

## CONTRIBUTIONS TO SEVERAL KEY INEQUALITIES, INCLUDING SOME OF THE CARLSON TYPE

Christophe Chesneau

**ABSTRACT.** The literature contains many key integral inequalities of reference. Among these are those proven by S. Barza and E. C. Popa in 1998. In this article, we present five new contributions to these inequalities, including an alternative proof, sharper inequalities, and new extensions. All the necessary details are provided, making the article self-contained.

### 1. INTRODUCTION

Integral inequalities form a fundamental part of analysis and have a wide range of applications in mathematics. Many such inequalities have been developed over the years, significantly contributing to our understanding of integral operators and their properties. One of the most well-known results of this kind is the Cauchy-Schwarz integral inequality, which is stated below. Let  $f, g : [0, +\infty) \mapsto [0, +\infty)$  be two functions. Then we have

$$\int_0^{+\infty} f(x)g(x)dx \leq \sqrt{\int_0^{+\infty} f^2(x)dx} \sqrt{\int_0^{+\infty} g^2(x)dx},$$

provided that the integrals of the upper bound converge. The article is motivated by another famous result: the Carlson integral inequality, which is stated below.

*2020 Mathematics Subject Classification.* 26D15, 33E20.

*Key words and phrases.* integral inequalities, Carlson integral inequality, Hölder integral inequality, beta function.

*Submitted:* 30.09.2025; *Accepted:* 16.10.2025; *Published:* 17.10.2025.

Let  $f, g : [0, +\infty) \mapsto [0, +\infty)$  be two functions. Then we have

$$\left[ \int_0^{+\infty} f(x) dx \right]^2 \leq \pi \sqrt{\int_0^{+\infty} f^2(x) dx} \sqrt{\int_0^{+\infty} x^2 f^2(x) dx},$$

provided that the integrals of the upper bound converge. The constant factor  $\pi$  is known to be optimal, and equality is achieved with the function  $f(x) = 1/(1+x^2)$ . Further details can be found in [5]. This inequality and its discrete counterpart, have inspired a large body of research, including the works in [1–4, 6–14].

The results established by S. Barza and E. C. Popa in [4] (1998) represent a significant advancement in the study of Carlson-type integral inequalities. Building on this foundational work, the present article makes five new contributions that extend and refine two of their theorems. These include an alternative proof, refined versions of the original inequalities and several new generalizations. All necessary details are provided to make the presentation self-contained and accessible to readers with a background in real analysis and inequality theory.

The structure of the paper is as follows: In Section 2, we revisit and analyze Carlson-type integral inequalities. Section 3 is devoted to specific variations of these inequalities. Finally, Section 4 presents some concluding remarks.

## 2. CARLSON-TYPE INTEGRAL INEQUALITIES

**2.1. Extension of an existing theorem.** The theorem below was established in [4]. Thanks to the incorporation of an auxiliary function  $g$ , it can be presented as an extension of the Carlson integral inequality.

**Theorem 2.1.** [4, Theorem 2.1] *Let  $f, g : [0, +\infty) \mapsto [0, +\infty)$  be two functions. We suppose that  $g$  is differentiable and non-decreasing with  $g(0) = 0$ ,  $\lim_{x \rightarrow +\infty} g(x) = +\infty$  and*

$$0 < m = \inf_{x \in [0, +\infty)} g'(x) < +\infty.$$

*Then we have*

$$\left[ \int_0^{+\infty} f(x) dx \right]^2 \leq \frac{\pi}{m} \sqrt{\int_0^{+\infty} f^2(x) dx} \sqrt{\int_0^{+\infty} g^2(x) f^2(x) dx},$$

*provided that the integrals of the upper bound converge.*

In our first contribution, we relax some of the assumptions made on  $g$  and refine the constant in the factor, i.e.,  $\pi/m$ . The details are given in the theorem below.

