed by setting $\|u\|_{s,p} = [u]_{s,p}$.

Definition 2.2. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Given $A \subset \Omega$, for any $0 < s < 1$ and $p \geq 1$, we define the Gagliardo $s-$capacity of A relatively to Ω as*

$$cap_s(A, \Omega) = \inf \left\{ [u]_s^p : \ u \in \mathcal{C}(\Omega), \ u \geq 0, \ A \subset \{u \geq 1\} \right\},$$

where

$$[u]_s^p = \int_\Omega \int_\Omega \frac{| u(x) - u(y) |^p}{| x - y |^{N+ps}} dxdy.$$

Definition 2.3. *A subset A of Ω is a s-quasi open set if there exists a decreasing sequence $\{w_k\}_{k\in\mathbb{N}}$ of open subsets of Ω such that $cap_s(w_k,\Omega)\to 0$, as $k\to+\infty$, and $A\cup w_k$ is an open set for all $k\in\mathbb{N}$.*

Now, we provide a definition of $(s-\gamma)-$convergence. This definition is inspired by $\gamma-$ convergence, see for instance [3, 12].

Definition 2.4. *Let $\{A_k\}_{k\in\mathbb{N}}\subset\mathcal{A}_s(\Omega)$ and $A\in\mathcal{A}_s(\Omega)$. We say that $A_k\xrightarrow{\gamma_s}A$ if $u^s_{A_k}\longrightarrow u^s_A$ strongly in $L^2(\Omega)$.*

Definition 2.5. *Let $0<s<1$ be fixed and let $F_s:\mathcal{A}_s(\Omega)\longrightarrow\mathbb{R}$ be such that: F_s is lower semi continuous with respect to the $(s-\gamma)-$ convergence; that is*

$$A_k\xrightarrow{\gamma_s}A \ \ \textit{implies} \ \ F_s(A)\leq\lim_{k\longrightarrow+\infty}\inf F_s(A_k).$$

F_s is decreasing with respect to set inclusion; that is $F_s(A)\geq F_s(B)$ whenever $A\subset B$.

Definition 2.6. *Let $p\in]2,+\infty)$ and $s\in(0,1)$, the fractional p-laplacian is defined by*

$$\begin{aligned}(-\Delta)^s_p u(x) &= 2\lim_{\epsilon\longrightarrow 0}\int_{\mathbb{R}^N\setminus B_\epsilon(x)}\frac{\mid u(x)-u(y)\mid^{p-2}(u(x)-u(y))}{\mid x-y\mid^{N+ps}}dy\\ &= 2Vp(u)\int_{\mathbb{R}^N}\frac{\mid u(x)-u(y)\mid^{p-2}(u(x)-u(y))}{\mid x-y\mid^{N+ps}}dy.\end{aligned}$$

where $V_p(u)$ is the principal value of u.

Remark 2.1. *For $p\neq 2$, the fractional p-laplacian $(-\Delta)^s_p$ is non-linear and the value of $(-\Delta)^s_p u(x)=(-\Delta)^s_p u(x,\Omega,N)$. This definition is a generalization of the fractional Laplacian operator for $p=2$, witch is defined by:*

$$(-\Delta)^s u(x)=C(N,s)Vp(u)\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{\mid x-y\mid^{N+2s}}dy.$$

Theorem 2.1. *Let $p\in[2,+\infty)$ and $s\in(0,1)$. Then, the application*

$$(-\Delta)^s_p: W^{s,p}_0(\Omega)\longrightarrow(W^{s,p}_0(\Omega))'$$

$$u_\Omega\longrightarrow(-\Delta)^s_p u_\Omega.$$

is well defined. Moreover

1. $\forall u, v \in W_0^{s,p}(\Omega)$ *we have:*

$$\langle(-\Delta)_p^s u, v\rangle = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mid u(x)-u(y)\mid^{p-2}(u(x)-u(y))(v(x)-v(y))}{\mid x-y\mid^{N+ps}}dxdy.$$

2. $\forall u, v \in W_0^{s,p}$ *we have:*

$$\langle(-\Delta)_p^s u, v\rangle \leq [u]_{s,p}^{p-1}[v]_{s,p}.$$

Proof. See [22]. □

This theorem will be useful in the following.

Theorem 2.2. *Let $s \in (0,1)$ and $p \in [2,+\infty)$, $q \in [1,p]$, $\Omega \subset \mathbb{R}^N$ be an open bounded subset domain for $W^{s,p}(\Omega)$ and T be a bounded subset of $L^p(\Omega)$. Suppose that*

$$\sup_{f\in T}\left(\int_\Omega\int_\Omega \frac{\mid f(x)-f(y)\mid^p}{\mid x-y\mid^{N+ps}}dxdy\right) < +\infty.$$

Then T is pre-compact in $L^q(\Omega)$.

Proof. For the proof, we refer to [15]. □

Theorem 2.3. *Let K be a compact and B a bounded open of $\mathbb{R}^N$. Let Ω_n be a sequence of open with $\overline{\Omega}_n \subset K \subset B$, verifying the ownership of the $\epsilon-$ cône. Then there is an open Ω verifying the ownership of the $\epsilon-$ cône and an extracted sequence Ω_{n_k} such as*

$$\Omega_{n_k} \xrightarrow{H} \Omega, \quad \chi_{\Omega_{n_k}} \xrightarrow{L^1 p.p} \chi_\Omega,$$

$$\overline{\Omega}_{n_k} \xrightarrow{H} \overline{\Omega}, \quad \partial\Omega_{n_k} \xrightarrow{H} \partial\Omega.$$

Proof. See [1]. □

This result which will allow us to characterize the existence of solution.

3. Existence of solution for the non local Dirichlet problem.

We recall that this part was established by Fall and al in [22]. For more information, the reader can consult this article.

The referred types of problems that we will considered in this section are studied by Caffarelli and Sylvestre [13] in the special case $s = \frac{1}{2}$. Moreover, the regularity of the solution of this problem is studied in [23] and [28], In this work, we propose to generalize the above case $0 < s < 1$, by using a variational approach. In this section, we are concerned with the extension result of the problem given by

$$\begin{cases} (-\Delta)_p^s u = f \text{ in } \Omega \\ u = 0 \text{ on } \mathbb{R}^N \setminus \Omega \\ p \geq 2. \end{cases} \tag{3.1}$$

To prove the existence result, We use the Euler Lagrange equation associated (3.1) in order to transform it into a functional $F(u)$.

Theorem 3.1. *Let Ω be an open subset of $\mathbb{R}^N$, $N > 1$ of class $\mathcal{C}^2$, $s \in (0,1)$ and $p \in [2, +\infty)$. Then there exists a unique weak solution $u \in W^{s,p}(\Omega)$ of the problem* (3.1). *Moreover this solution minimizes the problem*

$$\inf_{u \in W^{s,p}(\Omega)} (F(u,u)).$$

with

$$F(u,v) = \langle (-\Delta)_{p-1}^s u, v \rangle_{W^{s,p}(\Omega)} - \int_\Omega f(x)v(x)dx.$$

The following lemma 3.1 shows us sequence (u_k) is bounded in $W^{s,p}(\Omega)$. That is, $[u]_{s,p} < +\infty$. It is useful for the proof of the Theorem 3.1.

Lemma 3.1. *Since $(u_k)_{k\geq 1} \subset W^{s,p}(\Omega)$ is a minimizing sequence of F that is*

$$\lim_{k \longrightarrow +\infty} F(u_k, u_k) = \inf_{v \in W^{s,p}(\Omega)} F(v,v) = m$$

then $(u_k)_{k\geq 1}$ is bounded in $W^{s,p}(\Omega)$.

Proof. By hypothesis, $\{u_k\}$ is a minimizing sequence of the function $F(u_k, u_k)$ and m its limit, there exists a rank k, from which we have

$$m \leq F(u_k, u_k) \leq m + \frac{1}{k}, \ \forall k \geq 1.$$

$$F(u,v) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{| u(x) - u(y) |^{p-2} (u(x) - u(y))(v(x) - v(y))}{| x - y |^{N+ps}} dxdy - \int_\Omega f(x)v(x)dx.$$

Therefore, we have

$$F(u_k, u_k) \geq \ [u_k]_{s,p}^{p-1} [u_k]_{s,p} - \frac{p-1}{p} \parallel f \parallel_{L^2(\Omega)}^{\frac{p}{p-1}} - \frac{1}{p} \parallel u_k \parallel_{L^2(\Omega)}^p .$$

Then,

$$[u_k]_{s,p}^{p-1}[u_k]_{s,p} \leq F(u_k,u_k) + \frac{p-1}{p} \parallel f \parallel_{L^2(\Omega)}^{\frac{p}{p-1}} + \frac{1}{p} \parallel u_k \parallel_{L^2(\Omega)}^{p} . \tag{3.2}$$

Taking into account the fact that $f \in L^p(\Omega)$, the sequence $u_k \in L^p(\Omega)$ and the functional $F(u_k,u_k) \leq m + \frac{1}{k}$, then for k large enough, we show that the quantity on the right hand side of (3.2) is bounded, then the left hand side of (3.2), which implies that the norm $[.]_{s,p}$ is bounded by a constant which depends only on f and m. $\square$

Proof. **of Theorem 3.1** Let $v \in W_0^{s,p}(\Omega)$, multiplying the first equation of (3.1) by $v \in W_0^{s,p}(\Omega)$ and by integrating on Ω we have:

$$\int_\Omega (-\Delta)_p^s u\, v dx = \int_\Omega f(x)v(x)dx. \tag{3.3}$$

By using definition of the scalar product in $W_0^{s,p}(\Omega)$, we have:

$$\int_{\mathbb{R}^{2N}} (-\Delta)_p^s u\, v = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mid u(x)-u(y)\mid^{p-2} (u(x)-u(y))(v(x)-v(y))}{\mid x-y\mid^{N+ps}} dxdy.$$

and equation (3.3) becomes:

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mid u(x)-u(y)\mid^{p-2} (u(x)-u(y))(v(x)-v(y))}{\mid x-y\mid^{N+ps}} dxdy \\ &= \int_\Omega f(x)v(x)dx. \end{aligned} \tag{3.4}$$

In what follows, let F be the functional defined as

$$\begin{aligned} F(u,v) = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mid u(x)-u(y)\mid^{p-2} (u(x)-u(y))(v(x)-v(y))}{\mid x-y\mid^{N+ps}} dxdy \\ &- \int_\Omega f(x)v(x)dx, \end{aligned}$$

in others words,

$$F(u,v) = \langle (-\Delta)_p^s u, v\rangle - \int_\Omega f(x)v(x)dx. \tag{3.5}$$

We have to show that the functional F is lower and is greater than $-\infty$. From Hölder's inequality, it follows

$$| F(u,v) | \leq \left(\int_{\mathbb{R}^{2N}} \frac{| u(x)-u(y) |^p}{| x-y |^{N+ps}} dxdy \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^{2N}} \frac{| (v(x)-v(y) |^p}{| x-y |^{N+ps}} dxdy \right)^{\frac{1}{p}} + \int_\Omega |f(x)v(x)| dx,$$

giving directly

$$| F(u,v) | \leq [u]_{s,p}^{p-1} [v]_{s,p} + \| f \|_{L^2(\Omega)} \| \| \|_{L^2(\Omega)} .$$

Since $u,\ v \in W^{s,p}(\Omega)$, we derive that the functional $F(u,v)$ is bounded.

Since $\Omega \in \mathcal{O}_\epsilon$, the Theorem 2.2 shows that $[.]_{s,p}$ is pre-compact in $L^p(\Omega)$.

Since $(u_k)_{k\geq 1}$ is bounded in $W^{s,p}(\Omega)$, then there exists an subsequence $(u_{k_l})_{l\geq 1}$ of $(u_k)_{k\geq 1}$ such that: $u_{k_l} \rightharpoonup u \in W^{s,p}$,$u_{k_l} \longrightarrow u \in L^p(\Omega)$ and $u_{k_l} \rightharpoonup u \in L^p(\Omega)$, when, $\longrightarrow +\infty$. It follows that

$$F(u_{k_l}, u_{k_l}) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{| u_{k_l}(x) - u_{k_l}(y) |^{p-2} (u_{k_l}(x) - u_{k_l}(y))(u_{k_l}(x) - u_{k_l}(y))}{| x-y |^{N+ps}} dxdy - \int_\Omega f(x) u_{k_l}(x) dx \leq m + \epsilon, \ \ \forall \epsilon \geq 0.$$

It follows from the above inequality, that

$$\int_{\mathbb{R}^{2N}} \frac{| (u_{k_l}(x) - u_{k_l}(y)) |^p}{| x-y |^{N+ps}} dxdy \leq \int_\Omega f(x) u_{k_l}(x) dx + m + \epsilon, \ \ \forall \epsilon \geq 0.$$

Applying Fatou's Lemma, we obtain:

$$\int_{\mathbb{R}^{2N}} \liminf_{l\longrightarrow +\infty} \frac{| (u_{k_l}(x) - u_{k_l}(y)) |^p}{| x-y |^{N+ps}} dxdy \leq \liminf_{l\longrightarrow +\infty} \int_\Omega f u_{k_l} dx + m + \epsilon, \ \ \forall \epsilon \geq 0$$

$$\int_{\mathbb{R}^{2N}} \frac{| u(x)-u(y) |^p}{| x-y |^{N+ps}} dxdy \leq \liminf_{l\longrightarrow +\infty} \int_\Omega f u_{k_l} dx + m + \epsilon, \ \ \forall \epsilon \geq 0.$$

By the weak convergence, $u \in L^p(\Omega)$, we have:

$$\liminf_{l\longrightarrow +\infty} \int_\Omega f u_{k_l} dx = \lim_{l\longrightarrow +\infty} \int_\Omega f u_{k_l} dx = \int_\Omega f u dx.$$

$$\int_{\mathbb{R}^{2N}} \frac{| u(x)-u(y) |^p}{| x-y |^{N+ps}} dxdy \leq \int_\Omega f u dx + m + \epsilon, \ \ \forall \epsilon \geq 0.$$

$$F(u,u) = \int_{\mathbb{R}^{2N}} \frac{| u(x)-u(y) |^p}{| x-y |^{N+ps}} dxdy - \int_\Omega f u dx \leq m + \epsilon, \ \ \forall \epsilon \geq 0.$$

This implies $F(u,u) \leq m \Longrightarrow F(u,u) = m.$ $\square$

In the following section, we establish the main result of optimal form existence. In [22] the existence of optimal form was obtained using s-gamma convergence. But this time we use the epsilon cone property to determine it.