**Theorem 2.2.** *Let  $f : [0, +\infty) \mapsto [0, +\infty)$  and  $g : [0, +\infty) \mapsto \mathbb{R}$  be two functions. We suppose that  $g$  is differentiable and non-decreasing. We set*

$$\ell = \lim_{x \rightarrow 0} g(x), \quad L = \lim_{x \rightarrow +\infty} g(x),$$

*mentioning that they can be infinite. Then we have*

$$\begin{aligned} & \left[ \int_0^{+\infty} f(x) dx \right]^2 \\ & \leq 2 \arctan \left[ \frac{(L - \ell) \sqrt{\int_0^{+\infty} [1/g'(x)] f^2(x) dx} \int_0^{+\infty} [g^2(x)/g'(x)] f^2(x) dx}{\int_0^{+\infty} [g^2(x)/g'(x)] f^2(x) dx + L\ell \int_0^{+\infty} [1/g'(x)] f^2(x) dx} \right] \\ & \quad \times \sqrt{\int_0^{+\infty} \frac{1}{g'(x)} f^2(x) dx \int_0^{+\infty} \frac{g^2(x)}{g'(x)} f^2(x) dx}, \end{aligned}$$

*provided that the integrals of the upper bound converge.*

*Proof.* In order to facilitate future developments, we set

$$U = \int_0^{+\infty} \frac{1}{g'(x)} f^2(x) dx, \quad V = \int_0^{+\infty} \frac{g^2(x)}{g'(x)} f^2(x) dx.$$

Then, decomposing suitably the integral of  $f$ , introducing  $\alpha$  and  $\beta$  to be chosen later, applying the Cauchy-Schwarz integral inequality, using the arctangent primitive and the formula  $\arctan(x) - \arctan(y) = \arctan[(x - y)/(1 + xy)]$  for  $x, y > 0$ , we obtain

$$\begin{aligned} & \left[ \int_0^{+\infty} f(x) dx \right]^2 \\ & = \left[ \int_0^{+\infty} \sqrt{\frac{g'(x)}{\alpha + \beta g^2(x)}} \sqrt{\frac{\alpha + \beta g^2(x)}{g'(x)}} f(x) dx \right]^2 \\ & \leq \left[ \int_0^{+\infty} \frac{g'(x)}{\alpha + \beta g^2(x)} dx \right] \left[ \int_0^{+\infty} \frac{\alpha + \beta g^2(x)}{g'(x)} f^2(x) dx \right] \end{aligned}$$

$$\begin{aligned}
&= \left[ \frac{1}{\sqrt{\alpha\beta}} \arctan \left[ \sqrt{\frac{\beta}{\alpha}} g(x) \right] dx \right]_{x \rightarrow 0}^{x \rightarrow +\infty} (\alpha U + \beta V) \\
&= \left[ \arctan \left[ \sqrt{\frac{\beta}{\alpha}} L \right] - \arctan \left[ \sqrt{\frac{\beta}{\alpha}} \ell \right] \right] \left[ \sqrt{\frac{\alpha}{\beta}} U + \sqrt{\frac{\beta}{\alpha}} V \right] \\
&\leq \arctan \left[ \frac{(L - \ell) \sqrt{\alpha\beta}}{\alpha + L\ell\beta} \right] \left[ \sqrt{\frac{\alpha}{\beta}} U + \sqrt{\frac{\beta}{\alpha}} V \right].
\end{aligned}$$

With the choices  $\alpha = V$  and  $\beta = U$ , this inequality becomes

$$\left[ \int_0^{+\infty} f(x) dx \right]^2 \leq 2 \arctan \left[ \frac{(L - \ell) \sqrt{\alpha\beta}}{\alpha + L\ell\beta} \right] \sqrt{\alpha\beta},$$

which can also be expressed in the following expanded form:

$$\begin{aligned}
&\left[ \int_0^{+\infty} f(x) dx \right]^2 \\
&\leq 2 \arctan \left[ \frac{(L - \ell) \sqrt{\int_0^{+\infty} [1/g'(x)] f^2(x) dx \int_0^{+\infty} [g^2(x)/g'(x)] f^2(x) dx}}{\int_0^{+\infty} [g^2(x)/g'(x)] f^2(x) dx + L\ell \int_0^{+\infty} [1/g'(x)] f^2(x) dx} \right] \\
&\times \sqrt{\int_0^{+\infty} \frac{1}{g'(x)} f^2(x) dx \int_0^{+\infty} \frac{g^2(x)}{g'(x)} f^2(x) dx}.
\end{aligned}$$

This ends the proof of Theorem 2.2. □

Some inequalities derived from Theorem 2.2 are discussed below.