4. EXISTENCE OF A SOLUTION BY THE $\epsilon-$ CÔNE PROPERTY

Theorem 4.1. *Let $\mathcal{O}_{ad} \subset \mathcal{O}_\epsilon$ be a set open bounded domain of $\mathbb{R}^n$. Then there exists on open set $\Omega \in \mathcal{O}_{ad}$ satisfying*

$$F(\Omega) = \min_{\Omega \in \mathcal{O}_{ad}} F(\Omega).$$

Proof. Let us show that F is bounded:

$$F(\Omega) = \frac{C(N,s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\mid u_\Omega(x) - u_\Omega(y) \mid^{p-2} (u(x) - u(y))}{\mid x - y \mid^{N+ps}} dxdy$$

$$\mid F(\Omega) \mid = \frac{C(N,s)}{2} \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\mid u_\Omega(x) - u_\Omega(y) \mid^{p-2} (u(x) - u(y))}{\mid x - y \mid^{N+ps}} dxdy \right| > -\infty.$$

So F is reduced, and

$$\mid F(\Omega) \mid \leq \frac{C(N,s)}{2} \left(\int_{\mathbb{R}^{2N}} \frac{\mid u(x) - u(y) \mid^p}{\mid x - y \mid^{N+ps}} dxdy \right)^{\frac{p-2}{p}} \left(\int_{\mathbb{R}^{2N}} \frac{\mid (u(x) - u(y) \mid^p}{\mid x - y \mid^{N+ps}} dxdy \right)^{\frac{1}{p}}$$

$$\mid F(\Omega) \mid \leq \frac{C(N,s)}{2} [u]_{s,p}^{p-2} [u]_{s,p}$$

$$\mid F(\Omega) \mid \leq \frac{C(N,s)}{2} [u]_{s,p}^{p-1}.$$

So F is increased. Hence F is bounded.

Let

$$m = \inf_{\Omega \in \mathcal{O}_\epsilon \; or \; \mathcal{O}_{ad}} F(\Omega). \tag{4.1}$$

Then according to the properties of the lower bound, there exists a minimizing sequence (Ω_n) of $\mathcal{O}_{ad}$ such that

$$F(\Omega_n) \longrightarrow m = \inf_{\Omega \in \mathcal{O}_\epsilon \; or \; \mathcal{O}_{ad}} F(\Omega).$$

Let $\Omega_n \in \mathcal{O}_{ad}$ then according to the compactness theorem 2.3 there is an open $\Omega \in \mathcal{O}_\epsilon$ and an extracted sequence Ω_{n_k} which converges to Ω in the sense of Hausdorff.

Like $\Omega_n \in \mathcal{O}_{ad} \subset \mathcal{O}_\epsilon$, so the sequence Ω_{n_k} checks the property of the $\epsilon-$ cône. According to the theorem 2.3, we can extract from the sequence Ω_{n_k}, a sub-sequence Ω_{n_k} which verifies the following convergences:

$$\Omega_{n_k} \xrightarrow{\quad H \quad} \Omega, \chi_{\Omega_{n_k}} \xrightarrow{\quad L^1 p.p \quad} \chi_\Omega, \overline{\Omega}_{n_k} \xrightarrow{\quad H \quad} \overline{\Omega}, \partial\Omega_{n_k} \xrightarrow{\quad H \quad} \partial\Omega,$$

with Ω checking the ownership of the $\epsilon-$ cône. It remains to show that:

$$\lim F(\Omega_{n_k}) = F(\Omega) = \inf_{\Omega \in \mathcal{O}_\epsilon \text{ } or \text{ } \mathcal{O}_{ad}} F(\Omega).$$

From the variational formulation (3.4) we have:

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mid u_\Omega(x) - u_\Omega(y) \mid^{p-2} (u(x)-u(y))(v(x)-v(y))}{\mid x-y \mid^{N+ps}} dxdy \\ &= \int_\Omega f(x)v(x)dx. \end{aligned} \tag{4.2}$$

So, in Ω_{n_k} we get the following formulation:

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mid u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y) \mid^{p-2} (u_{\Omega_{n_k}}(x)-u_{\Omega_{n_k}}(y))(v_{\Omega_{n_k}}(x)-v_{\Omega_{n_k}}(y))}{\mid x-y \mid^{N+ps}} dxdy \\ &= \int_{\Omega_{n_k}} f(x)v_{\Omega_{n_k}}(x)dx. \end{aligned} \tag{4.3}$$

In addition we have also shown in the variational formulation part (3.2) that

$$\left(\frac{p-C^2}{p}\right)[u_k]^p_{s,p} \leq m + \frac{1}{k} + \frac{p-1}{p} \parallel f \parallel^{\frac{p}{p-1}}_{L^2(\Omega)} .$$

So the sequence $u_{\Omega_{n_k}}$ is bounded in $W^{s,p}(\Omega_{n_k})$. Like $\Omega \in \mathcal{O}_\epsilon$, Sobolev's inequality theorem 2.2 tells us that $W^{s,p}(\Omega_{n_k})$ is precompact in $L^p(\Omega_{n_k})$. Since $(u_{\Omega_{n_k}})$ is bounded in $W^{s,p}(\Omega_{n_k})$, then we can find an extracted sequence $(u_{\Omega_{n_k}})_{k\geq 1}$ of $(u_{\Omega_{n_k}})$ noted also $(u_{\Omega_{n_k}})_{k\geq 1}$ such that:

$$(u_{\Omega_{n_k}})_{k\geq 1} \rightharpoonup u^*_\Omega \in W^{s,p}, (u_{\Omega_{n_k}})_{k\geq 1} \longrightarrow u^*_\Omega \in L^p(\Omega),$$

and $(u_{\Omega_{n_k}})_{k\geq 1} \rightharpoonup u^*_\Omega \in L^p(\Omega)$, when $k \longrightarrow \infty$.

We have $\varphi \in W^{s,p}(\Omega_n)$ so $\varphi \in W^{s,p}(\Omega_{n_k})$ from a certain rank. From the relation (4.3), we obtain

$$\int_{\mathbb{R}^{2N}}\int_{\mathbb{R}^N} \frac{\mid u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y) \mid^{p-2} (u_{\Omega_{n_k}}(x)-u_{\Omega_{n_k}}(y))(\varphi_{\Omega_{n_k}}(x)-\varphi_{\Omega_{n_k}}(y))}{\mid x-y \mid^{N+ps}} dxdy$$

$$= \int_{\Omega_{n_k}} f(x)\varphi_{\Omega_{n_k}}(x)dx. \tag{4.4}$$

Then by making k tend towards infinity and using weak convergence we have:

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mid u^*_\Omega(x) - u^*_\Omega(y) \mid^{p-2} (u^*(x) - u^*(y))(\varphi(x) - \varphi(y))}{\mid x - y \mid^{N+ps}} dxdy \\ &= \int_\Omega f(x)\varphi(x)dx \ \ \forall \varphi \in W^{s,p}. \end{aligned} \tag{4.5}$$

The first term of the equality (4.5) represents the weak formulation of

$$\int_\Omega (-\Delta)^s_p u^*_\Omega \varphi(x)dx = \int_\Omega f(x)\varphi(x)dx \ \ \forall \varphi \in W^{s,p}.$$

So we have:

$$\begin{cases} (-\Delta)^s_p u^*_\Omega = f \text{ in } \Omega \\ u^*_\Omega = 0 \text{ on } \mathbb{R}^N \setminus \Omega \end{cases}.$$