- Using the classical upper bound  $\arctan(x) \leq \pi/2$  for any  $x \geq 0$ , we have

$$\begin{aligned}
(2.1) \quad &\left[ \int_0^{+\infty} f(x) dx \right]^2 \\
&\leq 2 \times \frac{\pi}{2} \times \sqrt{\int_0^{+\infty} \frac{1}{g'(x)} f^2(x) dx \int_0^{+\infty} \frac{g^2(x)}{g'(x)} f^2(x) dx}
\end{aligned}$$

$$(2.2) \quad = \pi \sqrt{\int_0^{+\infty} \frac{1}{g'(x)} f^2(x) dx \int_0^{+\infty} \frac{g^2(x)}{g'(x)} f^2(x) dx}.$$

Moreover, if we assume that  $0 < m = \inf_{x \in [0, +\infty)} g'(x) < +\infty$ , then we immediately deduce

$$\left[ \int_0^{+\infty} f(x) dx \right]^2 \leq \frac{\pi}{m} \sqrt{\int_0^{+\infty} f^2(x) dx} \sqrt{\int_0^{+\infty} g^2(x) f^2(x) dx}.$$

This is the result in Theorem 2.1, i.e., [4, Theorem 2.1]. In particular, setting  $g(x) = x$ , we have

$$\left[ \int_0^{+\infty} f(x) dx \right]^2 \leq \pi \sqrt{\int_0^{+\infty} f^2(x) dx} \sqrt{\int_0^{+\infty} x^2 f^2(x) dx}.$$

This is the Carlson integral inequality. Our result is thus much more general.

- Using the classical upper bound  $\arctan(x) \leq x$  for any  $x \geq 0$ , we have

$$\begin{aligned} & \left[ \int_0^{+\infty} f(x) dx \right]^2 \\ & \leq 2 \frac{(L - \ell) \sqrt{\int_0^{+\infty} [1/g'(x)] f^2(x) dx} \sqrt{\int_0^{+\infty} [g^2(x)/g'(x)] f^2(x) dx}}{\int_0^{+\infty} [g^2(x)/g'(x)] f^2(x) dx + L\ell \int_0^{+\infty} [1/g'(x)] f^2(x) dx} \\ & \quad \times \sqrt{\int_0^{+\infty} \frac{1}{g'(x)} f^2(x) dx} \sqrt{\int_0^{+\infty} \frac{g^2(x)}{g'(x)} f^2(x) dx} \\ & = 2 \frac{(L - \ell) \int_0^{+\infty} [1/g'(x)] f^2(x) dx \int_0^{+\infty} [g^2(x)/g'(x)] f^2(x) dx}{\int_0^{+\infty} [g^2(x)/g'(x)] f^2(x) dx + L\ell \int_0^{+\infty} [1/g'(x)] f^2(x) dx}. \end{aligned}$$

While this upper bound may be more precise than that in Equation (2.1) for certain functions  $f$  and  $g$ , it is clearly more complex.

**2.2. A new theorem.** Our second contribution is a new theorem proposing a Carlson-type integral inequality with an original functional structure. The details are given in the statement below.

**Theorem 2.3.** *Let  $f : [0, +\infty) \mapsto [0, +\infty)$  and  $g : [0, +\infty) \mapsto \mathbb{R}$  be two functions. We suppose that  $g$  is differentiable and non-decreasing. We set*

$$\ell = \lim_{x \rightarrow 0} g(x), \quad L = \lim_{x \rightarrow +\infty} g(x),$$

mentioning that they can be infinite. Then we have

$$\begin{aligned} & \left[ \int_0^{+\infty} f(x) dx \right]^2 \\ & \leq \frac{2}{\gamma} \log \left[ \frac{\int_0^{+\infty} [1/g'(x)] f^2(x) dx + e^{-\gamma \ell} \int_0^{+\infty} [e^{\gamma g(x)}/g'(x)] f^2(x) dx}{\int_0^{+\infty} [1/g'(x)] f^2(x) dx + e^{-\gamma L} \int_0^{+\infty} [e^{\gamma g(x)}/g'(x)] f^2(x) dx} \right] \\ & \times \int_0^{+\infty} \frac{1}{g'(x)} f^2(x) dx, \end{aligned}$$

provided that the integrals of the upper bound converge.

*Proof.* (Proof of Theorem 2.3) In order to facilitate future developments, we set

$$W = \int_0^{+\infty} \frac{1}{g'(x)} f^2(x) dx, \quad Z = \int_0^{+\infty} \frac{1}{g'(x)} e^{\gamma g(x)} f^2(x) dx.$$

Then, decomposing suitably the integral of  $f$ , introducing  $\alpha$  and  $\beta$  to be chosen later, applying the Cauchy-Schwarz integral inequality, using the logarithmic primitive and the formula  $\log(x) - \log(y) = \log(x/y)$  for  $x, y > 0$ , we obtain

$$\begin{aligned} & \left[ \int_0^{+\infty} f(x) dx \right]^2 = \left[ \int_0^{+\infty} \sqrt{\frac{g'(x)}{\alpha + \beta e^{\gamma g(x)}}} \sqrt{\frac{\alpha + \beta e^{\gamma g(x)}}{g'(x)}} f(x) dx \right]^2 \\ & \leq \left[ \int_0^{+\infty} \frac{g'(x)}{\alpha + \beta e^{\gamma g(x)}} dx \right] \left[ \int_0^{+\infty} \frac{\alpha + \beta e^{\gamma g(x)}}{g'(x)} f^2(x) dx \right] \\ & = \left[ -\frac{1}{\alpha \gamma} \log [\beta + \alpha e^{-\gamma g(x)}] \right]_{x \rightarrow 0}^{x \rightarrow +\infty} (\alpha W + \beta Z) \\ & = \frac{1}{\alpha \gamma} \log \left( \frac{\beta + e^{-\gamma \ell} \alpha}{\beta + e^{-\gamma L} \alpha} \right) (\alpha W + \beta Z). \end{aligned}$$

With the choices  $\alpha = Z$  and  $\beta = W$ , this inequality becomes

$$\left[ \int_0^{+\infty} f(x) dx \right]^2 \leq \frac{2}{\alpha \gamma} \log \left( \frac{\beta + e^{-\gamma \ell} \alpha}{\beta + e^{-\gamma L} \alpha} \right) \alpha \beta = \frac{2}{\gamma} \log \left( \frac{\beta + e^{-\gamma \ell} \alpha}{\beta + e^{-\gamma L} \alpha} \right) \beta,$$

which can also be expressed as follows:

$$\begin{aligned} & \left[ \int_0^{+\infty} f(x) dx \right]^2 \leq \frac{2}{\gamma} \log \left[ \frac{\int_0^{+\infty} [1/g'(x)] f^2(x) dx + e^{-\gamma \ell} \int_0^{+\infty} [e^{\gamma g(x)}/g'(x)] f^2(x) dx}{\int_0^{+\infty} [1/g'(x)] f^2(x) dx + e^{-\gamma L} \int_0^{+\infty} [e^{\gamma g(x)}/g'(x)] f^2(x) dx} \right] \\ & \times \int_0^{+\infty} \frac{1}{g'(x)} f^2(x) dx. \end{aligned}$$

$$\times \int_0^{+\infty} \frac{1}{g'(x)} f^2(x) dx.$$

This ends the proof of Theorem 2.3.  $\square$

In particular, setting  $g(x) = x$ , so that  $L = +\infty$  and  $\ell = 0$ , we have

$$\left[ \int_0^{+\infty} f(x) dx \right]^2 \leq \frac{2}{\gamma} \log \left[ 1 + \frac{\int_0^{+\infty} e^{\gamma x} f^2(x) dx}{\int_0^{+\infty} f^2(x) dx} \right] \int_0^{+\infty} f^2(x) dx.$$

To the best of our knowledge, this result is new in the literature.

### 3. SPECIAL INTEGRAL INEQUALITIES

In [4], in addition to [4, Theorem 2.1], several new integral inequalities are proposed, with detailed proofs. In this section, we revisit one of these inequalities, offering alternative proofs and further extensions.

**3.1. Revisit of an existing theorem.** The result below, which is from [4], deals with the weighted integral of the main function in the upper bound.

**Theorem 3.1.** [4, Theorem 2.3] Let  $\kappa, \theta \in (0, 2)$  and  $f : [0, 1] \mapsto [0, +\infty)$  be a function. Then we have

$$\left[ \int_0^1 f(x) dx \right]^2 \leq B \left( 1 - \frac{\kappa}{2}, 1 - \frac{\theta}{2} \right) \sqrt{\int_0^1 x^\kappa f^2(x) dx} \sqrt{\int_0^1 (1-x)^\theta f^2(x) dx},$$

provided that the integrals of the upper bound converge, where  $B(a, b)$  is the beta function classically defined by  $B(a, b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt$  with  $a, b > 0$ .