Finally by taking $\varphi = u_{\Omega_{n_k}}$ in (4.3), and $\varphi = u_\Omega$ in (4.2) then that gives:

$$\begin{aligned} &\lim \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mid u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y) \mid^p}{\mid x - y \mid^{N+ps}} dxdy \right) \\ &= \lim \left(\int_{\Omega_{n_k}} f(x) u_{\Omega_{n_k}}(x)dx \right) \\ &= \int_\Omega f(x)u^*_\Omega = \int_{\mathbb{R}^{2N}} \frac{[u^*_\Omega(x) - u^*_\Omega(y)]^p}{\mid x - y \mid^{N+ps}} dxdy. \end{aligned}$$

By taking the limit to infinity we have:

$$\int_{\mathbb{R}^{2N}} \frac{[(u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y)) - (u_\Omega(x) - u_\Omega(y))]^p}{\mid x - y \mid^{N+ps}} = 0.$$

So,

$$\lim \int_{\Omega_{n_k}} f\left(u_{\Omega_{n_k}} - u_\Omega\right) = 0.$$

So we get:

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} \frac{[(u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y)) - (u_\Omega(x) - u_\Omega(y))]^p}{\mid x - y \mid^{N+ps}} \\ &= \int_{\mathbb{R}^{2N}} \frac{(u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y)) - (u_\Omega(x) - u_\Omega(y))}{\mid x - y \mid^{N+ps}} = 0, \end{aligned}$$

$$\int_{\Omega_{n_k}} f\left(u_{\Omega_{n_k}} - u_\Omega\right) = 0.$$

And in the same way we show that

$$\lim \int_{\Omega_{n_k}} f\left(u_{\Omega_{n_k}} - u_\Omega\right) = 0.$$

So we get:

$$u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y) \xrightarrow{\ L^p\ } u_\Omega(x) - u_\Omega(y)$$
$$u_{\Omega_{n_k}} \xrightarrow{\ L^p\ } u_\Omega.$$

And so we have:

$$\begin{aligned} F(\Omega_{n_k}) &= \frac{C(N,s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\mid u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y) \mid^{p-2} (u_{\Omega_{n_k}}(x) - u_{\Omega_{n_k}}(y))}{\mid x-y \mid^{N+ps}} dxdy \\ \to F(\Omega) &= \frac{C(N,s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\mid u_\Omega(x) - u_\Omega(y) \mid^{p-2} (u_\Omega(x) - u_\Omega(y))}{\mid x-y \mid^{N+ps}} dxdy. \end{aligned}$$

So we can conclude that there is an open Ω^* which minimizes F. □

5. Derivation of the topological derivative

5.1. Some preliminary results. In this subsection, we describe how to calculate the topological derivative using the min-max approach, see [8,27]. To begin with, we will look at the following definitions and notations.

Definition 5.1. *A Lagrangian function is a function of the form*

$$(t,x,y) \mapsto L(t,x,y) : [0,\tau] \times X \times Y \to \mathbb{R} \quad \tau > 0,$$

where X is a vector espace, Y a non empty subset of vector space and the function $y \mapsto L(t,x,y)$ is affine.

Associate with the parameter t the parametrized minimax

$$t \mapsto \mathrm{g}(t) = \inf_{x\in X} \sup_{y \in Y} L(t,x,y) : [0,\tau] \to \mathbb{R} \;\text{ and }\; d\mathrm{g}(0) = \lim_{t\to 0^+} \frac{\mathrm{g}(t)-\mathrm{g}(0)}{t}.$$

When the limits exist, we will use the following notations

$$\begin{aligned} d_t L(0,x,y) &= \lim_{t\to 0^+} \frac{L(t,x,y)-L(0,x,y)}{t} \\ \varphi\in X,\ \ d_x L(t,x,y;\varphi) &= \lim_{\theta\to 0^+} \frac{L(t,x+\theta\varphi,y)-L(t,x,y)}{\theta} \\ \phi\in Y\ \ d_y L(t,x,y;\phi) &= \lim_{\theta\to 0^+} \frac{L(t,x,y+\theta\phi)-L(t,x,y)}{\theta}. \end{aligned}$$

Since $L(t,x,y)$ is affine in y, for all $(t,x)\in[0,\tau]\times X$,

(5.1) $\forall\, y,\psi\in Y\ \ d_y L(t,x,y;\psi) = L(t,x,\psi)-L(t,x,0) = d_y L(t,x,0,\psi).$

The state equation at $t\geq 0$

(5.2) Find $x^t\in X$ such that for all $\psi\in Y,\ \ d_y L(t,x^t,0;\psi)=0.$

The set of states x^t at $t\geq 0$ is denoted

(5.3) $$E(t)=\left\{x^t\in X,\ \ \forall\,\psi\in Y,\ \ d_y L(t,x^t,0;\psi)=0\right\}.$$

The adjoint equation at $t\geq 0$ is

(5.4) Find $p^t\in Y$ such that for all $\varphi\in X,\ \ d_x L(t,x^t,p^t;\varphi)=0.$

The set of solutions p^t at $t\geq 0$ is denoted

(5.5) $$Y(t,x^t)=\left\{p^t,\in Y,\ \ \forall\,\varphi\in X,\ \ d_x L(t,x^t,p^t;\varphi)=0\right\}.$$

Finally the set of minimisers for the minimax is given by

(5.6) $$X(t)=\left\{x^t\in X,\ \ \mathrm{g}(t)=\inf_{x\in X}\sup_{y\in Y} L(t,x,y)=\sup_{y\in Y} L(t,x^t,y)\right\}.$$

Lemma 5.1. (Constrained infimum and minimax) *We have the following assertions:*

(i) $\inf_{x\in X}\sup_{y\in Y} L(t,x,y)=\inf_{x\in E(t)} L(t,x,0)$.

(ii) *The minimax* $\mathrm{g}(t)=+\infty$ *if and only if* $E(t)=\emptyset$. *And in this case we have* $X(t)=X$.

(iii) *If* $E(t)\neq\emptyset$, *then*

$$X(t)=\left\{x^t\in E(t):\ \ L(t,x^t,0)=\inf_{x\in E(t)} L(t,x,0)\right\}\subset E(t)$$

and $\mathrm{g}(t)<+\infty$.

Proof. See [8–10]. □

To end this subsection, we give definitions and theorems on d-dimensional Minkowski content and d-rectifiability.

Definition 5.2. *Let E be a subset of a metric space X. $E \subset X$ is d-rectifiable if it is the image of a compact subset K of $\mathbb{R}^d$ by a continuous lipschitzian function $f : \mathbb{R}^d \to X$.*

Let E be a closed compact set of $\mathbb{R}^N$ and $r \geq 0$, the distance function d_E and the r-dilatation E_r of E are defined as follows:

$$d_E(x) = \inf_{x_0 \in E} |x - x_0|, \quad E_r = \{x \in \mathbb{R}^N : \, d_E(x) \leq r\}.$$

Definition 5.3. *Given d, $0 \leq d \leq N$ the upper and lower d-dimensional Minkowski contents of a set E are defined by an r-dilatation of this set as follows*

$$M^{*d}(E) = \limsup_{r \to 0^+} \frac{m_N(E_r)}{\alpha_{N-d} r^{N-d}}; \quad M_*^d(E) = \liminf_{r \to 0^+} \frac{m_N(E_r)}{\alpha_{N-d} r^{N-d}}$$

where m_N is the Lebesgue measure in $\mathbb{R}^N$ and α_{N-d} is the volume of the ball of radius 1 in $\mathbb{R}^{N-d}$.