A key to the proof in [4] is incorporating an adjustable parameter,  $\lambda$ , to decompose the integral of  $f$  using the Cauchy-Schwarz integral inequality (one time), the arithmetic-geometric inequality, and selecting  $\lambda$  appropriately.

The theorem below is our third contribution. It revisits this approach in a more direct way, proving a sharper inequality from which Theorem 3.1 can be derived.

**Theorem 3.2.** Let  $\kappa, \theta \in (0, 2)$  and  $f : [0, 1] \mapsto [0, +\infty)$  be a function. Then we have

$$\left[ \int_0^1 f(x) dx \right]^2 \leq B \left( 1 - \frac{\kappa}{2}, 1 - \frac{\theta}{2} \right) \int_0^1 x^{\kappa/2} (1-x)^{\theta/2} f^2(x) dx,$$

provided that the integrals of the upper bound converge. This inequality implies that in Theorem 3.1.

*Proof.* By decomposing the integral of  $f$  suitably, applying the Cauchy-Schwarz integral inequality, and recognizing the beta function, we obtain

$$\begin{aligned}
 \left[ \int_0^1 f(x) dx \right]^2 &= \left[ \int_0^1 x^{-\kappa/4} (1-x)^{-\theta/4} x^{\kappa/4} (1-x)^{\theta/4} f(x) dx \right]^2 \\
 &\leq \left[ \int_0^1 x^{-\kappa/2} (1-x)^{-\theta/2} dx \right] \left[ \int_0^1 x^{\kappa/2} (1-x)^{\theta/2} f^2(x) dx \right] \\
 &= \left[ \int_0^1 x^{(1-\kappa/2)-1} (1-x)^{(1-\theta/2)-1} dx \right] \left[ \int_0^1 x^{\kappa/2} (1-x)^{\theta/2} f^2(x) dx \right] \\
 (3.1) \quad &= B \left( 1 - \frac{\kappa}{2}, 1 - \frac{\theta}{2} \right) \int_0^1 x^{\kappa/2} (1-x)^{\theta/2} f^2(x) dx.
 \end{aligned}$$

Then, based on the integral of this upper bound, decomposing the integrand suitably and applying the Cauchy-Schwarz integral inequality, we get

$$\begin{aligned}
 \int_0^1 x^{\kappa/2} (1-x)^{\theta/2} f^2(x) dx &= \int_0^1 x^{\kappa/2} f(x) (1-x)^{\theta/2} f(x) dx \\
 (3.2) \quad &\leq \sqrt{\int_0^1 x^{\kappa} f^2(x) dx} \sqrt{\int_0^1 (1-x)^{\theta} f^2(x) dx}.
 \end{aligned}$$

Combining Equations (3.1) and (3.2), we obtain

$$\left[ \int_0^1 f(x) dx \right]^2 \leq B \left( 1 - \frac{\kappa}{2}, 1 - \frac{\theta}{2} \right) \sqrt{\int_0^1 x^{\kappa} f^2(x) dx} \sqrt{\int_0^1 (1-x)^{\theta} f^2(x) dx}.$$

We recognize Theorem 3.1, that is [4, Theorem 2.3]. This concludes the proof of Theorem 3.2.  $\square$

We have thus improved and significantly clarified the proof of [4, Theorem 2.3] by applying the Cauchy-Schwarz integral inequality twice instead of dealing with an additional parameter. Moreover, our approach can be extended in a more general framework, as developed in the next subsection.

**3.2. Some extensions.** The result below is our fourth contribution. It proposes a direct extension of Theorem 3.2 that introduces an adjustable parameter  $p$ . The application of the Hölder integral inequality is a key part of the proof.

**Theorem 3.3.** Let  $p > 1$ ,  $q = p/(p-1)$ ,  $\kappa, \theta \in (0, 2p/q)$  and  $f : [0, 1] \mapsto [0, +\infty)$  be a function. Then we have

$$\left[ \int_0^1 f(x) dx \right]^p \leq B^{p/q} \left( 1 - \frac{\kappa q}{2p}, 1 - \frac{\theta q}{2p} \right) \int_0^1 x^{\kappa/2} (1-x)^{\theta/2} f^p(x) dx$$

and

$$\left[ \int_0^1 f(x) dx \right]^p \leq B^{p/q} \left( 1 - \frac{\kappa q}{2p}, 1 - \frac{\theta q}{2p} \right) \sqrt{\int_0^1 x^{\kappa} f^p(x) dx} \sqrt{\int_0^1 (1-x)^{\theta} f^p(x) dx},$$

provided that the integrals of the upper bound converge.

*Proof.* By decomposing the integral of  $f$  suitably, applying the Hölder integral inequality, and recognizing the beta function taking into account that  $\kappa, \theta \in (0, 2p/q)$ , we obtain

$$\begin{aligned} (3.3) \quad & \left[ \int_0^1 f(x) dx \right]^p \\ &= \left[ \int_0^1 x^{-\kappa/(2p)} (1-x)^{-\theta/(2p)} x^{\kappa/(2p)} (1-x)^{\theta/(2p)} f(x) dx \right]^p \\ &\leq \left[ \int_0^1 x^{-\kappa q/(2p)} (1-x)^{-\theta q/(2p)} dx \right]^{p/q} \left[ \int_0^1 x^{\kappa/2} (1-x)^{\theta/2} f^p(x) dx \right] \\ &= \left[ \int_0^1 x^{[1-\kappa q/(2p)]-1} (1-x)^{[1-\theta q/(2p)]-1} dx \right]^{p/q} \left[ \int_0^1 x^{\kappa/2} (1-x)^{\theta/2} f^p(x) dx \right] \\ (3.4) \quad &= B^{p/q} \left( 1 - \frac{\kappa q}{2p}, 1 - \frac{\theta q}{2p} \right) \int_0^1 x^{\kappa/2} (1-x)^{\theta/2} f^p(x) dx. \end{aligned}$$

The first inequality is established.

For the second inequality, based on the integral of this upper bound, decomposing the integrand suitably and applying the Cauchy-Schwarz integral inequality, we get

$$\begin{aligned} & \int_0^1 x^{\kappa/2} (1-x)^{\theta/2} f^p(x) dx = \int_0^1 x^{\kappa/2} f^{p/2}(x) (1-x)^{\theta/2} f^{p/2}(x) dx \\ (3.5) \quad & \leq \sqrt{\int_0^1 x^{\kappa} f^p(x) dx} \sqrt{\int_0^1 (1-x)^{\theta} f^p(x) dx}. \end{aligned}$$

Combining Equations (3.3) and (3.5), we obtain

$$\left[ \int_0^1 f(x) dx \right]^p \leq B^{p/q} \left( 1 - \frac{\kappa q}{2p}, 1 - \frac{\theta q}{2p} \right) \sqrt[p]{\int_0^1 x^\kappa f^p(x) dx} \sqrt[p]{\int_0^1 (1-x)^\theta f^p(x) dx}.$$

This completes the proof of Theorem 3.3.  $\square$

Clearly, setting  $p = 2$ , Theorem 3.3 reduces to Theorem 3.2.

The theorem below is our fifth and last contribution. It is a modified version of Theorem 3.3, dealing with  $|1 - x|$  instead of  $1 - x$  and for  $x \in (0, +\infty)$  instead of  $x \in (0, 1)$ .

**Theorem 3.4.** *Let  $p > 1$ ,  $q = p/(p - 1)$ ,  $\eta, \xi \in (0, 2p/q)$  such that  $\eta + \xi > \frac{2p}{q}$ , and  $f : [0, +\infty) \mapsto [0, +\infty)$  be a function. Then we have*

$$\begin{aligned} \left[ \int_0^{+\infty} f(x) dx \right]^p &\leq \left[ B \left( 1 - \frac{\eta q}{2p}, 1 - \frac{\xi q}{2p} \right) + B \left( \frac{(\eta + \xi)q}{2p} - 1, 1 - \frac{\xi q}{2p} \right) \right]^{p/q} \\ &\quad \times \left[ \int_0^{+\infty} x^{\eta/2} |1 - x|^{\xi/2} f^p(x) dx \right] \end{aligned}$$

and

$$\begin{aligned} \left[ \int_0^{+\infty} f(x) dx \right]^p &\leq \left[ B \left( 1 - \frac{\eta q}{2p}, 1 - \frac{\xi q}{2p} \right) + B \left( \frac{(\eta + \xi)q}{2p} - 1, 1 - \frac{\xi q}{2p} \right) \right]^{p/q} \\ &\quad \times \sqrt[p]{\int_0^{+\infty} x^\eta f^p(x) dx} \sqrt[p]{\int_0^{+\infty} |1 - x|^\xi f^p(x) dx}, \end{aligned}$$

provided that the integrals of the upper bound converge.