Both concepts can be found in [8, 9].

We need the following assumption for everything that follows:

Hypothesis (H0)

Let X be a vector space.

(i) For all $t \in [0, \tau]$, $x^0 \in X(0)$, $x^t \in X(t)$ and $y \in Y$, the function $\theta \mapsto L(t, x^0 + \theta(x^t - x^0), y) : [0, 1] \to \mathbb{R}$ is absolutely continuous. This implies that for almost all θ the derivative exists and is equal to $d_x L(t, x^0 + \theta(x^t - x^0), y; x^t - x^0)$ and it is the integral of its derivative. In particular

$$L(t, x^s, y) = L(t, x^0, y) + \int_0^1 d_x L(t, x^0 + \theta(x^t - x^0), y; x^t - x^0) \, d\theta.$$

(ii) For all $t \in [0, \tau]$, $x^0 \in X(0)$, $x^t \in X(s)$ and $y \in Y$, $\phi \in X$ and for almost all $\theta \in [0, 1]$, $d_x L(t, x^0 + \theta(x^t - x^0), y; \phi)$ exists and the functions $\theta \mapsto d_x L(t, x^0 + \theta(x^t - x^0), y; \phi)$ belong to $L^1[0, 1]$.

Definition 5.4. *Given $x^0 \in X(0)$ and $x^t \in X(t)$, the averaged adjoint equation is:*

$$\text{Find } \; y^t \in Y \;\; \forall\, \phi \in X, \quad \int_0^1 d_x L(t, x^0 + \theta(x^t - x^0), y; \phi)\, d\theta = 0.$$

and the set of solutions is noted $Y(t, x^0, x^t)$. Clearly $Y(0, x^0, x^0)$ reduces to the set of standard adjoint states $Y(0, x^0)$ at $t = 0$.

Theorem 5.1. *Consider the Lagrangian functional*

$$(t, x, y) \mapsto L(t, x, y) : [0, \tau] \times X \times Y \to \mathbb{R}, \quad \tau > 0,$$

where X and Y are vector spaces and the function $y \mapsto L(t, x, y)$ is affine. Assume that

(H0) *and the following hypotheses are satisfied.*
(H1) *for all $t \in [0, \tau]$, g(t) is finite, $X(t) = \{x^t\}$ and $Y(0, x^0) = \{p^0\}$ are singletons,*
(H2) *$d_t L(0, x^0, y^0)$ exists,*
(H3) *The following limit exists*

$$R(x^0, y^0) = \lim_{t \to 0^+} \int_0^1 d_x L\left(t, x^0 + \theta(x^t - x^0), p^0; \frac{x^t - x^0}{t}\right) d\theta.$$

Then, dg(0) exists and dg$(0) = d_t L(0, x^0, p^0) + R(x^0, p^0)$.

Proof. See [8, 9]. □

Corollary 5.1. *Consider the Lagrangian functional*

$$(t, x, y) \mapsto L(t, x, y) : [0, \tau] \times X \times Y \to \mathbb{R}, \quad \tau > 0,$$

where X and Y are vector spaces and the function $y \mapsto L(t, x, y)$ is affine. Assume that

(H0) *and the following assumptions are satisfied:*
(H1a) *for all $t \in [0, \tau]$, $X(s) \neq \emptyset$, $g(t)$ is finite, and for each $x \in X(0)$, $Y(0, x) \neq \emptyset$,*
(H2a) *for all $x \in X(0)$ and $p \in Y(0, x)$ $d_t L(0, x, p)$ exists,*
(H3a) *there exist $x^0 \in X(0)$ and $p^0 \in Y(0, x^0)$ such that the following limit exists*

$$R(x^0, p^0) = \lim_{t \to 0^+} \int_0^1 d_x L\left(t, x^0 + \theta(x^t - x^0), p^0; \frac{x^t - x^0}{t}\right) d\theta.$$

Then, dg(0) exists and there exist $x^0 \in X(0)$ and $p^0 \in Y(0, x^0)$ such that dg$(0) = d_t L(0, x^0, p^0) + R(x^0, p^0)$.

In what follows, we are interested in the main result of the topological derivative of the functional. For more information on this part, the reader can consult the papers of [8, 9, 27].

5.2. **The topological derivative of the functional.** Let us consider the functionnal defined in Ω_t by

$$F(\Omega_t) = \frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\mid u_\Omega(x)-u_\Omega(y)\mid^{p-2}(u_\Omega(x)-u_\Omega(y))}{\mid x-y\mid^{N+ps}}dxdy. \tag{5.7}$$

where u_{Ω_t} be the solution to the following p- laplacian operator

$$\begin{cases}(-\Delta)^s_p u = f \text{ in } \Omega_t,\\ u = 0 \text{ on } \mathbb{R}^N\setminus\Omega_t,\\ p\geq 2.\end{cases} \tag{5.8}$$

Let us consider as shape functional F define by

$$F(\Omega) = \frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\mid u_\Omega(x)-u_\Omega(y)\mid^{p-2}(u_\Omega(x)-u_\Omega(y))}{\mid x-y\mid^{N+ps}}dxdy \tag{5.9}$$

and $u_\Omega \in W^{s,p}(\Omega)$ is solution to the variational problem

$$\begin{aligned}&\int_{\mathbb{R}^{2N}}\frac{\mid u_\Omega(x)-u_\Omega(y)\mid^{p-2}(u_\Omega(x)-u_\Omega(y))(v_\Omega(x)-v_\Omega(y))}{\mid x-y\mid^{N+ps}}dxdy\\ &=\int_\Omega f(x)v_\Omega(x)dx \;\; \forall v\in W^{s,p}(\Omega).\end{aligned} \tag{5.10}$$

We aim to compute the topological derivative of the functional $F(\Omega_t)$

$$dF = \lim_{t\longrightarrow 0}\frac{F(\Omega_t)-F(\Omega)}{\alpha_{N-d}r^{N-d}}.$$

Thus, the Lagrangian dependent on t will be written in the form:

$$\begin{aligned}L(t,\phi,\Phi) &= \frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\mid u_\Omega(x)-u_\Omega(y)\mid^{p-2}(u_\Omega(x)-u_\Omega(y))}{\mid x-y\mid^{N+ps}}dxdy\\ &+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\mid u_\Omega(x)-u_\Omega(y)\mid^{p-2}(u_\Omega(x)-u_\Omega(y))(v_\Omega(x)-v_\Omega(y))}{\mid x-y\mid^{N+ps}}dxdy\\ &-\int_\Omega f(x)v_\Omega(x)dx\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
& L(t,\phi,\Phi) \\
= \;& \frac{C(N,s)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(-1)^{p-2}\left[u_\Omega(x)-u_\Omega(y)\right]^{p-2}\left(u_\Omega(x)-u_\Omega(y)\right)}{\mid x-y\mid^{N+ps}}dxdy \\
+ \;& \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(-1)^{p-2}\left[u_\Omega(x)-u_\Omega(y)\right]^{p-2}\left(u_\Omega(x)-u_\Omega(y)\right)(v_\Omega(x)-v_\Omega(y))}{\mid x-y\mid^{N+ps}}dxdy \\
- \;& \int_\Omega f(x)v_\Omega(x)dx \\
F(\Omega_t) = \;& \inf_{\phi\in W_0^{s,p}(\Omega)}\sup_{\Phi\in W_0^{s,p}(\Omega)} L(t,\phi,\Phi).
\end{aligned}$$