*Proof.* By decomposing the integral of  $f$  suitably and applying the Hölder integral inequality, we obtain

$$\begin{aligned} (3.6) \quad &\left[ \int_0^{+\infty} f(x) dx \right]^p \\ &= \left[ \int_0^{+\infty} x^{-\eta/(2p)} |1 - x|^{-\xi/(2p)} x^{\eta/(2p)} |1 - x|^{\xi/(2p)} f(x) dx \right]^p \end{aligned}$$

$$(3.7) \quad \leq \left[ \int_0^{+\infty} x^{-\eta q/(2p)} |1 - x|^{-\xi q/(2p)} dx \right]^{p/q} \left[ \int_0^{+\infty} x^{\eta/2} |1 - x|^{\xi/2} f^p(x) dx \right].$$

Let us now develop the first integral of this upper bound. Using the Chasles integral relation at the splitting value  $x = 1$ , performing the change of variables

$x = 1/y$ , and recognizing the beta function two times taking into account that  $\eta, \xi \in (0, 2p/q)$  and  $\eta + \xi > 2p/q$ , we find that

$$\begin{aligned}
 & \int_0^{+\infty} x^{-\eta q/(2p)} |1-x|^{-\xi q/(2p)} dx \\
 &= \int_0^1 x^{-\eta q/(2p)} (1-x)^{-\xi q/(2p)} dx + \int_1^{+\infty} x^{-\eta q/(2p)} (x-1)^{-\xi q/(2p)} dx \\
 (3.8) \quad &= \int_0^1 x^{-\eta q/(2p)} (1-x)^{-\xi q/(2p)} dx \\
 & \quad + \int_1^0 \left(\frac{1}{y}\right)^{-\eta q/(2p)} \left(\frac{1}{y}-1\right)^{-\xi q/(2p)} \left(-\frac{1}{y^2}\right) dy \\
 &= \int_0^1 x^{-\eta q/(2p)} (1-x)^{-\xi q/(2p)} dx + \int_0^1 y^{(\eta+\xi)q/(2p)-2} (1-y)^{-\xi q/(2p)} dy \\
 (3.9) \quad &= \int_0^1 x^{[1-\eta q/(2p)]-1} (1-x)^{[1-\xi q/(2p)]-1} dx \\
 & \quad + \int_0^1 y^{[(\eta+\xi)q/(2p)-1]-1} (1-y)^{[1-\xi q/(2p)]-1} dy \\
 (3.10) \quad &= B\left(1 - \frac{\eta q}{2p}, 1 - \frac{\xi q}{2p}\right) + B\left(\frac{(\eta+\xi)q}{2p} - 1, 1 - \frac{\xi q}{2p}\right).
 \end{aligned}$$

Combining Equations (3.6) and (3.8), we get

$$\begin{aligned}
 & \left[ \int_0^{+\infty} f(x) dx \right]^p \leq \left[ B\left(1 - \frac{\eta q}{2p}, 1 - \frac{\xi q}{2p}\right) + B\left(\frac{(\eta+\xi)q}{2p} - 1, 1 - \frac{\xi q}{2p}\right) \right]^{p/q} \\
 (3.11) \quad & \quad \times \left[ \int_0^{+\infty} x^{\eta/2} |1-x|^{\xi/2} f^p(x) dx \right].
 \end{aligned}$$

The first inequality is established.