From this, we can now evaluate the derivative of the Lagrangian, dependent on t, with respect to ϕ:

$$\begin{aligned}
& d_\phi L(t,\phi,\Phi,\phi') \\
= \;& \frac{C(N,s)}{2}\int_{\mathbb{R}^{2N}}\frac{(p-2)\left(\phi'(x)-\phi'(y)\right)(-1)^{p-2}\left[\phi_\Omega(x)-\phi_\Omega(y)\right]^{p-3}\left(\phi(x)-\phi(y)\right)}{\mid x-y\mid^{N+ps}} \\
+ \;& \frac{C(N,s)}{2}\int_{\mathbb{R}^{2N}}\frac{(\phi'(x)-\phi'(y))(-1)^{p-2}\left[\phi_\Omega(x)-\phi_\Omega(y)\right]^{p-2}}{|x-y|^{N+ps}}dxdy \\
+ \;& \int_{\mathbb{R}^{2N}}\frac{(p-2)\left(\phi'(x)-\phi'(y)\right)(-1)^{p-2}\left[\phi_\Omega(x)-\phi_\Omega(y)\right]^{p-3}\left(\phi(x)-\phi(y)\right)\left(\Phi(x)\right)}{\mid x-y\mid^{N+ps}} \\
- \;& \int_{\mathbb{R}^{2N}}\frac{(p-2)\left(\phi'(x)-\phi'(y)\right)(-1)^{p-2}\left[\phi_\Omega(x)-\phi_\Omega(y)\right]^{p-3}\left(\phi(x)-\phi(y)\right)\left(\Phi(y)\right)}{\mid x-y\mid^{N+ps}} \\
+ \;& \int_{\mathbb{R}^{2N}}\frac{(\phi'(x)-\phi'(y))(-1)^{p-2}\left[\phi_\Omega(x)-\phi_\Omega(y)\right]^{p-2}\left(\Phi(x)-\Phi(y)\right)}{|x-y|^{N+ps}}dxdy.
\end{aligned}$$

The initial adjoint state p_{Ω_0} is a solution of $d_\phi L(0,u_{\Omega_0},p_{\Omega_0},\phi')=0$ for all ϕ' for $t=0$. Thus the variational formulation of the adjoint equation of state is given by

$$\begin{aligned}
& \frac{C(N,s)}{2}\int_{\mathbb{R}^{2N}}\frac{(p-2)\left(\phi'(x)-\phi'(y)\right)(-1)^{p-2}\left[u_{\Omega_0}(x)-u_{\Omega_0}(y)\right]^{p-3}\left(u_{\Omega_0}(x)-u_{\Omega_0}(y)\right)}{\mid x-y\mid^{N+ps}}dxdy \\
& +\int_{\mathbb{R}^{2N}}\frac{(\phi'(x)-\phi'(y))(-1)^{p-2}\left[u_{\Omega_0}(x)-u_{\Omega_0}(y)\right]^{p-2}}{|x-y|^{N+ps}}dxdy \\
& +\int_{\mathbb{R}^{2N}}\frac{(p-2)\left(\phi'(x)-\phi'(y)\right)(-1)^{p-2}\left[u_{\Omega_0}(x)-u_{\Omega_0}(y)\right]^{p-3}}{\mid x-y\mid^{N+ps}} \\
& \cdot\left(u_{\Omega_0}(x)-u_{\Omega_0}(y)\right)\left(p_{\Omega_0}(x)-p_{\Omega_0}(y)\right)dxdy
\end{aligned}$$

$$+ \int_{\mathbb{R}^{2N}} \frac{(\phi'(x)-\phi'(y))(-1)^{p-2}\left[u_{\Omega_0}(x)-u_{\Omega_0}(y)\right]^{p-2}(p_{\Omega_0}(x)-p_{\Omega_0}(y))}{|x-y|^{N+ps}}dxdy = 0.$$

And we have

$$\int_{\mathbb{R}^{2N}} \frac{\left[(p-2)\left(\phi'(x)-\phi'(y)\right)(-1)^{p-2}\left[u_{\Omega_0}(x)-u_{\Omega_0}(y)\right]^{p-3}\left(u_{\Omega_0}(x)-u_{\Omega_0}(y)\right)\right]}{\mid x-y\mid^{N+ps}} \cdot \left[p_{\Omega_0}(x)-p_{\Omega_0}(y)\right]dxdy$$
$$+\int_{\mathbb{R}^{2N}} \frac{\left[(p-2)\left(\phi'(x)-\phi'(y)\right)(-1)^{p-2}\left[u_{\Omega_0}(x)-u_{\Omega_0}(y)\right]^{p-3}\left(u_{\Omega_0}(x)-u_{\Omega_0}(y)\right)\right]}{\mid x-y\mid^{N+ps}} \cdot \left[\frac{C(N,s)}{2}\right]dxdy$$
$$+\int_{\mathbb{R}^{2N}} \frac{(\phi'(x)-\phi'(y))(-1)^{p-2}\left[u_{\Omega_0}(x)-u_{\Omega_0}(y)\right]^{p-2}\left[(p_{\Omega_0}(x)-p_{\Omega_0}(y))+1\right]}{|x-y|^{N+ps}}dxdy = 0.$$

And finally it can be rewritten in the following form

$$\int_{\mathbb{R}^{2N}} \frac{\left[(p-2)\left(\phi'(x)-\phi'(y)\right)(-1)^{p-2}\left[u_{\Omega_0}(x)-u_{\Omega_0}(y)\right]^{p-3}\left(u_{\Omega_0}(x)-u_{\Omega_0}(y)\right)\right]}{\mid x-y\mid^{N+ps}} \cdot \left[\frac{C(N,s)}{2}\right]dxdy$$
$$+\int_{\mathbb{R}^{2N}} \frac{\left[(p-2)\left(\phi'(x)-\phi'(y)\right)(-1)^{p-2}\left[u_{\Omega_0}(x)-u_{\Omega_0}(y)\right]^{p-3}\left(u_{\Omega_0}(x)-u_{\Omega_0}(y)\right)\right]}{\mid x-y\mid^{N+ps}} \cdot \left[p_{\Omega_0}(x)-p_{\Omega_0}(y)\right]dxdy$$
$$+\int_{\mathbb{R}^{2N}} \frac{(\phi'(x)-\phi'(y))\mid u_{\Omega_0}(x)-u_{\Omega_0}(y)\mid^{p-2}\left[(p_{\Omega_0}(x)-p_{\Omega_0}(y))+1\right]}{|x-y|^{N+ps}}dxdy = 0.$$

Next, we derive the Lagrangian with respect to Φ:

$$\begin{aligned} & d_\Phi L(t,\phi,\Phi,\Phi') \\ = & \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\mid \phi_\Omega(x)-\phi_\Omega(y)\mid^{p-2}(\phi_\Omega(x)-\phi_\Omega(y))(\Phi'(x)-\Phi'(y))}{|x-y|^{N+ps}}dxdy \\ - & \int_{\Omega_t} f(x)\Phi'(x)dx. \end{aligned}$$