For the second inequality, based on the integral of this upper bound, decomposing the integrand suitably and applying the Cauchy-Schwarz integral inequality, we obtain

$$\begin{aligned}
 & \int_0^{+\infty} x^{\eta/2} |1-x|^{\xi/2} f^p(x) dx = \int_0^{+\infty} x^{\eta/2} f^{p/2}(x) |1-x|^{\xi/2} f^{p/2}(x) dx \\
 (3.12) \quad & \leq \sqrt{\int_0^{+\infty} x^{\eta} f^p(x) dx} \sqrt{\int_0^{+\infty} |1-x|^{\xi} f^p(x) dx}.
 \end{aligned}$$

Combining Equations (3.11) and (3.12), we get

$$\left[ \int_0^{+\infty} f(x) dx \right]^p \leq \left[ B \left( 1 - \frac{\eta q}{2p}, 1 - \frac{\xi q}{2p} \right) + B \left( \frac{(\eta + \xi)q}{2p} - 1, 1 - \frac{\xi q}{2p} \right) \right]^{p/q} \\ \times \sqrt{\int_0^{+\infty} x^\eta f^p(x) dx} \sqrt{\int_0^{+\infty} |1 - x|^\xi f^p(x) dx}.$$

This concludes the proof of Theorem 3.4.  $\square$

To the best of our knowledge, this result is a new addition to the literature on this topic.

#### 4. CONCLUSION

In this article, we have revisited two key theorems published by S. Barza and E. C. Popa in 1998. In particular, we presented five new contributions that reinforce and extend their findings. Our alternative proofs and refined inequalities provide a more in-depth understanding of the structure of Carlson-type integral inequalities. These findings may stimulate further research into related integral inequalities, including extensions to weighted settings, multidimensional cases, and non-Euclidean domains. Future work could also examine how these refined inequalities can be applied in fields such as functional analysis, harmonic analysis, and probability theory.

#### ACKNOWLEDGMENT

The author would like to thank the reviewers for their constructive comments.

#### REFERENCES

- [1] N. ALP, M.Z. SARIKAYA, H. BUDAK: *Hölder and Carlson type inequalities for the interval-valued functions*, Authorea, (2020), 1–13.
- [2] L.E. AZAR: *A relation between Hilbert and Carlson inequalities*, J. Inequal. Appl., **2012** (2012), 1–10.
- [3] S. BARZA, J. PECARIĆ, L.-E. PERSSON: *Carlson type inequalities*, J. Inequal. Appl., **2**(2) (1998), 121–135.

- [4] S. BARZA, E.C. POPA: *Inequalities related with Carlson's inequality*, Tamkang J. Math., **29**(1) (1998), 59–64.
- [5] F. CARLSON: *Une inégalité*, Ark. Mat. Astr. Fysik, **25B** (1934), 1–5.
- [6] B. DARABY: *Generalizations of related Fritz Carlson type inequalities for fuzzy integrals*, Sahand Commun. Math. Anal., **18**(4) (2021), 131–153.
- [7] G.H. HARDY: *A note on two inequalities*, J. London Math. Soc., **11** (1936), 167–170.
- [8] M. HLEILI, K. BRAHIM: *On Nash and Carlson's inequalities for symmetric  $q$ -integral transforms*, Complex Anal. Oper. Theory, **10** (2016), 1339–1350.
- [9] A. KAMALY: *Fritz Carlson's inequality and its application*, Math. Scand., **86**(1) (2000), 100–108.
- [10] L. LARSSON: *Carlson type inequalities and embeddings of interpolation spaces*, Proc. Amer. Math. Soc., **132**(8) (2004), 2351–2356.
- [11] L. LARSSON, L. MALIGRANDA, J. PECARIĆ, L.-E. PERSSON: *Multiplicative Inequalities of Carlson Type and Interpolation*, World Scientific Publication Co. Pty., Hackensack, NJ, (2006).
- [12] V.I. LEVIN: *Exact constants in inequalities of the Carlson type*, Doklady Akad. Nauk SSSR (N.S.), **59** (1948), 635–638. (in Russian)
- [13] M. LUO, K.R. RAINA: *A new extension of Carlson's inequality*, Math. Inequal. Appl., **19**(2) (2016), 417–424.
- [14] B. TSERENDORJ, L.E. AZAR: *A new Carlson type sharp inequality*, J. Math. Ext., **17**(5) (2021), 1–12.

DEPARTMENT OF MATHEMATICS

UNIVERSITY OF CAEN-NORMANDIE

UFR DES SCIENCES - CAMPUS 2, CAEN

FRANCE.

*Email address:* christophe.chesneau@gmail.com