The initial state u_{Ω_0} is a solution of $d_\Phi L(0,u_{\Omega_0},0,\Phi'_{\Omega_0}) = 0 \ \forall \ \Phi'_{\Omega_0} \in W^{s,p}(\Omega)$ and in this case, we have:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\mid u_{\Omega_0}(x)-u_{\Omega_0}(y)\mid^{p-2}(u_{\Omega_0}(x)-u_{\Omega_0}(y))(\Phi'(x)-\Phi'(y))}{|x-y|^{N+ps}}dxdy$$

$$-\quad\int_{\Omega_t}f(x)\Phi'(x)dx=0.$$

And we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\mid u_{\Omega_0}(x)-u_{\Omega_0}(y)\mid^{p-2}(u_{\Omega_0}(x)-u_{\Omega_0}(y))(\Phi'(x)-\Phi'(y))}{|x-y|^{N+ps}}dxdy$$

$$=\quad\int_{\Omega_t}f(x)\Phi'(x)dx,$$

$$L(t,\phi,\Phi)-L(0,\phi,\Phi)=\int_{\Omega_t}f(x)\Phi(x)dx-\int_{\Omega}f(x)\Phi(x)dx,$$

$$L(t,\phi,\Phi)-L(0,\phi,\Phi)=\int_{\Omega_t}f(x)\Phi(x)dx-\int_{E_t}f(x)\Phi(x)dx-\int_{\Omega_t}f(x)\Phi(x)dx,$$

$$L(t,\phi,\Phi)-L(0,\phi,\Phi)=-\int_{E_t}f(x)\Phi(x)dx,$$

$$d_sL(0,\phi,\Phi)=-\lim_{s\longrightarrow 0}\frac{1}{|B(x_0,s)|}\left[\int_{B(x_0,s)}f(x)\Phi(x)\right]dx,$$

$$d_sL(0,\phi,\Phi)=-f(x_0)\Phi(x_0).$$

We will now define $R(t)$ by

$$\mathcal{R}(t)=\int_0^1 d_\phi L\left(t,u_{\Omega_0}+\Psi\left(u_{\Omega_t}-u_{\Omega_0}\right),p_{\Omega_0},\left(\frac{u_{\Omega_t}-u_{\Omega_0}}{t}\right)\right)\,d\Psi.$$

By substituting $\phi'=\frac{u_{\Omega_t}-u_{\Omega_0}}{t}$ and $\Psi=\frac{u_{\Omega_t}-u_{\Omega_0}}{2}$ into the adjoint equation for p_{Ω_0}, we obtain:

$$\mathcal{R}(t)=\int_{\mathbb{R}^{2N}}\frac{C(N,s)(p-2)\left|\left(\frac{u_{\Omega_t}-u_{\Omega_0}}{t}\right)(x)-\left(\frac{u_{\Omega_t}-u_{\Omega_0}}{t}\right)(y)\right|}{2\mid x-y\mid^{N+ps}}$$
$$\cdot\left|\left(\frac{u_{\Omega_t}+u_{\Omega_0}}{2}\right)(x)-\left(\frac{u_{\Omega_t}+u_{\Omega_0}}{2}\right)(y)\right|^{p-2}dxdy$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^{2N}} \frac{C(N,s)\left|\left(\frac{u_{\Omega_t}-u_{\Omega_0}}{t}\right)(x)-\left(\frac{u_{\Omega_t}-u_{\Omega_0}}{t}\right)(y)\right|}{2\mid x-y\mid^{N+ps}}\\
&\cdot\left|\left(\frac{u_{\Omega_t}+u_{\Omega_0}}{2}\right)(x)-\left(\frac{u_{\Omega_t}+u_{\Omega_0}}{2}\right)(y)\right|^{p-2} dxdy\\
&+\int_{\mathbb{R}^{2N}} \frac{\left|\left(\frac{u_{\Omega_t}-u_{\Omega_0}}{t}\right)(x)-\left(\frac{u_{\Omega_t}-u_{\Omega_0}}{t}\right)(y)\right|\left|\left(\frac{u_{\Omega_t}+u_{\Omega_0}}{2}\right)(x)-\left(\frac{u_{\Omega_t}+u_{\Omega_0}}{2}\right)(y)\right|^{p-2}}{\mid x-y\mid^{N+ps}}\\
&\cdot\left[(p-2)p_{\Omega_0}(x)\right] dxdy\\
&+\int_{\mathbb{R}^{2N}} \frac{\left|\left(\frac{u_{\Omega_t}-u_{\Omega_0}}{t}\right)(x)-\left(\frac{u_{\Omega_t}-u_{\Omega_0}}{t}\right)(y)\right|\left|\left(\frac{u_{\Omega_t}+u_{\Omega_0}}{2}\right)(x)-\left(\frac{u_{\Omega_t}+u_{\Omega_0}}{2}\right)(y)\right|^{p-2}}{\mid x-y\mid^{N+ps}}\\
&\cdot\left[-(p-2)p_{\Omega_0}(y)\right]\\
&+\int_{\mathbb{R}^{2N}} \frac{\left|\left(\frac{u_{\Omega_t}-u_{\Omega_0}}{t}\right)(x)-\left(\frac{u_{\Omega_t}-u_{\Omega_0}}{t}\right)(y)\right|\left|\left(\frac{u_{\Omega_t}+u_{\Omega_0}}{2}\right)(x)-\left(\frac{u_{\Omega_t}+u_{\Omega_0}}{2}\right)(y)\right|^{p-2}}{\mid x-y\mid^{N+ps}}\\
&\cdot\left(p_{\Omega_0}(x)-p_{\Omega_0}(y)\right) dxdy.
\end{aligned}
$$

Theorem 5.2. *Let $0\le d<N,E$. Verify **Hypothesis H1** and $t=\alpha_{N-d}r^{N-d}$. The topological derivative exists if the function $\mathcal{R}(t)$ has a finite limit. Therfore, the topological derivative of the function is given by the expression:*

$$
dF=\lim_{t\longrightarrow 0}\frac{F(\Omega_t)-F(\Omega)}{\alpha_{N-d}r^{N-d}},\qquad dF=\mathcal{R}(x_0,p_{\Omega_0})-f(x_0)\ p_{\Omega_0}(x_0).
$$

where $p_{\Omega_0},u_{\Omega_0}$ are solutions of systems

$$
\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{\left[(p-2)\left(\phi'(x)-\phi'(y)\right)(-1)^{p-2}\left[u_{\Omega_0}(x)-u_{\Omega_0}(y)\right]^{p-3}\left(u_{\Omega_0}(x)-u_{\Omega_0}(y)\right)\right]}{\mid x-y\mid^{N+ps}}\\
&\cdot\left[\frac{C(N,s)}{2}\right] dxdy\\
&+\int_{\mathbb{R}^{2N}} \frac{\left[(p-2)\left(\phi'(x)-\phi'(y)\right)(-1)^{p-2}\left[u_{\Omega_0}(x)-u_{\Omega_0}(y)\right]^{p-3}\left(u_{\Omega_0}(x)-u_{\Omega_0}(y)\right)\right]}{\mid x-y\mid^{N+ps}}\\
&\cdot\left[p_{\Omega_0}(x)-p_{\Omega_0}(y)\right] dxdy\\
&+\int_{\mathbb{R}^{2N}} \frac{\left(\phi'(x)-\phi'(y)\right)\mid u_{\Omega_0}(x)-u_{\Omega_0}(y)\mid^{p-2}\left[\left(p_{\Omega_0}(x)-p_{\Omega_0}(y)\right)+1\right]}{|x-y|^{N+ps}}dxdy=0.
\end{aligned}
$$

6. Conclusion and perspectives

The present work deals with topological optimization for a fractional p-Laplacian operator problem. We first looked at the existence of weak solutions for the fractional p-Laplacian problem, a part that was essential in the topological derivative part for the application of the theorems used. Next we established the existence of optimal forms, but using the $\epsilon-$ cône property. The topological derivative follows, using the minmax method.

In what follows, we plan not only to carry out numerical simulations, but also to give a generalization of the problem studied by combining both the shape derivative and the topological derivative.

Conflict of interest

The authors declare that they have no conflict ot interest.

References

[1] A. Henrot, M. Pierre: *Variation et optimisation de formes*, Mathématiques et Applications (Berlin), Springer, Berlin, **48**, 2005.

[2] A. Lischke , G. Pang, M. Gulian, F. Song , Ch. Glusa , X. Zheng , Zh. Mao , W. Cai , M. M. Meerschaert , M. Ainsworth , G. E. Karniadakis: *What is the fractional Laplacian?, A comparative review with new results*. Journal of Computational Physics, **404** (2020), art. no. 109009.

[3] G. Buttazzo, G. D. G. Maso: *An existence result for a class of shape optimization problems*, Arch. Ration. Mech. Anal., **122**(2) (1993), 183-195.

[4] G. Buttazzo and H. Shrivastava : *Optima shapes for general integral funtionals*, Annales Henri Lebesgue, **3** (2020), 261-272.

[5] L. A. Caffarelli: *Nonlocal equations, drifts and games, Nonlinear Partial Differential Equations,* Abel Symposia, **7** (2012), 37-52.

[6] L. M. Del-Pezzo, A. Quaas: *Global bifurcation for fractional p-Laplacian and application*, Z. Anal. Anwend. **35**(4) (2016), 411-447.

[7] L. M. Del Pezzo, A. Quaas: *Non-resonant Fredholm alternative and anti-maximum principle for the fractional p-Laplacian*, J. Fixed Point Theory Appl. **19**(2017), 939-958.

[8] M. C. Delfour: *Topological derivative: A semidifferential via the Minkowski content*, J. Convex Anal., **3** (2018), 957-982.

[9] M. C. Delfour: *Control, shape, and topological derivatives via minimax differentiability of Lagrangians*, in Numerical Methods for Optimal Control Problems Springer INdAM Ser. 29,

M. Falcone, R. Ferretti, L. Grüne, and W. McEneaney, eds., Springer, Cham, Switzerland, 2018, 137-164.

[10] M. C. DELFOUR: *Topological derivatives via one-sided derivative of parametrized minima and minimax*, Eng. Comput., (2021).

[11] J.F. BONDER, A. RITORTO, A. M. SALORT: *Shape optimization problems for nonlocal operators*, arXiv preprint arXiv: 1612.08717, 2016.

[12] J. F. BONDER, A. RITORTO, A. M. SALORT: *A class of shape optimization problems for some non local operators*, Advances in Calculus of Variations, **11**(4), 2018, 373-386.

[13] L. CAFFARELLI, L. SILVESTRE: *An extension problem related to the fractional Laplacian*, Communications in partial differential equations, **32**(8) (2007), 1245-1260.

[14] A. L. DALIBARD, G. VARET: *On Shape optimization problems involving the fractional laplacian*, ESAIM, **19** (2013), 976-1013.

[15] E. DI NEZZA, G. PALATUCCI, E. VALDINOCI: *Hitchhiker ' s guide to the fractional Sobolev spaces*. Bull. Sci. math. **136**(5) (2012), 521-573.

[16] F. GIOVANNI, P. GIAMPIERO: *Fractional p-eigenvalues*, Riv. Math. Univ. Parma **5**(1) (2014), 373-386.

[17] M. KWASNICKI: *Ten equivalent definitions of the fractional Laplace operator*, Fractional Calculus and Applied Analysis. **20**(1) (2017), 7-51.

[18] A. DEVI, A. KUMAR: *Hyers-Ulam stability and existence of solution for hybrid fractional differential equation with p-Laplacian operator* Chaos, Solitons & Fractals, **156** (2022), art. no. 111859.

[19] S. DHANIYA, A. KUMAR, A. K. TH. ABDELJAWAD, M. A. ALQUDAH: *Existence Results of Langevin Equations with Caputo-Hadamard Fractional Operator*, Journal of Mathematics, 2023, Article ID 2288477, 12 pages.

[20] P. BEDI, A. KHAN, A. KUMAR, TH. ABDELJAWAD: *Computational Study of fractional-order vector bornd diseases model*, Fractals, **30**(05) (2022), art. no. 2240149.

[21] M. FALL, I. FAYE, A. SY, D. SECK: *On Shape Optimisation Theory with Fractional Laplacian*, Applied and Computational Mathematics, **10**(3) (2021), 56-68.

[22] M. FALL, A. SY, I. FAYE, D. SECK: *On Shape Optimization Theory with Fractional p-Laplacian Opeartors*, Abstract and Applied Analysis, 2025, Article ID 1932719, 12 pages.

[23] M. M. FALL: *Regularity estimates for non local Schroidinger equations*, arXiv preprint arXiv:1711.02206, 2017.

[24] R. METZLER, J. KLAFTER: *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, Physics Reports, **339**(1) (2000), 1-77.

[25] R. METZLER, J. KLAFTER: *Boundary value problems for fractional diffusion equations*, Physica A, **278** (2000), art. no. 107?125.

[26] M. CHRISTOF, Z.N. SAKELLARIS: *Global dissipative half-harmonic flows into spheres: small data in critical Sobolev spaces*. Communications in Partial Differential Equations, **44**(5) (2018), 397-415.

[27] M. G. NGOM, I. FAYE, D. SECK: *A minmax method on shape and topological derivatives and homogenization: the case of helmholtz equation*, Nonlinear studies, **30**(1) (2023), 213-247.

[28] A. NIANG: *Régularité de Solutions d'équations aux dérivées partielles elliptiques avec conditions aux bords mixtes*, thèse de Doctorat de Mathématiques, UCAD, 2020.

[29] X. ROS-OTON, J. SERRA: *The Dirichlet problem for the fractional Laplacian: regularity up to the boundary*, Journal de Mathématiques Pures et Appliquées, **101**(3) (2014), 275-302.

[30] L. M. DEL-PEZZO, A. QUAAS: *Global bifurcation for fractional p-Laplacian and application*, Z. Anal. Anwend, **35**(4) (2016), 411-447.

[31] X. ROS-OTON, J. SERRA: *Boundary regularity for fully nonlinear integro-differential equations*, Duke Mathematical Journal, **165**(11) (2016), 2079–2154.

[32] X. ROS-OTON, J. SERRA: *Regularity theory for general stable operators, Journal of Differential Equations*, **260**(12) (2016), 8675–8715.

[33] M. SRATI: *$P-$Laplacien fractionnaire $(-\Delta)_p^s$*, M.Sc. thesis, 2017, Univ S. M. B. Abdellah, Maroco.

[34] WETTSTEIN, D. JEROME: *Half-harmonic gradient flow: aspects of a non-local geometric PDE*. Mathematics in Engineering, **5**(3) (2023), 1-38.

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